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Moving Estimates Test for Jumps in Time Series Models[1]

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Abstract

In this paper, we consider the problem of testing for a change of the parameter function $\theta(t)$ that may have a discontinuity at some unknown point τ. We introduce a varying-h moving estimate to test the null hypothesis that $\theta(t)$ is continuous against the alternative that $\theta(\tau-) \neq \theta(\tau+)$. Simulation results are provided for illustration.

Keywords : CUSUM test; MOSUM test; Varying-h Moving Estimates; Jump points.

1. Introduction

The problem of testing for parameter changes in statistical models has a long history. It was originally issued in the quality control context and then has been extended to various areas such as economics, finance, medicine, and seismic signal analysis. For a general review of the change point problem, see Csörgő and Horváth (1997), Lee, Ha, Na and Na (2003), Lee and Na (2005) and Lee, Na and Na (2003). Recently, Horvath and Kokoszaka (2002) considered the problem of testing whether or not the mean function is continuous over time, rather than a single parameter change, in which they considered the regression model $y_i = f(x_i) + \epsilon_i , i = 1,...,n$, where the function f or its p-th derivative $f^{(p)}$ may have a discontinuity at some unknown point τ. By fitting local polynomials, they tested for the continuity of $f^{(p)}$ and obtained a Darling-Erodos type limit theorem for the test statistic. However, they only considered the mean function case, so here we consider the case of a general parameter function $\theta(\cdot)$. Moreover, we

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employ the moving sum test instead of the local polynomial test.

In fact, the recursive-MOSUM (moving sum) test was originally introduced by Bauer and Hackl (1979). Although it complements the CUSUM test, it has a drawback to produce conservative critical values as indicated by Bauer and Hackl (1979). Chu, Hornik and Kuan (1995a,b) and Kuan and Hornik (1995) characterized the limiting process of fixed-h moving sums of recursive residuals in terms of the increments of a standard Wiener process, and the limiting process of moving sums of least squares residuals in terms of the increments of a Brownian bridge.

In the fixed-h ME (moving estimates) test, it is not an easy task to find a proper bandwidth. Thus we investigate the effect of the size of window that is determined as a functional of sample size. We introduce a varying-h ME test to test for the continuity of the null parameter function. Based on the result of Lee and Na (2005), we demonstrate through a simulation study that the varying-h ME test performs adequately. The paper is organized as follows. In Section 2, we introduce the varying-h ME test and characterize its asymptotic null distribution. In Section 3, we perform a simulation to examine the power performance of the ME test. Chapter 4 includes concluding remarks.

2. ME (moving estimates) test

In this section, we introduce the ME test and summarize the result on the asymptotic behavior of the ME test. The following result is due to Lee and Na (2005).

First, to illustrate the ME test for the continuity of parameter function, consider the following model :

$$y_i = \mu(i/n) + \epsilon_i,\ i = 1,\ \ldots\ ,n \tag{1}$$

where $\mu : [0,1] \to \mathbb{R}$ is a mean function and $\epsilon_1, \ldots, \epsilon_n$ are i.i.d. random variables with mean 0 and variance $0 < \sigma^2 < \infty$. We wish to test the null hypothesis

$$H_0\ :\ \mu \text{ is continuous}$$

against the alternative

$$H_1\ :\ \text{There exists a jump at } \tau \in (0,1)$$

based on moving averages $\hat{\mu}_k = [nh]^{-1} \sum_{i=k+1}^{k+[nh]} y_i$, $k = 0,1,\ldots,n-[nh]$, where $[nh]$ denotes the integer part of nh and h is a sequence of positive real numbers s.t.

