

**ADDITIVE REVERSES OF THE CONTINUOUS  
TRIANGLE INEQUALITY FOR BOCHNER INTEGRAL  
OF VECTOR-VALUED FUNCTIONS IN BANACH  
SPACES**

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ABSTRACT. Some additive reverses of the continuous triangle inequality for Bochner integral of vector-valued functions in Banach spaces are given. Applications for complex-valued functions are considered as well.

**1. Introduction**

Let  $f : [a, b] \rightarrow \mathbb{K}$ ,  $\mathbb{K} = \mathbb{C}$  or  $\mathbb{R}$  be a Lebesgue integrable function. The following inequality, which is the continuous version of the *triangle inequality*

$$(1.1) \quad \left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx,$$

plays a fundamental role in Mathematical Analysis and its applications.

It appears, see [11, p. 492], that the first reverse inequality for (1.1) was obtained by J. Karamata in his book from 1949, [9]. It can be stated as

$$(1.2) \quad \cos \theta \int_a^b |f(x)| dx \leq \left| \int_a^b f(x) dx \right|$$

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provided

$$-\theta \leq \arg f(x) \leq \theta, \quad x \in [a, b]$$

for given  $\theta \in (0, \frac{\pi}{2})$ .

This result has recently been extended by the author for the case of Bochner integrable functions with values in a Hilbert space  $H$ . If by  $L([a, b]; H)$ , we denote the space of Bochner integrable functions with values in a Hilbert space  $H$ , i.e., we recall that  $f \in L([a, b]; H)$  if and only if  $f : [a, b] \rightarrow H$  is Bochner measurable on  $[a, b]$  and the Lebesgue integral  $\int_a^b \|f(t)\| dt$  is finite, then [7]

$$(1.3) \quad \int_a^b \|f(t)\| dt \leq K \left\| \int_a^b f(t) dt \right\|,$$

provided that  $f$  satisfies the condition

$$(1.4) \quad \|f(t)\| \leq K \operatorname{Re} \langle f(t), e \rangle \quad \text{for a.e. } t \in [a, b],$$

where  $e \in H$ ,  $\|e\| = 1$  and  $K \geq 1$  are given. The case of equality holds in (1.4) if and only if

$$(1.5) \quad \int_a^b f(t) dt = \frac{1}{K} \left( \int_a^b \|f(t)\| dt \right) e.$$

As some natural consequences of the above results, we have noted in [7] that, if  $\rho \in (0, 1)$  and  $f \in L([a, b]; H)$  are such that

$$(1.6) \quad \|f(t) - e\| \leq \rho \quad \text{for a.e. } t \in [a, b],$$

then

$$(1.7) \quad \sqrt{1 - \rho^2} \int_a^b \|f(t)\| dt \leq \left\| \int_a^b f(t) dt \right\|$$

with equality if and only if

$$\int_a^b f(t) dt = \sqrt{1 - \rho^2} \left( \int_a^b \|f(t)\| dt \right) \cdot e.$$

Also, for  $e$  as above and if  $M \geq m > 0$ ,  $f \in L([a, b]; H)$  such that either

$$(1.8) \quad \operatorname{Re} \langle Me - f(t), f(t) - me \rangle \geq 0$$

or, equivalently,

$$(1.9) \quad \left\| f(t) - \frac{M+m}{2}e \right\| \leq \frac{1}{2}(M-m)$$

for a.e.  $t \in [a, b]$ , then

$$(1.10) \quad \int_a^b \|f(t)\| dt \leq \frac{M+m}{2\sqrt{mM}} \left\| \int_a^b f(t) dt \right\|,$$

with equality if and only if

$$\int_a^b f(t) dt = \frac{2\sqrt{mM}}{M+m} \left( \int_a^b \|f(t)\| dt \right) \cdot e.$$

The above results have been generalised by the author to the case of Banach spaces [4].

Let  $F$  be a continuous linear functional of unit norm on the real or complex Banach space  $(X, \|\cdot\|)$ . If the function  $f : [a, b] \rightarrow X$  is Bochner integrable on  $[a, b]$  and there exists a  $r \geq 0$  such that

$$(1.11) \quad r \|f(t)\| \leq \operatorname{Re} [F(f(t))] \quad \text{for a.e. } t \in [a, b],$$

then

$$(1.12) \quad r \int_a^b \|f(t)\| dt \leq \left\| \int_a^b f(t) dt \right\|,$$

with equality if and only if both

$$(1.13) \quad \begin{aligned} F \left( \int_a^b f(t) dt \right) &= r \int_a^b \|f(t)\| dt \quad \text{and} \\ F \left( \int_a^b f(t) dt \right) &= \left\| \int_a^b f(t) dt \right\|. \end{aligned}$$

