East Asian Math. J. **22** (2006), No. 1, pp. 53–59

SOME BOUNDARY BEHAVIOR OF CONFORMAL MAPPINGS

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Abstract. We introduce the resistant length and examine its properties. And consider the applications of resistant length to the boundary behavior of conformal mappings. We obtain the theorems in the connection with the resistant length zero and the fundamental sequences. The method of resistant length leads a simple proofs of theorems.

1. Resistant length

Resistant length was introduced as a conformally invariant measure of curve families. Throughout this paper, $\mathbb{C}$ will denote the finite complex plane, D is a domain in $\mathbb{C}$.

Let $\{\gamma\}$ be a family whose elements γ are locally rectifiable curves (simply, curves or arcs) in D. We shall introduce a geometric guantity $\lambda(\gamma)$, called the resistant length of Γ.

Let $\rho(z)$ be a non-negative real-valued function defined on D. We set

$$A_D(\rho) = A(\rho) = \iint_D \rho^2(z)\, dxdy.$$

If there is countable sequence $\{\gamma_i\}$ of disjoint rectifiable arcs, which are parameterized by their arc lengths (i.e., the arc γ_i is given by $z = z_i(s_i)$, where $0 \leq s_i \leq l_i$, and l_i is the length of γ_i), such that $(\gamma) = \cup_i^\infty(\gamma_i)$, we set

Received June 15, 2002.

2000 Mathematics Subject Classification: 30C35, 30C62.

Key words and phrases: Resistant length, fundamental sequence.

$$l_\rho(\gamma) = \sum_{i=1}^{\infty} \int_{\gamma_i} \rho(z_i(s_i))\, ds_i.$$

We introduce

$$L(\rho) = L_{\{\gamma\}}(\rho) = \inf_{\gamma \in \{\gamma\}} l_\rho(\gamma),$$

where $L_{\{\gamma\}}(\rho) = \infty$, the family $\{\gamma\}$ is empty.

DEFINITION 1.1 ([1]). The function $\rho(z)$ shall be called *admissible* with respect to D and $\{\gamma\}$, if $A(\rho)$ and $L(\rho)$ are not both zero or both infinite.

DEFINITION 1.2 ([1]). The quantity

$$\lambda_D(\gamma) = \lambda(\gamma) = \sup \frac{L^2(\rho)}{A(\rho)}, \tag{1}$$

where the supremum is taken over all admissible functions $\rho(z)$, is called the *resistant length* of the family of curves $\{\gamma\}$ with respect to the domain D.

PROPOSITION 1.3 ([9]). *Let B be the interior of the annulus formed by two concentric circles of radii r_1 and r_2 ($r_1 < r_2$). The family $\{\gamma\}$ of curves of B which connect the two circles has resistant length*

$$\lambda(\gamma) = \frac{1}{2\pi} \log \frac{r_2}{r_1}, \tag{2}$$

while the family $\{\gamma^*\}$ of simple closed curves of B which separate the two boundary components has resistant length

$$\lambda(\gamma^*) = 2\pi (\log \frac{r_2}{r_1})^{-1}. \tag{3}$$

The following theorem is an immediate consequence of the definition.

THEOREM 1.4 ([10]). (Conformal invariance of resistant length) *Let $z^* = f(z)$ be a 1-1 conformal mapping on D upon a domain D^* and $\{\gamma\}$ be a family of curves in D, then*

$$\lambda(\gamma) = \lambda(f(\gamma)).$$

THEOREM 1.5. *Let $\{\gamma\}$ and $\{\gamma'\}$ be two families of curves of a domain D such that each $\gamma \in \{\gamma\}$ contains a $\gamma' \in \{\gamma'\}$. Then*

$$\lambda(\gamma) \geq \lambda(\gamma').$$

Proof. Let $\varepsilon > 0$ be given and let $\rho'(z)$ be admissible for the family $\{\gamma'\}$, subject to the condition that

$$\frac{L^2_{\{\gamma'\}}(\rho')}{A(\rho')} > \lambda(\gamma') - \varepsilon. \tag{4}$$

Define a function $\rho_0(z)$ in D by

$$\rho_0(z) = \begin{cases} \rho'(z) & z \text{ belongs to some } \gamma' \\ 0 & \text{otherwise} \end{cases}.$$