$$h \to 0 \text{ and } nh \to \infty,\ as\ n \to 0. \tag{2}$$

Let

$$J_n(\mu) = \sqrt{\frac{[nh]}{2\sigma^2}}\, max_{[nh] \le k \le n-[nh]} |\hat{\mu}_k - \hat{\mu}_{k-[nh]}|$$

$$= \sqrt{\frac{1}{2\sigma^2[nh]}}\, max_k \left| \sum_{i=k+1}^{k+[nh]} \mu\left(\frac{i}{n}\right) - \sum_{i=k-[nh]+1}^{k} \mu\left(\frac{i}{n}\right) + \sum_{i=k+1}^{k+[nh]} \epsilon_i - \sum_{i=k-[nh]+1}^{k} \epsilon_i \right|$$

and note that if $E|\epsilon_1|^\nu < \infty$ for some $\nu > 3$ and

$$\limsup_{n\to\infty} n^{1/\nu - 1/2}(h^{-1}\log h^{-1})^{1/2} < \infty \tag{3}$$

then

$$\lim_{n\to\infty} P\left\{ A_n \sqrt{\frac{1}{2\sigma^2[nh]}}\, max_k \left| \sum_{i=k+1}^{k+[nh]} \epsilon_i - \sum_{i=k-[nh]+1}^{k} \epsilon_i \right| - D_n \le x \right\} = \exp\left(-2e^{-x}\right)$$

for all $x \in \mathbb{R}$, where

$$A_n = \sqrt{2\log(n/[nh]-2)} \tag{4}$$

and

$$D_n = 2\log(n/[nh]-2) + 1/2\log\log(n/[nh]-2) - 1/2\log\pi + \log 1.5 \tag{5}$$

(cf. Lemma 6.3 in Lee and Na (2005)).

If μ is Lipschitz continuous, then we have

$$\left| J_n(\mu) - \sqrt{\frac{1}{2\sigma^2[nh]}}\, max_{[nh] \le k \le n-[nh]} \left| \sum_{i=k+1}^{k+[nh]} \epsilon_i - \sum_{i=k-[nh]+1}^{k} \epsilon_i \right| \right|$$

$$\le \sqrt{\frac{1}{2\sigma^2[nh]}}\, max_{[nh] \le kn-[nh]} \sum_{i=k+1}^{k+[nh]} \left| \mu\left(\frac{i}{n}\right) - \mu\left(\frac{i-[nh]}{n}\right) \right|$$

$$\le \sqrt{\frac{1}{2\sigma^2[nh]}}\, \frac{[nh]^2}{n} M$$

for some $0 < M < \infty$. Therefore, if

$$\lim_{n\to\infty} nh^3 \log h^{-1} = 0, \tag{6}$$

then

$$J_n(\mu) = \sqrt{\frac{1}{2\sigma^2[nh]}}\, max_{[nh] \le k \le n-[nh]} \left| \sum_{i=k+1}^{k+[nh]} \epsilon_i - \sum_{i=k-[nh]+1}^{k} \epsilon_i \right| + o(A_n^{-1}),\ as\ n\to\infty$$

and so

$$\lim_{n\to\infty} P\{A_n J_n(\mu) - D_n \le x\} = \exp\left(-2e^{-x}\right),\ \forall x \in \mathbb{R}.$$

If

$$\mu(t) = \mu_1(t) I_{[0,\tau]}(t) + \mu_2(t) I_{(\tau,1]}(t)$$

for some $\tau \in (0,1)$, where μ_1 and μ_2 are Lipschitz continuous and $\mu_1(\tau) \ne \mu_2(\tau)$, then

$$J_n(\mu) \geq \sqrt{\frac{[nh]}{2\sigma^2}}\,|\mu_1(\tau)-\mu_2(\tau)|$$

$$- \sqrt{\frac{1}{2\sigma^2[nh]}}\sum_{i=[n\tau]+1}^{[n\tau]+[nh]}\left|\mu_2\left(\frac{i}{n}\right)-\mu_2(\tau)\right| - \sqrt{\frac{1}{2\sigma^2[nh]}}\sum_{i=[n\tau]-[nh]+1}^{[n\tau]}\left|\mu_1\left(\frac{i}{n}\right)-\mu_1(\tau)\right|$$

$$- \sqrt{\frac{1}{2\sigma^2[nh]}}\, max_{[nh]\leq k\leq n-[nh]}\left|\sum_{i=k+1}^{k+[nh]}\epsilon_i - \sum_{i=k-[nh]+1}^{k}\epsilon_i\right|$$

$$= \sqrt{\frac{[nh]}{2\sigma^2}}\,|\mu_1(\tau)-\mu_2(\tau)| - O(\sqrt{nh^3}) - O_P(\sqrt{\log h^{-1}})$$

$$\to\infty$$

as $n\to\infty$, provided $\lim_{n\to\infty}\log h^{-1}/nh = 0$. Therefore, we have that rejecting H_0 versus H_1 for large values of $J_n(\mu)$ constitutes a consistent test.