If  $(X, \|\cdot\|)$  is a strictly convex Banach space and  $[\cdot, \cdot]$  is a semi-inner product [2] generating the norm  $\|\cdot\|$  while  $e \in X$  is such that  $\|e\| = 1$  and

$$(1.14) \quad r \|f(t)\| \leq \operatorname{Re} [f(t), e] \quad \text{for a.e. } t \in [a, b],$$

then the reverse for the continuous triangle inequality (1.12) holds true with equality if and only if

$$(1.15) \quad \int_a^b f(t) dt = r \left( \int_a^b \|f(t)\| dt \right) \cdot e.$$

For other reverses in terms of  $m$  bounded linear functionals, see Section 3 of [4].

The aim of this paper is to provide a different approach to the problem of reversing the continuous triangle inequality. Namely, we are interested in finding upper bounds for the positive difference

$$\int_a^b \|f(t)\| dt - \left\| \int_a^b f(t) dt \right\|$$

under various assumptions for the Bochner integrable function  $f : [a, b] \rightarrow X$ .

Applications for complex-valued functions are given as well.

## 2. Reverses of the Continuous Triangle Inequality

In the following we provide an alternative of the reverse for the continuous triangle inequality established in [4].

**THEOREM 1.** *Let  $(X, \|\cdot\|)$  be a Banach space over the real or complex number field  $\mathbb{K}$  and  $F : X \rightarrow \mathbb{K}$  be a continuous linear functional of unit norm on  $X$ . Suppose that the function  $f : [a, b] \rightarrow X$  is Bochner integrable on  $[a, b]$  and there exists a Lebesgue integrable function  $k : [a, b] \rightarrow [0, \infty)$  such that*

$$(2.1) \quad \|f(t)\| - \operatorname{Re} F[f(t)] \leq k(t)$$

for a.e.  $t \in [a, b]$ . Then we have the inequality

$$(2.2) \quad (0 \leq) \int_a^b \|f(t)\| dt - \left\| \int_a^b f(t) dt \right\| \leq \int_a^b k(t) dt.$$

The equality holds in (2.2) if and only if both

$$(2.3) \quad F \left( \int_a^b f(t) dt \right) = \left\| \int_a^b f(t) dt \right\|$$

and

$$(2.4) \quad F\left(\int_a^b f(t) dt\right) = \int_a^b \|f(t)\| dt - \int_a^b k(t) dt.$$

*Proof.* Since the norm of  $F$  is unity, then

$$|F(x)| \leq \|x\| \quad \text{for any } x \in X.$$

Applying this inequality for the vector  $\int_a^b f(t) dt$ , we get

$$(2.5) \quad \begin{aligned} \left\|\int_a^b f(t) dt\right\| &\geq \left|F\left(\int_a^b f(t) dt\right)\right| \\ &\geq \left|\operatorname{Re}F\left(\int_a^b f(t) dt\right)\right| \\ &= \left|\int_a^b \operatorname{Re}F[f(t)] dt\right| \geq \int_a^b \operatorname{Re}F[f(t)] dt. \end{aligned}$$

Integrating (2.1), we have

$$(2.6) \quad \int_a^b \|f(t)\| dt - \operatorname{Re}F\left(\int_a^b f(t) dt\right) \leq \int_a^b k(t) dt.$$

Now, making use of (2.5) and (2.6), we deduce (2.2).

Obviously, if the equality hold in (2.3) and (2.4), then it holds in (2.2) as well. Conversely, if the equality holds in (2.2), then it must hold in all the inequalities used to prove (2.2). Therefore, we have

$$\int_a^b \|f(t)\| dt = \operatorname{Re}\left[F\left(\int_a^b f(t) dt\right)\right] + \int_a^b k(t) dt.$$

and

$$\operatorname{Re}\left[F\left(\int_a^b f(t) dt\right)\right] = \left|F\left(\int_a^b f(t) dt\right)\right| = \left\|\int_a^b f(t) dt\right\|$$

which imply (2.3) and (2.4).  $\square$

Before we state some corollaries for semi-inner products, we recall some concepts and results that are needed.

In 1961, G. Lumer [10] introduced the following concept.