For $\gamma \in \{\gamma\}$, let γ' be one of the elements of $\{\gamma'\}$ contained in γ. Then

$$l_{\rho_0}(\gamma) \geq L_{\{\gamma'\}}(\rho'), \tag{5}$$

since either $l_{\rho_0}(\gamma) = \infty$, or

$$l_{\rho_0}(\gamma) = \int_\gamma \rho_0 \, ds \geq \int_{\gamma'} \rho' \, ds = l_{\rho'}(\gamma') \geq L_{\{\gamma'\}}(\rho').$$

For each $\gamma \in \{\gamma\}$, (5) holds, then

$$L_{\{\gamma\}}(\rho_0) \geq L_{\{\gamma'\}}(\rho'). \tag{6}$$

Also, we have

$$A(\rho_0) \leq A(\rho'). \tag{7}$$

Show that ρ_0 is admissible for the family $\{\gamma\}$. If $\lambda(\gamma') = 0$, the theorem holds trivially. Assume that $A(\rho') < \infty$, and $A(\rho_0) < \infty$.

If $A(\rho_0) = 0$, then either (i) $A(\rho') = 0$, or (ii) $A(\rho') > 0$. In case (i), $L_{\{\gamma'\}}(\rho') > 0$, since ρ' is admissible for $\{\gamma'\}$, while in case (ii), we choose ε so that $0 < \varepsilon < \lambda(\gamma')$ in (4), and we conclude from that formula that $L_{\{\gamma'\}}(\rho') > 0$. In either case, therefore, $A(\rho_0) = 0$ implies, by (6), that $L_{\{\gamma\}}(\rho_0) > 0$; hence, ρ_0 is admissible for $\{\gamma\}$. By (6) and (7), we obtain

$$\frac{L^2_{\{\gamma\}}(\rho_0)}{A(\rho_0)} \geq \frac{L^2_{\{\gamma\}}(\rho')}{A(\rho')},$$

and therefore, by (4), $\lambda(\gamma) > \lambda(\gamma') - \varepsilon$, then

$$\lambda(\gamma) \geq \lambda(\gamma').$$

This completes the proof of the theorem. □

Briefly, the set $\{\gamma\}$ of fewer or longer arcs has the larger resistant length,([8]).

COROLLARY 1.6 ([7]). *In particular, if $\{\gamma\}$ is contained in $\{\gamma'\}$, then*

$$\lambda(\gamma) \geq \lambda(\gamma').$$

2. Boundary behavior

An important application of resistant length is to the boundary correspondence between two simply connected domains which are mapped conformally on each other.

DEFINITION 2.1 ([4]). A *crosscut* of D is a Jordan curve γ in D which in both directions tends to a boundary point.

It is well known that $D - \gamma$ consists of two simply connected components.

DEFINITION 2.2 ([6]). Choose a fixed $z_0 \in D$ and consider sequences $P = \{p_n\}$ of points in D. With the sequence P, we associate the family $\mathbf{\Gamma}_p$ of all clusters of crosscuts of D which separate z_0 from almost all p_n. The sequence P is said to be *fundamental* if $\lambda(\Gamma_p) = 0$.
The definition is independent of the choice of z_0.

Recall that $\lambda(\mathbf{\Gamma}_p) = 0$ if and only if $L(\rho) = 0$ for all ρ with $A(\rho) < \infty$.

LEMMA 2.3 ([5]). *Let D and D^* be simply connected domains in the complex plane, and consider sequence $P = \{p_n\}$ of points in D. Let f be a conformal mapping from D to D^*. Then the sequences $\{p_n\}$ and $\{f(p_n)\}$ are simultaneously fundamental.*

LEMMA 2.4. *Let $P = \{p_n\}$ be a fundamental sequence in D. Then all accumulation points of the $P = \{p_n\}$ lie on the boundary of D.*