Now, consider the general parameter case. Let y_i, $i=1,\ldots,n$ be the random sample with marginal distribution $F(\theta_i)$, where θ_i is a d-dimensional parameter and the parameter function $\theta(\cdot):[0,1]\to\mathbb{R}^d$, such that

$$\theta\left(\frac{i}{n}\right)=\theta_i,\; i=1,\ldots,n. \tag{7}$$

We are interested in testing the null hypothesis

$$H_0 : \theta(\cdot) \text{ is continuous on } (0,1)$$

against the alternative

$$H_1 : \text{There exists a jump at } \tau\in(0,1) \text{ such that } \theta(\tau-)\neq\theta(\tau+)$$

based on moving estimates.

Let $\hat{\theta}_k$ be the moving estimate based on $y_{k+1},\ldots,y_{k+[nh]}$ and suppose that $\hat{\theta}_k$ satisfies the following

$$\hat{\theta}_k = \frac{1}{[nh]}\sum_{i=k+1}^{k+[nh]}\theta_i + \frac{1}{[nh]}\sum_{i=k+1}^{k+[nh]} l_i + R_k, \tag{8}$$

where $l_1,\ldots,l_n$ are i.i.d. random vectors with mean 0, nonsingular covariance matrix Σ and $E\|l_i\|^{\nu}<\infty$ for some $\nu>3$ and $R_k=(R_{k1},\ldots,R_{kd})'$. Based on this, we construct the test statistic

$$J_n(\theta) = \sqrt{\frac{[nh]}{2}}\, max_{[nh]\leq k\leq n-[nh]}\|\Sigma^{-1/2}(\hat{\theta}_k-\hat{\theta}_{k-[nh]})\|, \tag{9}$$

where $\|\cdot\|$ denotes the maximum norm. In applications, a consistent estimator $\hat{\Sigma}$ will replace Σ when it is unknown.

In order to investigate its limiting distribution, we assume that

$$H_0' : \theta(\cdot) \text{ is Lipschitz continuous on } (0,1). \tag{10}$$

Assuming that

$$\max_{0 \le k \le n-[nh]} \|R_k\| = o_P\left(\frac{1}{\sqrt{nh \log h^{-1}}}\right). \tag{11}$$

Since

$$\hat{\theta}_k - \hat{\theta}_{k-[nh]} = \frac{1}{[nh]} \sum_{i=k+1}^{k+[nh]} \left\{\theta\left(\frac{i}{n}\right) - \theta\left(\frac{i-[nh]}{n}\right)\right\}$$

$$+ \frac{1}{[nh]} \sum_{i=k+1}^{k+[nh]} (l_i - l_{i-[nh]}) + (R_k - R_{k-[nh]}),$$

it holds that under H_0',

$$\left| J_n(\theta) - \sqrt{\frac{1}{2[nh]}}\, max_{[nh] \le k \le n-[nh]} \left\| \Sigma^{-1/2} \left(\sum_{i=k+1}^{k+[nh]} l_i - \sum_{i=k-[nh]+1}^{k} l_i \right) \right\| \right|$$

$$\le C_1 \sqrt{\frac{1}{[nh]}}\, max_k \sum_{i=k+1}^{k+[nh]} \left\| \theta\left(\frac{i}{n}\right) - \theta\left(\frac{i-[nh]}{n}\right) \right\| + C_2 \sqrt{[nh]}\, max_k \|R_k\|$$

$$= O(\sqrt{nh^3}) + o_P\left(\frac{1}{\sqrt{\log h^{-1}}}\right).$$

Therefore, from Lemma 6.3 in Lee and Na (2005), we have that under H_0', (2), (3), (6) and (11),

$$\lim_{n \to \infty} P\{A_n J_n(\theta) - D_n \le x\} = \exp(-2de^{-x}), \ \forall x \in \mathbb{R}, \tag{12}$$

where d is a dimension of parameter and A_n and D_n are defined in (4) and (5), respectively.