**DEFINITION 1.** Let  $X$  be a linear space over the real or complex number field  $\mathbb{K}$ . The mapping  $[\cdot, \cdot] : X \times X \rightarrow \mathbb{K}$  is called a *semi-inner product* on  $X$ , if the following properties are satisfied (see also [2, p. 17]):

- (i)  $[x + y, z] = [x, z] + [y, z]$  for all  $x, y, z \in X$ ;
- (ii)  $[\lambda x, y] = \lambda [x, y]$  for all  $x, y \in X$  and  $\lambda \in \mathbb{K}$ ;
- (iii)  $[x, x] \geq 0$  for all  $x \in X$  and  $[x, x] = 0$  imply  $x = 0$ ;
- (iv)  $|[x, y]|^2 \leq [x, x][y, y]$  for all  $x, y \in X$ ;
- (v)  $[x, \lambda y] = \lambda [x, y]$  for all  $x, y \in X$  and  $\lambda \in \mathbb{K}$ .

It is well known that the mapping  $X \ni x \mapsto [x, x]^{\frac{1}{2}} \in \mathbb{R}$  is a norm on  $X$  and for any  $y \in X$ , the functional  $X \ni x \mapsto [x, x]^{\frac{1}{2}} \in \mathbb{K}$  is a continuous linear functional on  $X$  endowed with the norm  $\|\cdot\|$  generated by  $[\cdot, \cdot]$ . Moreover, one has  $\|\varphi_y\| = \|y\|$  (see for instance [2, p. 17]).

Let  $(X, \|\cdot\|)$  be a real or complex normed space. If  $J : X \rightarrow 2^{X^*}$  is the *normalised duality mapping* defined on  $X$ , i.e., we recall that (see for instance [2, p. 1])

$$J(x) = \{\varphi \in X^* \mid \varphi(x) = \|\varphi\| \|x\|, \|\varphi\| = \|x\|\}, \quad x \in X,$$

then we may state the following representation result (see for instance [2, p. 18]):

*Each semi-inner product  $[\cdot, \cdot] : X \times X \rightarrow K$  that generates the norm  $\|\cdot\|$  of the normed linear space  $(X, \|\cdot\|)$  is of the form*

$$[x, y] = \left\langle \tilde{J}(y), x \right\rangle \quad \text{for any } x, y \in X,$$

where  $\tilde{J}$  is a selection of the normalised duality mapping and  $\langle \varphi, x \rangle := \varphi(x)$  for  $\varphi \in X^*$  and  $x \in X$ .

**COROLLARY 1.** Let  $(X, \|\cdot\|)$  be a Banach space,  $[\cdot, \cdot] : X \times X \rightarrow \mathbb{K}$  a semi-inner product which generates its norm. If  $e \in X$  is such that  $\|e\| = 1$ ,  $f : [a, b] \rightarrow X$  is Bochner integrable on  $[a, b]$  and there exists a Lebesgue integrable function  $k : [a, b] \rightarrow [0, \infty)$  such that

$$(2.7) \quad (0 \leq) \|f(t)\| - \operatorname{Re} [f(t), e] \leq k(t),$$

for a.e.  $t \in [a, b]$ , then

$$(2.8) \quad (0 \leq) \int_a^b \|f(t)\| dt - \left\| \int_a^b f(t) dt \right\| \leq \int_a^b k(t) dt.$$

where equality holds in (2.8) if and only if both

$$(2.9) \quad \left[ \int_a^b f(t) dt, e \right] = \left\| \int_a^b f(t) dt \right\| \quad \text{and} \\ \left[ \int_a^b f(t) dt, e \right] = \left\| \int_a^b f(t) dt \right\| - \int_a^b k(t) dt.$$

The proof is obvious by Theorem 1 applied for the continuous linear functional of unit norm  $F_e : X \rightarrow \mathbb{K}$ ,  $F_e(x) = [x, e]$ .

In order to provide a simpler necessary and sufficient condition of equality in (2.8), we need to recall the concept of strictly convex spaces.

**DEFINITION 2.** A normed linear space  $(X, \|\cdot\|)$  is said to be strictly convex if for every  $x, y$  from  $X$  with  $x \neq y$  and  $\|x\| = \|y\| = 1$ , we have  $\|\lambda x + (1 - \lambda)y\| < 1$  for all  $\lambda \in (0, 1)$ .

The following characterisation of strictly convex spaces is useful in what follows (see [1], [8], [12] or [2, p. 21]).

**THEOREM 2.** Let  $(X, \|\cdot\|)$  be a normed linear space over  $\mathbb{K}$  and  $[\cdot, \cdot]$  a semi-inner product generating its norm. The following statements are equivalent:

- (i)  $(X, \|\cdot\|)$  is strictly convex;
- (ii) For every  $x, y \in X$ ,  $x, y \neq 0$  with  $[x, y] = \|x\| \|y\|$ , there exists a  $\lambda > 0$  such that  $x = \lambda y$ .

The following corollary may be stated.

