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MULTIPLE L_p ANALYTIC GENERALIZED FOURIER–FEYNMAN TRANSFORM ON A FRESNEL TYPE CLASS

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Abstract. In this paper, we define a class of functional defined on a very general function space $C_{a,b}[0,T]$ like a Fresnel class of an abstract Wiener space. We then define the multiple L_p analytic generalized Fourier–Feynman transform and the generalized convolution product of functionals on function space $C_{a,b}[0,T]$. Finally, we establish some relationships between the multiple L_p analytic generalized Fourier–Feynman transform and the generalized convolution product for functionals in $\mathcal{F}(C_{a,b}[0,T])$.

1. Introduction

Let $C_0[0,T]$ denote one-parameter Wiener space, that is the space of $\mathbb{R}$-valued continuous functions $x(t)$ on $[0,T]$ with $x(0) = 0$. The concept of an L_1 analytic Fourier–Feynman transform(FFT) for functionals on the Wiener space was introduced by Brue in [2]. In [3], Cameron and Storvick introduced an L_2 FFT on the Wiener space. In [10], Johnson and Skoug developed an L_p FFT theory for $1 \le p \le 2$ that extended the results in [2,3] and established various relationships between the L_1 and L_2 theories. In [11], Huffman, Park and Skoug developed an L_p FFT theory on certain classes of functionals defined on Wiener space and they defined a convolution product for functionals in the classes, and they obtained various results for the FFT and convolution product [11,13,12]. On the other hand, in [1], Ahn investigated the L_1 FFT theory on the Fresnel class $\mathcal{F}(B)$ of an abstract

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Wiener space, and in [4] Chang, Song and Yoo studied the FFT and the first variation on an abstract Wiener space and the Fresnel class $\mathcal{F}(B)$.

In [6], Chang and Choi studied the multiple L_p analytic GFFT on the Banach algebra.

In recent paper [7], Chang and Skoug established various results involving generalized Feynman integrals, and the generalized FFTs(GFFT) for functionals defined on a very general function space $C_{a,b}[0,T]$ rather than on the Wiener space $C_0[0,T]$. The function space was introduced by Chang and Chung in [5].

In [8], Chang and Lee studied the GFFT and CGFFT on a Fresnel type class $\mathcal{F}(C_{a,b}[0,T])$.

In Section 2 of this paper, we introduce the basic concepts and the notations for our research. In Section 3, we study the L_p analytic GFFT and the GCP on a function space $C_{a,b}[0,T]$. Finally, we establish some relationships between the multiple L_p analytic GFFT and the GCP for functionals in $\mathcal{F}(C_{a,b}[0,T])$.

2. Definitions and preliminaries

Let $D = [0,T]$ and let $(\Omega, \mathcal{B}, P)$ be a probability measure space. A real-valued stochastic process Y on $(\Omega, \mathcal{B}, P)$ and D is called *a generalized Brownian motion process* if $Y(0,\omega)$=0 almost everywhere and for $0 = t_0 < t_1 < \cdots < t_n \le T$, the n-dimensional random vector $(Y(t_1,\omega), \cdots, Y(t_n,\omega))$ is normally distributed with the density function

$$
\begin{aligned}
K(\vec{t},\vec{\eta}) = &\Big((2\pi)^n \prod_{j=1}^{n}(b(t_j) - b(t_{j-1}))\Big)^{-1/2} \\
&\cdot \exp\Big\{-\frac{1}{2}\sum_{j=1}^{n} \frac{((\eta_j - a(t_j)) - (\eta_{j-1} - a(t_{j-1})))^2}{b(t_j) - b(t_{j-1})}\Big\}
\end{aligned}
\tag{2.1}
$$

where $\vec{\eta} = (\eta_1, \cdots, \eta_n)$, $\eta_0 = 0$, $\vec{t} = (t_1, \cdots, t_n)$, $a(t)$ is an absolutely continuous real-valued function on $[0,T]$ with $a(0) = 0$, $a'(t) \in L^2[0,T]$, and $b(t)$ is a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0,T]$.