Proof. Let $\alpha \in D$ be an accumulation point of P. Then $\alpha \notin E$, since otherwise $\{\gamma\}$ would be empty, and $\lambda(\gamma) = \infty$, where E is the set of z_0 in definition 2.2. Therefore, the open set $D - E$ contains a closed disk F with center α such that $p_n \in F$ for an infinite number of indices. If ξ is a curve contained in D which connects α to a point β of E, then every crosscut γ of the family $\{\gamma\}$ intersects the continuum $T = \xi \cup F$. T has positive spherical distance δ from the boundary. Let $\rho^*(z) = \psi(z, \infty)$, where $\psi(z, w)$ is the spherical distance. Then, for each $\gamma \in \{\gamma\}$, $l_{\rho^*}(\gamma) \geq 2\delta$. Hence $L_{\{\gamma\}}(\rho^*) \geq 2\delta$. Since $0 < A(\rho^*) < \pi$, ρ^* is admissible. We have

$$\lambda(\gamma) \geq \frac{L^2(\rho^*)}{A(\rho^*)} \geq \frac{4}{\pi} \cdot \delta^2 > 0.$$

This contradiction shows that α cannot be an interior point of D. This completes the proof. □

LEMMA 2.5. *A sequence $P = \{p_n\}$ of points of $\Delta = \{z||z| < 1\}$ is a fundamental if and only if it converges to a point $\zeta = e^{i\theta}$ of the boundary of Δ.*

Proof. Let $\lim_{n\to\infty} p_n = \zeta$, and set $r_j = |p_j - \zeta|$, $C_j = \{z||z - \zeta| = r_j\}$. Then the sequence $\{r_j\}$ converges to 0. By restriction to a suitable subsequence of $\{p_n\}$, we assume that C_1 separates C_n, $(n > 1)$ from z_0. Let $\{\gamma\}$ be the family of crosscuts of Δ which separate z_0 from almost all p_n. For j, every simple closed curve, which separates the C_0 and C_j, contains a crosscut which separates z_0 from almost all p_n and which belongs to the family $\{\gamma\}$. Let $\{\delta_k\}$ be the family of all such simple closed curves by Proposition 1.3 and Theorem 1.5, we have

$$\lambda(\gamma) \leq \lambda(\delta_k) = 2\pi(\log \frac{r_0}{r_k})^{-1}.$$

The right side converses to zero as $k \to \infty$, and the result, $\lambda(\gamma) = 0$. Thus, $P = \{p_n\}$ is a fundamental sequence in D.

On the other hand, if $\zeta_1 = e^{i\theta_1}$ and $\zeta_2 = e^{i\theta_2}$ are distinct points of the boundary. Let $\{p_m\}$ and $\{q_n\}$ be sequences converging to ζ_1 and ζ_2, respectively. Let $z_0 = 0$. Each member of the family $\{\gamma\}$ of crosscuts of Δ which separate z_0 from almost all points of the join of our two sequences have Euclidean length at least $|\zeta_1 - \zeta_2|$. Denoting by ρ_0 the function $\rho_0(z) = 1$ for all z of Δ, we have $L_{\{\gamma\}}(\rho_0) \geq |\zeta_1 - \zeta_2|$, while $A(\rho_0) = \pi$; hence,

$$\lambda(\gamma) \geq \frac{1}{\pi}|\zeta_1 - \zeta_2|^2 > 0.$$

A sequence of points of Δ with more than one accumulation point on the boundary cannot be a fundamental sequence. This completes the proof. □

THEOREM 2.6. *Let D be a Jordan domain in the complex plane, and let $w = f(z)$ be a conformal mapping on $\Delta = \{z||z| < 1\}$ upon D. Let w_0 be a point on the boundary of D, and consider a sequence $P = \{w_n\}$ of points of D converging to w_0. Let $f^{-1}(P) = \{z_n\}$ be a inverse image of P. Then the sequence of points $\{z_n\}$ has no point of accumulation in the interior of Δ, and $\{z_n\}$ has one and only one point of accumulation z_0 on the boundary of Δ.*

Proof. Since w_0 is a point on the boundary of D, $P = \{w_n\}$ is a fundamental by Lemma 2.4. Thus, $\{z_n\}$ is a fundamental by Lemma 2.3. Therefore, point of accumulation z_0 of $\{z_n\}$ exist on the boundary of Δ, by Lemma 2.4.

On the other hand, z_0 is one and only one point of accumulation by Lemma 2.5. This completes the proof of the theorem. □

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