**COROLLARY 2.** Let  $(X, \|\cdot\|)$  be a strictly convex Banach space, and  $[\cdot, \cdot]$ ,  $e$ ,  $f$ ,  $k$  as in Corollary 1. Then the case of equality holds in (2.8) if and only if

$$(2.10) \quad \int_a^b \|f(t)\| dt \geq \int_a^b k(t) dt$$

and

$$(2.11) \quad \int_a^b f(t) dt = \left( \int_a^b \|f(t)\| dt - \int_a^b k(t) dt \right) e.$$

*Proof.* Suppose that (2.10) and (2.11) are valid. Taking the norm on (2.11) we have

$$\begin{aligned} \left\| \int_a^b f(t) dt \right\| &= \left| \int_a^b \|f(t)\| dt - \int_a^b k(t) dt \right| \|e\| \\ &= \int_a^b \|f(t)\| dt - \int_a^b k(t) dt, \end{aligned}$$

and the case of equality holds true in (2.8).

Now, if the equality case holds in (2.8), then obviously (2.10) is valid, and by Corollary 1,

$$\left[ \int_a^b f(t) dt, e \right] = \left\| \int_a^b f(t) dt \right\| \|e\|.$$

Utilising Theorem 2, we get

$$(2.12) \quad \int_a^b f(t) dt = \lambda e \quad \text{with } \lambda > 0.$$

Replacing  $\int_a^b f(t) dt$  with  $\lambda e$  in the second equation of (2.9) we deduce

$$(2.13) \quad \lambda = \int_a^b \|f(t)\| dt - \int_a^b k(t) dt,$$

and by (2.12) and (2.13) we deduce (2.11).  $\square$

**REMARK 1.** If  $X = H$ ,  $(H; \langle \cdot, \cdot \rangle)$  is a Hilbert space, then from Corollary 2 we deduce the additive reverse inequality obtained in [5]. For further similar results in Hilbert spaces, see [5] and [6].

### 3. Reverses for $m$ Functionals

The following result may be stated:

**THEOREM 3.** *Let  $(X, \|\cdot\|)$  be a Banach space over the real or complex number field  $\mathbb{K}$  and  $F_k : X \rightarrow \mathbb{K}$ ,  $k \in \{1, \dots, m\}$  continuous linear functionals on  $X$ . If  $f : [a, b] \rightarrow X$  is a Bochner integrable function on  $[a, b]$  and  $M_k : [a, b] \rightarrow [0, \infty)$ ,  $k \in \{1, \dots, m\}$  are Lebesgue integrable functions such that*

$$(3.1) \quad \|f(t)\| - \operatorname{Re} F_k[f(t)] \leq M_k(t)$$

for each  $k \in \{1, \dots, m\}$  and a.e.  $t \in [a, b]$ , then

$$(3.2) \quad \int_a^b \|f(t)\| dt \leq \left\| \frac{1}{m} \sum_{k=1}^m F_k \right\| \left\| \int_a^b f(t) dt \right\| + \frac{1}{m} \sum_{k=1}^m \int_a^b M_k(t) dt.$$

The case of equality holds in (3.2) if and only if both

$$(3.3) \quad \frac{1}{m} \sum_{k=1}^m F_k \left( \int_a^b f(t) dt \right) = \left\| \frac{1}{m} \sum_{k=1}^m F_k \right\| \left\| \int_a^b f(t) dt \right\|$$

and

$$(3.4) \quad \frac{1}{m} \sum_{k=1}^m F_k \left( \int_a^b f(t) dt \right) = \int_a^b \|f(t)\| dt - \frac{1}{m} \sum_{k=1}^m \int_a^b M_k(t) dt.$$

*Proof.* If we integrate on  $[a, b]$  and sum over  $k$  from 1 to  $m$ , we deduce

$$(3.5) \quad \int_a^b \|f(t)\| dt \leq \frac{1}{m} \sum_{k=1}^m \operatorname{Re} \left[ F_k \left( \int_a^b f(t) dt \right) \right] + \frac{1}{m} \sum_{k=1}^m \int_a^b M_k(t) dt.$$

Utilising the continuity property of the functionals  $F_k$  and the properties of the modulus, we have:

$$(3.6) \quad \begin{aligned} \sum_{k=1}^m \operatorname{Re} F_k \left( \int_a^b f(t) dt \right) &\leq \left| \sum_{k=1}^m \operatorname{Re} \left[ F_k \left( \int_a^b f(t) dt \right) \right] \right| \\ &\leq \left| \sum_{k=1}^m F_k \left( \int_a^b f(t) dt \right) \right| \\ &\leq \left\| \sum_{k=1}^m F_k \right\| \left\| \int_a^b f(t) dt \right\|. \end{aligned}$$

Now, by (3.5) and (3.6) we deduce (3.2).