As explained in [15,p.18-20], Y induces a probability measure μ on the measurable space $(\mathbb{R}^D, \mathcal{B}^D)$ where $\mathbb{R}^D$ is the space of all real valued functions $x(t)$, $t \in D$, and $\mathcal{B}^D$ is the smallest σ-algebra of subsets of $\mathbb{R}^D$ with respect to which all the coordinate evaluation maps $e_t(x) = x(t)$ defined on $\mathbb{R}^D$ are measurable. The triple $(\mathbb{R}^D, \mathcal{B}^D, \mu)$ is a probability measure space. This measure space is called the function space induced by the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$.

We note that the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t) = \min\{b(s), b(t)\}$. By Theorem 14.2 [15,p.187], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0) = 0$ under the sup norm). Hence $(C_{a,b}[0,T], \mathcal{B}(C_{a,b}[0,T]), \mu)$ is the function space induced by Y, where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel σ-algebra of $C_{a,b}[0,T]$.

A subset B of $C_{a,b}[0,T]$ is said to be scale-invariant measurable(s.i.m.)[9] provided ρB is $\mathcal{B}(C_{a,b}[0,T])$-measurable for all $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null set provided $\mu(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere(s-a.e.). If two functionals F and G defined on $C_{a,b}[0,T]$ are equal s-a.e., then we write $F \approx G$.

Let $L^2_{a,b}[0,T]$ be the set of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue-Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$; i.e.,

$$L^2_{a,b}[0,T] = \left\{ v : \int_0^T v^2(s)db(s) < \infty \text{ and } \int_0^T v^2(s)d|a|(s) < \infty \right\} \tag{2.2}$$

where $|a|(t)$ denotes the total variation of the function a on the interval $[0,t]$.

For $u, v \in L^2_{a,b}[0,T]$, let

$$(u,v)_{a,b} = \int_0^T u(t)v(t)d[b(t) + |a|(t)]. \tag{2.3}$$

Then $(\cdot,\cdot)_{a,b}$ is an inner product on $L^2_{a,b}[0,T]$ and $\|u\|_{a,b} = \sqrt{(u,u)_{a,b}}$ is a norm on $L^2_{a,b}[0,T]$. In particular, note that $\|u\|_{a,b} = 0$ if and only if $u(t) = 0$ a.e. on $[0,T]$. Furthermore, $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$ is a separable Hilbert space.

Let $\{\phi_j\}_{j=1}^{\infty}$ be a complete orthonormal set of real-valued functions of bounded variation on $[0,T]$ such that

$$(\phi_j,\phi_k)_{a,b} = \begin{cases} 0 & , \ j \neq k \\ 1 & , \ j = k \end{cases},$$

and for each $v \in L^2_{a,b}[0,T]$, let

$$v_n(t) = \sum_{j=1}^{n} (v,\phi_j)_{a,b}\phi_j(t) \tag{2.4}$$

for $n = 1,2,\cdots$. Then for each $v \in L^2_{a,b}[0,T]$, the Paley-Wiener-Zygmund (PWZ) stochastic integral $\langle v,x\rangle$ is defined by the formula

$$\langle v,x\rangle = \lim_{n\to\infty} \int_0^T v_n(t)dx(t) \tag{2.5}$$

for all $x \in C_{a,b}[0,T]$ for which the limit exists; one can show that for each $v \in L^2_{a,b}[0,T]$, the PWZ stochastic integral $\langle v,x\rangle$ exists for μ-a.e. $x \in C_{a,b}[0,T]$.

We denote the function space integral of a $\mathcal{B}(C_{a,b}[0,T])$-measurable functional F by

$$E[F] = \int_{C_{a,b}[0,T]} F(x)d\mu(x)$$

whenever the integral exists.

Throughout this paper, we will assume that each functional F we consider satisfies the conditions:

$$F : C_{a,b}[0,T] \to \mathbb{C} \text{ is s.i.m. and s-a.e. defined,} \tag{2.6}$$

and

$$E_x\big[|F(\rho x)|\big] < \infty \text{ for each } \rho > 0. \tag{2.7}$$

Next, we state the definition of the generalized analytic Feynman integral.