Remark. When there exists a jump at $\tau \in (0,1)$ with jump size Δ, we can estimate the time of jump by

$$\hat{\tau} = \frac{1}{n} arg\max_{[nh] \le k \le n-[nh]} \| \hat{\theta}_k - \hat{\theta}_{k-[nh]} \| \tag{13}$$

and the size of jump by

$$\hat{\Delta} = \hat{\theta}_{[n\hat{\tau}]} - \hat{\theta}_{[n\hat{\tau}]-[nh]}. \tag{14}$$

From the simulation result (cf. Section 3), we can expect that $\hat{\tau}$ and $\hat{\Delta}$ are consistent estimators of τ and Δ, respectively.

3. Simulation Results

In this section, we evaluate the varying-h ME test through a simulation study. For the task, we consider the location model in (1), where ϵ_i are i.i.d. standard normal random variables. Generally, since σ^2 is unknown, we employ the test statistic

$$\hat{J}_n(\mu) = \sqrt{\frac{[nh]}{2\hat{\sigma}^2}}\, max_{[nh] \le k \le n-[nh]} |\hat{\mu}_k - \hat{\mu}_{k-[nh]}|,$$

where $\hat{\mu}_k = [nh]^{-1} \sum_{i=k+1}^{k+[nh]} y_i$, $k = 0,1,\ldots,n-[nh]$ and

$$\hat{\sigma}^2 = \frac{1}{n-[nh]+1} \sum_{k=0}^{n-[nh]} \hat{\epsilon}_k^2, \tag{15}$$

where $\hat{\epsilon}_k = y_{k+[nh]/2} - \hat{\mu}_k$, $k = 0,1,\ldots,n-[nh]$. We use the bandwidth $h = n^{-0.4}$ for our test. The critical value at α is $-\log\{-0.5\log(1-\alpha)\}$ by (12). The empirical sizes are calculated at $\alpha = 0.05, 0.1$ and the empirical powers are calculated at $\alpha = 0.1$. The number of replication is 1000 in all cases.

Size simulations are based on the models

$$y_i = \epsilon_i \text{ and } y_i = \frac{i}{n} + \epsilon_i\,,\ i = 1,\ldots,n,$$

with n=200, 500, 1000, 2000, 5000, 10000. The results are summarized in Table 1. It can be seen that the empirical sizes are smaller than the nominal level for all cases.

Next, we examine the power. For the alternative of a single change, we consider

$$y_i = \begin{cases} \mu_i + \epsilon_i & \text{if } i \le [n\tau] \\ \mu_i + \Delta + \epsilon_i & \text{if } i > [n\tau] \end{cases},$$

where we consider the case τ=0.1, 0.3, 0.5, 0.7, 0.9. For alternative of a double structural changes, we consider

$$y_i = \begin{cases} \mu_i + \epsilon_i & \text{if } i \le [n\tau_1] \\ \mu_i + \Delta + \epsilon_i & \text{if } [n\tau_1] < i \le [n\tau_2], \\ \mu_i + \epsilon_i & \text{if } i > [n\tau_2] \end{cases}$$

where (τ_1, τ_2) with τ_1 =0.1, 0.3, 0.5, 0.7 and $\tau_2 = \tau_1 + 0.3 + 0.6$. For the single change test, we consider the four cases: the constant parameter case $\mu_i = 0$ with Δ =0.5, 1.0 and the linear parameter case of $\mu_i = i/n$ with Δ =0.5, 1.0. For double changes, we consider the four cases: the constant parameter case of $\mu_i = 0$ with Δ =0.5, 1.0 and the linear parameter case of $\mu_i = i/n$ with Δ =0.5, 1.0. We observe from Tables 2, 6, 10 and 11 that

(1) The performance of statistics is getting better as the jump size becomes larger.

(2) The performance of the varying-h ME test is not efficient when the sample size is small and the change occurs too early or too late. This

phenomenon is somewhat natural since the test itself was intended to work out in large samples. Also, it is natural for the test to behave best when the change occurs at the center of the series.