Obviously, if (3.3) and (3.4) hold true, then the case of equality is valid in (3.2).

Conversely, if the case of equality holds in (3.2), then it must hold in all the inequalities used to prove (3.2). Therefore, we have

$$\begin{aligned} \int_a^b \|f(t)\| dt &= \frac{1}{m} \sum_{k=1}^m \operatorname{Re} \left[ F_k \left( \int_a^b f(t) dt \right) \right] + \frac{1}{m} \sum_{k=1}^m \int_a^b M_k(t) dt, \\ \sum_{k=1}^m \operatorname{Re} \left[ F_k \left( \int_a^b f(t) dt \right) \right] &= \left\| \int_a^b f(t) dt \right\| \left\| \sum_{k=1}^m F_k \right\| \end{aligned}$$

and

$$\sum_{k=1}^m \operatorname{Im} \left[ F_k \left( \int_a^b f(t) dt \right) \right] = 0.$$

These imply that (3.3) and (3.4) hold true, and the theorem is completely proved.  $\square$

**REMARK 2.** If  $F_k$ ,  $k \in \{1, \dots, m\}$  are of unit norm, then, from (3.2) we deduce the inequality

$$(3.7) \quad \int_a^b \|f(t)\| dt \leq \left\| \int_a^b f(t) dt \right\| + \frac{1}{m} \sum_{k=1}^m \int_a^b M_k(t) dt,$$

which is obviously coarser than (3.2) but, perhaps more useful for applications.

The case of Hilbert spaces, in which one may provide a simpler condition for equality, is of interest in applications.

**THEOREM 4.** *Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space over the real or complex number field  $\mathbb{K}$  and  $e_k \in H$ ,  $k \in \{1, \dots, m\}$ . If  $f : [a, b] \rightarrow H$  is a Bochner integrable function on  $[a, b]$ ,  $f(t) \neq 0$  for a.e.  $t \in [a, b]$  and  $M_k : [a, b] \rightarrow [0, \infty)$ ,  $k \in \{1, \dots, m\}$  is a Lebesgue integrable function such that*

$$(3.8) \quad \|f(t)\| - \operatorname{Re} \langle f(t), e_k \rangle \leq M_k(t)$$

for each  $k \in \{1, \dots, m\}$  and for a.e.  $t \in [a, b]$ , then

$$(3.9) \quad \int_a^b \|f(t)\| dt \leq \left\| \frac{1}{m} \sum_{k=1}^m e_k \right\| \left\| \int_a^b f(t) dt \right\| + \frac{1}{m} \sum_{k=1}^m \int_a^b M_k(t) dt.$$

The case of equality holds in (3.9) if and only if

$$(3.10) \quad \int_a^b \|f(t)\| dt \geq \frac{1}{m} \sum_{k=1}^m \int_a^b M_k(t) dt$$

and

$$(3.11) \quad \int_a^b f(t) dt = \frac{m \left( \int_a^b \|f(t)\| dt - \frac{1}{m} \sum_{k=1}^m \int_a^b M_k(t) dt \right)}{\left\| \sum_{k=1}^m e_k \right\|^2} \sum_{k=1}^m e_k.$$

*Proof.* As in the proof of Theorem 3, we have

$$(3.12) \quad \int_a^b \|f(t)\| dt \leq \operatorname{Re} \left\langle \frac{1}{m} \sum_{k=1}^m e_k, \int_a^b f(t) dt \right\rangle + \frac{1}{m} \sum_{k=1}^m \int_a^b M_k(t) dt$$

and  $\sum_{k=1}^m e_k \neq 0$ .

On utilising Schwarz's inequality in Hilbert space  $(H, \langle \cdot, \cdot \rangle)$  for  $\int_a^b f(t) dt$  and  $\sum_{k=1}^m e_k$ , we have

$$(3.13) \quad \left\| \int_a^b f(t) dt \right\| \left\| \sum_{k=1}^m e_k \right\| \geq \left| \left\langle \int_a^b f(t) dt, \sum_{k=1}^m e_k \right\rangle \right| \\ \geq \left| \operatorname{Re} \left\langle \int_a^b f(t) dt, \sum_{k=1}^m e_k \right\rangle \right| \\ \geq \operatorname{Re} \left\langle \int_a^b f(t) dt, \sum_{k=1}^m e_k \right\rangle.$$

By (3.12) and (3.13), we deduce (3.9).