Definition 2.1. *Let* $\mathbb{C}$ *denote the complex numbers, let* $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : Re\lambda > 0\}$ *and let* $\tilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} : \lambda \neq 0 \text{ and } Re\lambda \geq 0\}$. *Let* F *satisfy conditions (2.6) and (2.7) above. If there exists a function* $J^*(\lambda)$ *analytic in* $\mathbb{C}_+$ *such that* $J^*(\lambda) = E_x[F(\lambda^{-1/2}x)]$ *for all* $\lambda > 0$, *then* $J^*(\lambda)$ *is defined to be the analytic function space integral of* F *over* $C_{a,b}[0,T]$ *with parameter* λ, *and for* $\lambda \in \mathbb{C}_+$ *we write*

$$E^{\text{an}_\lambda}[F] \equiv E_x^{\text{an}_\lambda}[F(x)] = J^*(\lambda). \tag{2.8}$$

Let $q \neq 0$ *be a real number and let* F *be a functional such that* $E^{\text{an}_\lambda}[F]$ *exists for all* $\lambda \in \mathbb{C}_+$. *If the following limit exists, we call it the generalized analytic Feynman integral of* F *with parameter* q *and we write*

$$E^{\text{anf}_q}[F] \equiv E_x^{\text{anf}_q}[F(x)] = \lim_{\lambda \to -iq} E^{\text{an}_\lambda}[F] \tag{2.9}$$

where $\lambda \to -iq$ *through values in* $\mathbb{C}_+$.

Next, we state the definition of the GFFT.

Definition 2.2. *For* $\lambda \in \mathbb{C}_+$ *and* $y \in C_{a,b}[0,T]$, *let*

$$T_\lambda(F)(y) = E_x^{\text{an}_\lambda}[F(y+x)]. \tag{2.10}$$

For $p \in (1,2]$, *we define the* L_p *analytic GFFT,* $T_q^{(p)}(F)$ *of* F, *by the formula* $(\lambda \in \mathbb{C}_+)$

$$T_q^{(p)}(F)(y) = l.i.m._{\lambda \to -iq} T_\lambda(F)(y) \tag{2.11}$$

if it exists; i.e., for each $\rho > 0$,

$$\lim_{\lambda \to -iq} \int_{C_{a,b}[0,T]} \left| T_\lambda(F)(\rho y) - T_q^{(p)}(F)(\rho y) \right|^{p'} d\mu(y) = 0$$

where $1/p + 1/p' = 1$. *We define the* L_1 *analytic GFFT,* $T_q^{(1)}(F)$ *of* F, *by the formula* $(\lambda \in \mathbb{C}_+)$

$$T_q^{(1)}(F)(y) = \lim_{\lambda \to -iq} T_\lambda(F)(y) \tag{2.12}$$

if it exists.

We note that for $1 \leq p \leq 2$, $T_q^{(p)}(F)$ is defined only s-a.e.. We also note that if $T_q^{(p)}(F)$ exists and if $F \approx G$, then $T_q^{(p)}(G)$ exists and $T_q^{(p)}(G) \approx T_q^{(p)}(F)$.

Definition 2.3. *Let F and G be measurable functionals on $C_{a,b}[0,T]$. For $\lambda \in \tilde{\mathbb{C}}_+$,we define their GCP $(F*G)_\lambda$ (if it exists) by*

$$(2.13)\quad (F*G)_\lambda(y) = \begin{cases} E_x^{\mathrm{an}_\lambda}[F(\frac{y+x}{\sqrt{2}})G(\frac{y-x}{\sqrt{2}})] & ,\lambda \in \mathbb{C}_+ \\ E_x^{\mathrm{anf}_q}[F(\frac{y+x}{\sqrt{2}})G(\frac{y-x}{\sqrt{2}})] & ,\lambda = -iq, q \in \mathbb{R}-\{0\} \end{cases}.$$

Remark 2.1. *(i) When $\lambda = -iq$, we denote$(F*G)_\lambda$ by $(F*G)_q$.*

(ii)For any real $q \neq 0$,we briefly describe F_q^* and ${}^*F_q$ of a functional F on $C_{a,b}[0,T]$ as follow:

$$(2.14)\qquad F_q^* = (F*1)_q \qquad \text{and} \qquad {}^*F_q = (1*F)_q.$$

The following generalized analytic Feynman integral formula is used several times in this paper.