Tables 3-5 and 7-9 show the estimated means and variances for the estimators in the Remark above, from which one can see that the estimators well estimate the time and size of jump and assert the consistency of the estimators. All these results demonstrate the adequacy of our test.

<Table 1> Empirical size of varying-h ME test

n	nominal level 5%		nominal level 10%	
	constant	linear	constant	linear
200	0.006	0.025	0.034	0.076
500	0.016	0.021	0.019	0.057
1000	0.013	0.027	0.044	0.066
2000	0.018	0.033	0.036	0.080
5000	0.019	0.037	0.036	0.092
10000	0.015	0.034	0.055	0.076

<Table 2> Power simulation under single structural change : constant parameter

Δ	n	τ				
		0.1	0.3	0.5	0.7	0.9
0.5	200	0.073	0.126	0.117	0.139	0.065
	500	0.229	0.212	0.247	0.234	0.241
	1000	0.386	0.391	0.382	0.384	0.401
	2000	0.575	0.599	0.605	0.577	0.610
	5000	0.888	0.903	0.901	0.891	0.881
	10000	0.986	0.987	0.986	0.986	0.993
1.0	200	0.381	0.679	0.679	0.685	0.366
	500	0.901	0.913	0.945	0.914	0.908
	1000	0.993	0.987	0.991	0.997	0.993
	2000	1.000	1.000	1.000	1.000	1.000
	5000	1.000	1.000	1.000	1.000	1.000
	10000	1.000	1.000	1.000	1.000	1.000

<Table 3> Mean of $\hat{\tau}$ under single structural change : constant parameter

Δ	n	τ				
		0.1	0.3	0.5	0.7	0.9
0.5	200	0.41489	0.42488	0.50515	0.59380	0.58432
	500	0.27040	0.39673	0.49525	0.62414	0.71373
	1000	0.20386	0.35486	0.49825	0.64991	0.79740
	2000	0.15927	0.32937	0.49924	0.67593	0.84530
	5000	0.10864	0.30923	0.49966	0.69674	0.88844
	10000	0.10116	0.30000	0.50012	0.69996	0.89942
1.0	200	0.22030	0.32160	0.50416	0.69418	0.77688
	500	0.11308	0.30657	0.50130	0.69724	0.89746
	1000	0.10186	0.30102	0.50116	0.70082	0.90074
	2000	0.10040	0.30047	0.50033	0.70030	0.90053
	5000	0.10018	0.30011	0.50017	0.70021	0.90020
	10000	0.10009	0.30007	0.50010	0.70009	0.90012

<Table 4> Variance of $\hat{\tau}$ under single structural change : constant parameter

Δ	n	τ				
		0.1	0.3	0.5	0.7	0.9
0.5	200	0.06764	0.04032	0.02539	0.03858	0.06673
	500	0.06230	0.03748	0.02680	0.03179	0.06808
	1000	0.04882	0.02610	0.01803	0.02528	0.04825
	2000	0.03168	0.01470	0.01251	0.01400	0.02706
	5000	0.00450	0.00517	0.00190	0.00154	0.00760
	10000	0.00064	0.00023	0.00012	0.00004	0.00023
1.0	200	0.03730	0.00711	0.00440	0.00544	0.04067
	500	0.00537	0.00245	0.00114	0.00216	0.00225
	1000	0.00050	0.00009	0.00011	0.00007	0.00007
	2000	1.6e-05	1.3e-05	1.6e-05	1.4e-05	1.5e-05
	5000	2.1e-06	2.2e-06	2.5e-06	2.0e-06	2.2e-06
	10000	5.9e-07	6.8e-07	6.9e-07	5.7e-07	6.1e-07

<Table 5> Mean(Variance) of $\hat{\Delta}$ under single structural change : constant parameter