Taking the norm on (3.11) and using (3.10), we have

$$\left\| \int_a^b f(t) dt \right\| = \frac{m \left( \int_a^b \|f(t)\| dt - \frac{1}{m} \sum_{k=1}^m \int_a^b M_k(t) dt \right)}{\left\| \sum_{k=1}^m e_k \right\|},$$

showing that the equality holds in (3.9).

Conversely, if the equality case holds in (3.9), then it must hold in all the inequalities used to prove (3.9). Therefore we have

$$(3.14) \quad \|f(t)\| = \operatorname{Re} \langle f(t), e_k \rangle + M_k(t)$$

for each  $k \in \{1, \dots, m\}$  and for a.e.  $t \in [a, b]$ ,

$$(3.15) \quad \left\| \int_a^b f(t) dt \right\| \left\| \sum_{k=1}^m e_k \right\| = \left| \left\langle \int_a^b f(t) dt, \sum_{k=1}^m e_k \right\rangle \right|$$

and

$$(3.16) \quad \operatorname{Im} \left\langle \int_a^b f(t) dt, \sum_{k=1}^m e_k \right\rangle = 0.$$

From (3.14) on integrating on  $[a, b]$  and summing over  $k$ , we get

$$(3.17) \quad \operatorname{Re} \left\langle \int_a^b f(t) dt, \sum_{k=1}^m e_k \right\rangle = m \int_a^b \|f(t)\| dt - \sum_{k=1}^m \int_a^b M_k(t) dt.$$

On the other hand, by the use of the following identity in inner product spaces:

$$\left\| u - \frac{\langle u, v \rangle v}{\|v\|^2} \right\|^2 = \frac{\|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2}{\|v\|^2}, \quad v \neq 0,$$

the relation (3.15) holds if and only if

$$\int_a^b f(t) dt = \frac{\left\langle \int_a^b f(t) dt, \sum_{k=1}^m e_k \right\rangle}{\left\| \sum_{k=1}^m e_k \right\|^2} \sum_{k=1}^m e_k,$$

giving, from (3.16) and (3.17), that (3.11) holds true.

If the equality holds in (3.9), then obviously (3.10) is valid and the theorem is proved.  $\square$

**REMARK 3.** If in the above theorem, the vectors  $\{e_k\}_{k \in \{1, \dots, m\}}$  are assumed to be orthogonal, then (3.9) becomes

$$(3.18) \quad \int_a^b \|f(t)\| dt \leq \frac{1}{m} \left( \sum_{k=1}^m \|e_k\|^2 \right)^{\frac{1}{2}} \left\| \int_a^b f(t) dt \right\| + \frac{1}{m} \sum_{k=1}^m \int_a^b M_k(t) dt.$$

Moreover, if  $\{e_k\}_{k \in \{1, \dots, m\}}$  is an orthonormal family, then (3.18) becomes

$$(3.19) \quad \int_a^b \|f(t)\| dt \leq \frac{1}{\sqrt{m}} \left\| \int_a^b f(t) dt \right\| + \frac{1}{m} \sum_{k=1}^m \int_a^b M_k(t) dt$$

which has been obtained in [3].

The following corollaries are of interest.

**COROLLARY 3.** Let  $(H; \langle \cdot, \cdot \rangle)$ ,  $e_k$ ,  $k \in \{1, \dots, m\}$  and  $f$  be as in Theorem 4. If  $r_k : [a, b] \rightarrow [0, \infty)$ ,  $k \in \{1, \dots, m\}$  are such that  $r_k \in L^2[a, b]$ ,  $k \in \{1, \dots, m\}$  and

$$(3.20) \quad \|f(t) - e_k\| \leq r_k(t),$$

for each  $k \in \{1, \dots, m\}$  and a.e.  $t \in [a, b]$ , then

$$(3.21) \quad \int_a^b \|f(t)\| dt \leq \left\| \frac{1}{m} \sum_{k=1}^m e_k \right\| \left\| \int_a^b f(t) dt \right\| + \frac{1}{2m} \sum_{k=1}^m \int_a^b r_k^2(t) dt.$$

The case of equality holds in (3.21) if and only if

$$\int_a^b \|f(t)\| dt \geq \frac{1}{2m} \sum_{k=1}^m \int_a^b r_k^2(t) dt$$

and

$$\int_a^b f(t) dt = \frac{m \left( \int_a^b \|f(t)\| dt - \frac{1}{2m} \sum_{k=1}^m \int_a^b r_k^2(t) dt \right)}{\left\| \sum_{k=1}^m e_k \right\|^2} \sum_{k=1}^m e_k.$$

The proof follows by Theorem 4 on using the following result that has been obtained in a similar form in [2, p. 27].