$$(2.15)\qquad E_x[\exp\{i\lambda^{-1/2}\langle v,x\rangle\}] = \exp\bigg\{-\frac{1}{2\lambda}(v^2,b') + i\lambda^{-1/2}(v,a')\bigg\}$$

for all $\lambda \in \tilde{\mathbb{C}}_+$ and $v \in L^2_{a,b}[0,T]$ where

$$(2.16)\qquad (v,a') = \int_0^T v(t)a'(t)dt = \int_0^T v(t)da(t)$$

and

$$(2.17)\qquad (v^2,b') = \int_0^T v^2(t)b'(t)dt = \int_0^T v^2(t)db(t).$$

In this paper, for each $\lambda \in \tilde{\mathbb{C}}_+$, $\lambda^{-1/2}$(or $\lambda^{1/2}$) is always chosen to have positive real part.

Let

(2.18)

$$C'_{a,b}[0,T] = \big\{w \in C_{a,b}[0,T] : w(t) = \int_0^t z(s)db(s) \text{ for some } z \in L^2_{a,b}[0,T]\big\}.$$

For $w \in C'_{a,b}[0,T]$, with $w(t) = \int_0^t z(s)db(s)$ for $t \in [0,T]$, let $D_t : C'_{a,b}[0,T] \to L^2_{a,b}[0,T]$ be defined by the formula

$$(2.19)\qquad D_t w = z(t) = \frac{w'(t)}{b'(t)}.$$

Then $C'_{a,b} \equiv C'_{a,b}[0,T]$ with inner product

$$(w_1, w_2)_{C'_{a,b}} = \int_0^T D_t w_1 D_t w_2 db(t) \tag{2.20}$$

is a separable Hilbert space. Furthermore, $(C'_{a,b}[0,T], C_{a,b}[0,T], \mu)$ is an abstract Wiener space. For more details, see [14].

Note that for all $w, w_1, w_2 \in C'_{a,b}[0,T]$,

$$((D_t w)^2, b') = \int_0^T (D_t w)^2 db(t) = \|w\|^2_{C'_{a,b}}, \tag{2.21}$$

$$(D_t w, a') = \int_0^T D_t w da(t) = \int_0^T D_t w D_t a db(t) = (w, a)_{C'_{a,b}} \tag{2.22}$$

and

$$\langle D_t w_1, w_2 \rangle = \int_0^T D_t w_1 dw_2(t) = \int_0^T D_t w_1 D_t w_2 db(t) = (w_1, w_2)_{C'_{a,b}}. \tag{2.23}$$

Next, we define a class of functionals on $C_{a,b}[0,T]$ like a Fresnel class of an abstract Wiener space. Note that the linear operator given by equation (2.19) is an isomorphism. In fact, the inverse operator $D_t^{-1} : L^2_{a,b}[0,T] \to C'_{a,b}[0,T]$ is given by the formula

$$D_t^{-1} z = \int_0^t z(s) db(s) \tag{2.24}$$

and D_t^{-1} is a bounded operator since

$$\begin{aligned} \|D_t^{-1} z\|_{C'_{a,b}} &= \left\| \int_0^t z(s) db(s) \right\|_{C'_{a,b}} = \left(\int_0^T z^2(t) db(t) \right)^{\frac{1}{2}} \\ &\le \left(\int_0^T z^2(t) d[b(t) + |a|(t)] \right)^{\frac{1}{2}} = \|z\|_{a,b}. \end{aligned} \tag{2.25}$$

Thus by open mapping theorem, D_t is also bounded and there exist positive real numbers α and β such that $\alpha \|w\|_{C'_{a,b}} \le \|D_t w\|_{a,b} \le \beta \|w\|_{C'_{a,b}}$ for all

$w \in C'_{a,b}[0,T]$. Hence we see that the Borel σ-algebra on $(C'_{a,b}[0,T], \|\cdot\|_{C'_{a,b}})$ is given by

$$\mathcal{B}(C'_{a,b}[0,T]) = \{D_t^{-1}(E) : E \in \mathcal{B}(L^2_{a,b}[0,T])\}.$$

Let σ be a complex Borel measure on $\mathcal{B}(L^2_{a,b}[0,T])$. Define a set function f on $\mathcal{B}(C'_{a,b}[0,T])$ by $f(B) = \sigma(D_t(B))$ for each $B \in \mathcal{B}(C'_{a,b}[0,T])$. Then f is a complex Borel measure on $\mathcal{B}(C'_{a,b}[0,T])$. Conversely, let f be a complex Borel measure on $\mathcal{B}(C'_{a,b}[0,T])$. Then the set function σ defined on $\mathcal{B}(L^2_{a,b}[0,T])$ by $\sigma(E) = f(D_t^{-1}(E))$ for each $E \in \mathcal{B}(L^2_{a,b}[0,T])$ is a complex Borel measure on $\mathcal{B}(L^2_{a,b}[0,T])$.