Δ	n	τ				
		0.1	0.3	0.5	0.7	0.9
0.5	200	0.300(0.495)	0.559(0.353)	0.591(0.308)	0.571(0.360)	0.285(0.483)
	500	0.482(0.270)	0.495(0.250)	0.525(0.236)	0.537(0.221)	0.454(0.298)
	1000	0.522(0.143)	0.541(0.125)	0.513(0.158)	0.511(0.153)	0.516(0.153)
	2000	0.525(0.077)	0.532(0.078)	0.523(0.083)	0.525(0.078)	0.545(0.066)
	5000	0.534(0.024)	0.542(0.019)	0.541(0.016)	0.538(0.017)	0.536(0.019)
	10000	0.523(0.007)	0.526(0.008)	0.527(0.008)	0.533(0.007)	0.526(0.008)
1.0	200	0.832(0.262)	1.085(0.135)	1.102(0.127)	1.116(0.099)	0.811(0.281)
	500	1.053(0.068)	1.053(0.070)	1.075(0.050)	1.061(0.065)	1.064(0.056)
	1000	1.043(0.026)	1.048(0.030)	1.039(0.030)	1.056(0.027)	1.048(0.028)
	2000	1.041(0.019)	1.036(0.018)	1.022(0.017)	1.033(0.018)	1.036(0.019)
	5000	1.016(0.011)	1.023(0.011)	1.018(0.010)	1.018(0.011)	1.023(0.011)
	10000	1.007(0.007)	1.012(0.008)	1.015(0.007)	1.010(0.007)	1.007(0.008)

<Table 6> Power simulation under single structural change : linear parameter

Δ	n	τ				
		0.1	0.3	0.5	0.7	0.9
0.5	200	0.153	0.333	0.269	0.317	0.164
	500	0.379	0.380	0.418	0.411	0.371
	1000	0.598	0.589	0.536	0.567	0.559
	2000	0.763	0.744	0.756	0.741	0.734
	5000	0.939	0.934	0.938	0.942	0.951
	10000	0.994	0.994	0.993	0.995	0.992
1.0	200	0.584	0.856	0.839	0.862	0.541
	500	0.969	0.963	0.984	0.966	0.973
	1000	0.999	0.998	1.000	0.997	0.998
	2000	1.000	1.000	1.000	1.000	1.000
	5000	1.000	1.000	1.000	1.000	1.000
	10000	1.000	1.000	1.000	1.000	1.000

<Table 7> Mean of $\hat{\tau}$ under single structural change : linear parameter

Δ	n	τ				
		0.1	0.3	0.5	0.7	0.9
0.5	200	0.38316	0.37693	0.50512	0.62321	0.62895
	500	0.23675	0.35450	0.49668	0.64496	0.74713
	1000	0.18118	0.33501	0.50392	0.66513	0.81164
	2000	0.14130	0.31899	0.49437	0.67751	0.85040
	5000	0.10792	0.30469	0.50014	0.69770	0.89298
	10000	0.10081	0.30029	0.49989	0.69969	0.89884
1.0	200	0.20868	0.31119	0.50357	0.70100	0.79598
	500	0.11000	0.30225	0.50162	0.70161	0.89538
	1000	0.10258	0.30109	0.50086	0.70175	0.90037
	2000	0.10064	0.30067	0.50050	0.70015	0.90042
	5000	0.10021	0.30027	0.50017	0.70015	0.90019
	10000	0.10006	0.30007	0.50004	0.70006	0.90006

<Table 8> Variance of $\hat{\tau}$ under single structural change : linear parameter

Δ	n	τ				
		0.1	0.3	0.5	0.7	0.9
0.5	200	0.06345	0.03000	0.02149	0.03198	0.06361
	500	0.05317	0.02543	0.01915	0.02640	0.05945
	1000	0.03958	0.01654	0.01296	0.01891	0.04202
	2000	0.02048	0.00927	0.00671	0.01160	0.02773
	5000	0.00477	0.00321	0.00139	0.00139	0.00416
	10000	0.00016	0.00001	0.00001	0.00019	0.00083
1.0	200	0.03149	0.00414	0.00389	0.00309	0.03139
	500	0.00312	0.00059	0.00047	0.00071	0.00317
	1000	0.00079	0.00006	0.00004	0.00008	0.00033
	2000	1.3e-05	1.4e-05	1.6e-05	4.0e-05	1.6e-05
	5000	2.3e-06	2.6e-06	2.3e-06	2.9e-06	2.2e-06
	10000	6.3e-07	6.5e-07	6.0e-07	7.9e-07	7.0e-07