LEMMA 1. Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over the real or complex number field  $\mathbb{K}$  and  $x, a \in H$ ,  $r > 0$ . If  $\|x - a\| \leq r$ , then we have the inequality

$$(3.22) \quad \|x\| \|a\| - \operatorname{Re} \langle x, a \rangle \leq \frac{1}{2} r^2.$$

The case of equality holds in (3.22) if and only if

$$\|x - a\| = r \quad \text{and} \quad \|x\| = \|a\|.$$

Finally, the following corollary may be stated.

COROLLARY 4. Let  $(H; \langle \cdot, \cdot \rangle)$ ,  $e_k$ ,  $k \in \{1, \dots, m\}$  and  $f$  be as in Theorem 4. If  $M_k, \mu_k : [a, b] \rightarrow \mathbb{R}$  are such that  $M_k \geq \mu_k > 0$  a.e. on  $[a, b]$ ,  $\frac{(M_k - \mu_k)^2}{M_k + \mu_k} \in L[a, b]$  and

$$\operatorname{Re} \langle M_k(t) e_k - f(t), f(t) - \mu_k(t) e_k \rangle \geq 0$$

for each  $k \in \{1, \dots, m\}$  and for a.e.  $t \in [a, b]$ , then

$$\begin{aligned} \int_a^b \|f(t)\| dt &\leq \left\| \frac{1}{m} \sum_{k=1}^m e_k \right\| \left\| \int_a^b f(t) dt \right\| \\ &\quad + \frac{1}{4m} \sum_{k=1}^m \|e_k\|^2 \int_a^b \frac{[M_k(t) - \mu_k(t)]^2}{M_k(t) + \mu_k(t)} dt. \end{aligned}$$

The proof follows by Theorem 4 on using the following result that has been established in [3, p. 28].

LEMMA 2. Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over the real or complex number field  $\mathbb{K}$  and  $x, y \in H$ ,  $M \geq m > 0$ . If either

$$(3.23) \quad \operatorname{Re} \langle My - x, x - my \rangle \geq 0$$

or, equivalently,

$$(3.24) \quad \left\| x - \frac{m+M}{2} y \right\| \leq \frac{1}{2} (M-m) \|y\|,$$

holds, then

$$(3.25) \quad \|x\| \|y\| - \operatorname{Re} \langle x, y \rangle \leq \frac{1}{4} \cdot \frac{(M-m)^2}{m+M} \|y\|^2.$$

The case of equality holds in (3.25) if and only if the equality case is realised in (3.23) and

$$\|x\| = \frac{M+m}{2} \|y\|.$$

#### 4. Applications for Complex-Valued Functions

Let  $\mathbb{C}$  be the field of complex numbers. If  $z = \operatorname{Re}z + i\operatorname{Im}z$ , then by  $|\cdot|_p : \mathbb{C} \rightarrow [0, \infty)$ ,  $p \in [1, \infty]$  we define the  $p$ -modulus of  $z$  as

$$|z|_p := \begin{cases} \max\{|\operatorname{Re}z|, |\operatorname{Im}z|\} & \text{if } p = \infty, \\ (|\operatorname{Re}z|^p + |\operatorname{Im}z|^p)^{\frac{1}{p}} & \text{if } p \in [1, \infty), \end{cases}$$

where  $|a|$ ,  $a \in \mathbb{R}$  is the usual modulus of the real number  $a$ .

For  $p = 2$ , we recapture the usual modulus of a complex number, i.e.,

$$|z|_2 = \sqrt{|\operatorname{Re}z|^2 + |\operatorname{Im}z|^2} = |z|, \quad z \in \mathbb{C}.$$

It is well known that  $(\mathbb{C}, |\cdot|_p)$ ,  $p \in [1, \infty]$  is a Banach space over the real number field  $\mathbb{R}$ .

In the following, we give some examples of inequalities for complex-valued functions that are Lebesgue integrable on using the general results obtained in Section 2.

Consider the Banach space  $(\mathbb{C}, |\cdot|_1)$  and  $F : \mathbb{C} \rightarrow \mathbb{C}$ ,  $F(z) = ez$  with  $e = \alpha + i\beta$  and  $|e|^2 = \alpha^2 + \beta^2 = 1$ , then  $F$  is linear on  $\mathbb{C}$  and

$$\|F\|_1 := \sup_{z \neq 0} \frac{|F(z)|}{|z|_1} = 1,$$

showing that  $F$  is a bounded linear functional of unit norm on  $(\mathbb{C}, |\cdot|_1)$ .

Therefore we can apply Theorem 1 to state the following result for complex-valued functions.