Definition 2.4. *Let*

$$\mathcal{M}(C'_{a,b}[0,T]) = \{f : f = \sigma\circ D_t, \sigma \text{ is a complex Borel measure on } L^2_{a,b}[0,T]\}.$$

The Banach algebra $\mathcal{F}(C_{a,b}[0,T])$ consists of those functionals F on $C_{a,b}[0,T]$ expressible in the form

$$F(x) = \int_{C'_{a,b}[0,T]} \exp\{i\langle D_t w, x\rangle\} df(w) \tag{2.26}$$

for s-a.e. $x \in C_{a,b}[0,T]$ where the associated measure f is an element $\mathcal{M}(C'_{a,b}[0,T])$. We call $\mathcal{F}(C_{a,b}[0,T])$ the Fresnel type class of the function space $C_{a,b}[0,T]$.

Remark 2.2. *(i) $\mathcal{M}(C'_{a,b}[0,T])$ is a Banach algebra under the total variation norm where convolution is taken as the multiplication.*

(ii) One can show that the correspondence $f \to F$ is injective, carries convolution into pointwise multiplication and that $\mathcal{F}(C_{a,b}[0,T])$ is a Banach algebra with norm

$$\|F\| = \|f\| = \int_{C'_{a,b}[0,T]} |df(w)|.$$

3. Transforms and convolutions

In this section, we obtain several results for the GCP of functionals in the class $\mathcal{F}(C_{a,b}[0,T])$.

REMARK 3.1. *Let F be an element of $\mathcal{F}(C_{a,b}[0,T])$ whose associated measure f satisfies the condition*

$$\int_{C'_{a,b}[0,T]} \exp\{|2q_0|^{-1/2}\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\}|df(w)| < \infty \tag{3.1}$$

for some $q_0 \in \mathbb{R}-\{0\}$. Then for all $q \in \mathbb{R}$ with $|q| \geq |q_0|$, the generalized analytic Feynman integral $E^{\mathrm{anf}_q}[F]$ of F exists and is given by the formula

$$E^{\mathrm{anf}_q}[F] = \int_{C'_{a,b}[0,T]} \exp\bigg\{-\frac{i}{2q}\|w\|^2_{C'_{a,b}} + i\Big(\frac{i}{q}\Big)^{\frac{1}{2}}(w,a)_{C'_{a,b}}\bigg\}df(w). \tag{3.2}$$

In next theorem, we obtain the L_p analytic GFFT $T_q^{(p)}(F)$ of a functional F in $\mathcal{F}(C_{a,b}[0,T])$.

THEOREM 3.1. *Let q_0 be a nonzero real number and let F be an element of $\mathcal{F}(C_{a,b}[0,T])$ whose associated measure f satisfies the condition (3.1) above. Then for all $p \in [1,2]$ and all real q with $|q| \geq |q_0|$, the L_p analytic GFFT of F, $T_q^{(p)}(F)$ exists and is given by the formula*

$$T_q^{(p)}(F)(y) = \int_{C'_{a,b}[0,T]} \exp\bigg\{i\langle D_t w, y\rangle - \frac{i}{2q}\|w\|^2_{C'_{a,b}} + i\Big(\frac{i}{q}\Big)^{\frac{1}{2}}(w,a)_{C'_{a,b}}\bigg\}df(w) \tag{3.3}$$

for s-a.e. $y \in C_{a,b}[0,T]$. Furthermore, $T_q^{(p)}(F)$ is an element of $\mathcal{F}(C_{a,b}[0,T])$ with associated measure ϕ defined by

$$\phi(B) = \int_B \exp\bigg\{-\frac{i}{2q}\|w\|^2_{C'_{a,b}} + i\Big(\frac{i}{q}\Big)^{\frac{1}{2}}(w,a)_{C'_{a,b}}\bigg\}df(w) \tag{3.4}$$

for $B \in \mathcal{B}(C'_{a,b}[0,T])$.