<Table 9> Mean(Variance) of $\hat{\Delta}$ under single structural change : linear parameter

Δ	n	τ				
		0.1	0.3	0.5	0.7	0.9
0.5	200	0.774(0.111)	0.907(0.062)	0.885(0.058)	0.893(0.071)	0.786(0.102)
	500	0.728(0.073)	0.744(0.060)	0.763(0.047)	0.757(0.051)	0.738(0.063)
	1000	0.690(0.038)	0.687(0.032)	0.675(0.028)	0.680(0.041)	0.674(0.045)
	2000	0.626(0.026)	0.624(0.023)	0.627(0.021)	0.623(0.021)	0.619(0.021)
	5000	0.576(0.010)	0.577(0.011)	0.576(0.009)	0.576(0.010)	0.581(0.011)
	10000	0.550(0.007)	0.552(0.007)	0.555(0.006)	0.556(0.006)	0.557(0.007)
1.0	200	1.058(0.071)	1.266(0.057)	1.238(0.063)	1.248(0.058)	1.040(0.072)
	500	1.152(0.039)	1.151(0.039)	1.158(0.034)	1.148(0.043)	1.152(0.043)
	1000	1.111(0.027)	1.114(0.027)	1.127(0.027)	1.108(0.027)	1.123(0.029)
	2000	1.085(0.017)	1.073(0.019)	1.080(0.017)	1.075(0.020)	1.080(0.020)
	5000	1.047(0.011)	1.051(0.012)	1.049(0.011)	1.051(0.012)	1.051(0.011)
	10000	1.033(0.008)	1.038(0.007)	1.037(0.007)	1.032(0.007)	1.032(0.008)

<Table 10> Power simulation under double structural changes : constant parameter

Δ	n	(τ_1,τ_2)				
		(0.1,0.4)	(0.1,0.7)	(0.3,0.6)	(0.3,0.9)	(0.5,0.8)
0.5	200	0.184	0.178	0.221	0.185	0.251
	500	0.368	0.375	0.371	0.392	0.401
	1000	0.591	0.641	0.613	0.603	0.604
	2000	0.825	0.818	0.823	0.824	0.834
	5000	0.989	0.991	0.990	0.984	0.991
	10000	0.998	1.000	1.000	1.000	1.000
1.0	200	0.781	0.800	0.891	0.788	0.884
	500	0.993	0.990	0.994	0.991	0.990
	1000	1.000	1.000	1.000	1.000	1.000
	2000	1.000	1.000	1.000	1.000	1.000
	5000	1.000	1.000	1.000	1.000	1.000
	10000	1.000	1.000	1.000	1.000	1.000

<Table 11> Power simulation under double structural changes : linear parameter

Δ	n	(τ_1,τ_2)				
		(0.1,0.4)	(0.1,0.7)	(0.3,0.6)	(0.3,0.9)	(0.5,0.8)
0.5	200	0.184	0.175	0.310	0.280	0.293
	500	0.435	0.455	0.429	0.447	0.437
	1000	0.631	0.671	0.649	0.664	0.654
	2000	0.832	0.855	0.848	0.853	0.851
	5000	0.993	0.984	0.992	0.993	0.984
	10000	0.999	1.000	1.000	1.000	1.000
1.0	200	0.762	0.772	0.908	0.863	0.906
	500	0.994	0.999	0.996	0.992	0.987
	1000	1.000	1.000	1.000	1.000	1.000
	2000	1.000	1.000	1.000	1.000	1.000
	5000	1.000	1.000	1.000	1.000	1.000
	10000	1.000	1.000	1.000	1.000	1.000

4. Concluding Remarks

Statistical inference for change point is of great concern in stochastic modeling. In this article, we considered the ME test with varying bandwidth and demonstrated that it can detect structural change and change point. We found that the ME test with varying bandwidth has some size distortions. It means that the speed of convergence to asymptotic distribution is somewhat slow. So, there must be a further study to remedy this drawback. On the other hand, the simulation result shows that the ME test is very efficient to find change points, which demonstrates the validity of the test. For future study, we will consider a modification of our test to remedy the size distortion problem.

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