**PROPOSITION 1.** *Let  $\alpha, \beta \in \mathbb{R}$  with  $\alpha^2 + \beta^2 = 1$ ,  $f, k : [a, b] \rightarrow \mathbb{C}$  Lebesgue integrable functions such that*

$$|\operatorname{Re}f(t)| + |\operatorname{Im}f(t)| \leq \alpha \operatorname{Re}f(t) - \beta \operatorname{Im}f(t) + k(t)$$

for a.e.  $t \in [a, b]$ . Then

$$\begin{aligned} (0 \leq) \int_a^b |\operatorname{Re}f(t)| dt + \int_a^b |\operatorname{Im}f(t)| dt \\ - \left[ \left| \int_a^b \operatorname{Re}f(t) dt \right| + \left| \int_a^b \operatorname{Im}f(t) dt \right| \right] \leq \int_a^b k(t) dt. \end{aligned}$$

Now, consider the Banach space  $(\mathbb{C}, |\cdot|_\infty)$ . If  $F(z) = dz$  with  $d = \gamma + i\delta$  and  $|d| = \frac{\sqrt{2}}{2}$ , i.e.,  $\gamma^2 + \delta^2 = \frac{1}{2}$ , then  $F$  is bounded linear on  $\mathbb{C}$  and

$$\|F\|_\infty := \sup_{z \neq 0} \frac{|F(z)|}{|z|_\infty} = 1.$$

Applying Theorem 1, we may state:

**PROPOSITION 2.** Let  $\gamma, \delta \in \mathbb{R}$  with  $\gamma^2 + \delta^2 = \frac{1}{2}$ ,  $f, k : [a, b] \rightarrow \mathbb{C}$  Lebesgue integrable functions on  $[a, b]$  such that

$$\max \{ |\operatorname{Re} f(t)|, |\operatorname{Im} f(t)| \} \leq \gamma \operatorname{Re} f(t) - \delta \operatorname{Im} f(t) + k(t)$$

for a.e.  $t \in [a, b]$ . Then

$$(0 \leq) \int_a^b \max \{ |\operatorname{Re} f(t)|, |\operatorname{Im} f(t)| \} dt \\ - \max \left\{ \left| \int_a^b \operatorname{Re} f(t) dt \right|, \left| \int_a^b \operatorname{Im} f(t) dt \right| \right\} \leq \int_a^b k(t) dt.$$

Finally, consider the Banach space  $(\mathbb{C}, |\cdot|_{2p})$  with  $p \geq 1$ . Let  $F : \mathbb{C} \rightarrow \mathbb{C}$ ,  $F(z) = cz$  with  $|c| = 2^{\frac{1}{2p}-\frac{1}{2}}$  ( $p \geq 1$ ). It is easy to see that  $F$  is a bounded linear functional with

$$\|F\|_{2p} := \sup_{z \neq 0} \frac{|F(z)|}{|z|_{2p}} = 1.$$

Utilising Theorem 1, we may state that:

**PROPOSITION 3.** Let  $\varphi, \phi \in \mathbb{R}$  with  $\varphi^2 + \phi^2 = 2^{\frac{1}{2p}-\frac{1}{2}}$  ( $p \geq 1$ ),  $f, k : [a, b] \rightarrow \mathbb{C}$  be Lebesgue integrable functions such that

$$[|\operatorname{Re} f(t)|^{2p} + |\operatorname{Im} f(t)|^{2p}]^{\frac{1}{2p}} \leq \varphi \operatorname{Re} f(t) - \phi \operatorname{Im} f(t) + k(t)$$

for a.e.  $t \in [a, b]$ . Then

$$(0 \leq) \int_a^b [|\operatorname{Re} f(t)|^{2p} + |\operatorname{Im} f(t)|^{2p}]^{\frac{1}{2p}} dt \\ - \left[ \left| \int_a^b \operatorname{Re} f(t) dt \right|^{2p} + \left| \int_a^b \operatorname{Im} f(t) dt \right|^{2p} \right]^{\frac{1}{2p}} \leq \int_a^b k(t) dt.$$

**REMARK 4.** If  $p = 1$  in the above proposition, then, from

$$|f(t)| \leq \varphi \operatorname{Re} f(t) - \psi \operatorname{Im} f(t) + k(t) \quad \text{for a.e. } t \in [a, b],$$

provided  $\varphi, \psi \in \mathbb{R}$  and  $\varphi^2 + \psi^2 = 1$ , we have the additive reverse of the classical continuous triangle inequality

$$(0 \leq) \int_a^b |f(t)| dt - \left| \int_a^b f(t) dt \right| \leq \int_a^b k(t) dt.$$

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