Proof. For $\lambda > 0$ and for s-a.e. $y \in C_{a,b}[0,T]$, using equation (2.10), the Fubini theorem and equation (2.15), we obtain

$$\begin{aligned} T_\lambda(F)(y) &= E_x[F(y+\lambda^{-1/2}x)] \\ &= \int_{C'_{a,b}[0,T]} E_x[\exp\{i\langle D_t w, y\rangle + i\lambda^{-1/2}\langle D_t w, x\rangle\}]df(w) \\ &= \int_{C'_{a,b}[0,T]} \exp\bigg\{i\langle D_t w, y\rangle - \frac{1}{2\lambda}((D_t w)^2, b') + \frac{i}{\sqrt{\lambda}}(D_t w, a')\bigg\}df(w) \end{aligned} \tag{3.5}$$

$$= \int_{C'_{a,b}[0,T]} \exp\left\{ i\langle D_t w, y\rangle - \frac{1}{2\lambda}\|w\|^2_{C'_{a,b}} + \frac{i}{\sqrt{\lambda}}(w,a)_{C'_{a,b}} \right\} df(w).$$

But the last expression above is analytic through $\mathbb{C}_+$ and is continuous on $\tilde{\mathbb{C}}_+$. Also, it is bounded on the region $\Gamma = \{\lambda \in \tilde{\mathbb{C}}_+ : |\mathrm{Im}(\lambda^{-1/2})| \le |2q_0|^{-1/2}\}$. Thus equation (3.3) is established. Let ϕ be a set function on $\mathcal{B}(C'_{a,b}[0,T])$ defined by equation (3.4). By using condition (3.1) we see that

$$\|\phi\| \le \int_{C'_{a,b}[0,T]} \exp\left\{ |2q_0|^{-1/2}\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}} \right\} |df(w)| < +\infty. \tag{3.6}$$

Hence we have the desired result. □

In our next theorem, we obtain the GCP of functionals in $\mathcal{F}(C_{a,b}[0,T])$

THEOREM 3.2. *Let q_0 be the nonzero real number and let F and G be elements of $\mathcal{F}(C_{a,b}[0,T])$ whose associated measures f and g satisfy the condition*

$$\int_{C'_{a,b}[0,T]} \exp\left\{ |4q_0|^{-1/2}\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}} \right\} [|df(w)| + |dg(w)|] < \infty. \tag{3.7}$$

Then their GCP $(F*G)_q$ exists for all $p \in [1,2]$ and all real q with $|q| \ge |q_0|$ and is given by the formula

$$\begin{aligned} &(F*G)_q(y) \\ &= \int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]} \exp\left\{ \frac{i}{\sqrt{2}}\langle D_t w_1 + D_t w_2, y\rangle \right. \\ &\qquad - \frac{i}{4q}(\|w_1\|^2_{C'_{a,b}} + \|w_2\|^2_{C'_{a,b}} - 2(w_1,w_2)_{C'_{a,b}}) \\ &\qquad \left. + i\left(\frac{i}{2q}\right)^{\frac{1}{2}}((w_1,a)_{C'_{a,b}} - (w_2,a)_{C'_{a,b}}) \right\} df(w_1)dg(w_2). \end{aligned} \tag{3.8}$$

Proof. By using (2.13), the Fubini theorem, and (2.15), we have that for $\lambda > 0$,

$$
\begin{aligned}
&(F*G)_\lambda(y)\\
&= E_x\Big[F\Big(\frac{y+\lambda^{-1/2}x}{\sqrt{2}}\Big)G\Big(\frac{y-\lambda^{-1/2}x}{\sqrt{2}}\Big)\Big]\\
&= \int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]}\exp\Big\{\frac{i}{\sqrt{2}}\langle D_t w_1 + D_t w_2, y\rangle\\
&\qquad - \frac{i}{4\lambda}(\|w_1\|^2_{C'_{a,b}} + \|w_2\|^2_{C'_{a,b}} - 2(w_1,w_2)_{C'_{a,b}})\\
&\qquad + i\Big(\frac{1}{2\lambda}\Big)^{\frac{1}{2}}((w_1,a)_{C'_{a,b}} - (w_2,a)_{C'_{a,b}})\Big\}df(w_1)dg(w_2)
\end{aligned} \tag{3.9}
$$

for s-a.e. $y\in C_{a,b}[0,T]$. But the last expression above is analytic throughout $\mathbb{C}_+$, and is continuous on $\tilde{\mathbb{C}}_+$. Thus we have the equation (3.10) above. Let a set function $h:\mathcal{B}(C'_{a,b}[0,T]\times C'_{a,b}[0,T])\longrightarrow\mathbb{C}$ be defined by

$$
\begin{aligned}
h(E) = \int_E \exp\Big\{ -\frac{i}{4q}(\|w_1\|^2_{C'_{a,b}} + \|w_2\|^2_{C'_{a,b}} + 2(w_1,w_2)_{C'_{a,b}})\\
+i\Big(\frac{i}{2q}\Big)^{\frac{1}{2}}((w_1,a)_{C'_{a,b}} - (w_2,a)_{C'_{a,b}})\Big\}df(w_1)dg(w_2)
\end{aligned} \tag{3.10}
$$

for each $E\in\mathcal{B}(C'_{a,b}[0,T]\times C'_{a,b}[0,T])$. Then h is a complex Borel measure on $\mathcal{B}(C'_{a,b}[0,T]\times C'_{a,b}[0,T])$. Now we define a function $\varphi : C'_{a,b}[0,T]\times C'_{a,b}[0,T]\longrightarrow C'_{a,b}[0,T]$ by

$$
\varphi(w_1,w_2) = \frac{1}{\sqrt{2}}(w_1+w_2). \tag{3.11}
$$

Then φ is continuous and so it is Borel measurable. Let $\tilde{h} = h\circ\varphi^{-1}$. By the condition (3.7) above, we that for real q with $|q|\ge|q_0|$

$$
\begin{aligned}
\|\tilde{h}\| &= \int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]}|dh(w_1,w_2)|\\
&\le \int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]}\Big|\exp\Big\{-\frac{i}{4q}(\|w_1\|^2_{C'_{a,b}} + \|w_2\|^2_{C'_{a,b}} - 2(w_1,w_2)_{C'_{a,b}})
\end{aligned}
$$

(3.12)

$$+\,i\Big(\frac{i}{2q}\Big)^{\frac12}((w_1,a)_{C'_{a,b}}-(w_2,a)_{C'_{a,b}})\Big\}\Big||df(w_1)||dg(w_2)|$$
$$\le\int_{C'_{a,b}[0,T]}\exp\Big\{\frac{1}{\sqrt{4q_0}}\int_0^T|w_1(s)|d|a(s)|\Big\}|df(w_1)|$$
$$\cdot\int_{C'_{a,b}[0,T]}\exp\Big\{\frac{1}{\sqrt{4q_0}}\int_0^T|w_2(s)|d|a(s)|\Big\}|dg(w_2)|<\infty.$$

Hence $\tilde{h}=h\circ\varphi^{-1}$ belongs to $\mathcal{M}(C'_{a,b}[0,T])$ and

$$(F*G)_q(y)=\int_{C'_{a,b}[0,T]}\exp\{i\langle r,y\rangle\}d\,h(r) \tag{3.13}$$

for s-a.e. $y\in C'_{a,b}[0,T]$. Hence $(F*G)_q$ exists and is given by (3.8) for all real q with $|q|\ge|q_0|$ and it belong to $\mathcal{F}(C_{a,b}[0,T])$. □

REMARK 3.2. *Let F, f, and q_0 be as in Theorem 3.2. Then for all real q with $|q|\ge|q_0|$, ${}^*F_q$ and F_q^* are in $\mathcal{F}(C_{a,b}[0,T])$.*

THEOREM 3.3. *Let F, G, f, g, and q_0 be as in Theorem 3.2. Then for all $p\in[1,2]$ and all real q with $|q|\ge|q_0|$,*

$$T_q^{(p)}((F*G)_q)(y)=\Big(T_q^{(p)}(F_q^*)(y)\Big)\Big(T_q^{(p)}({}^*G_q)(y)\Big) \tag{3.14}$$

for s-a.e. $y\in C_{a,b}[0,T]$, where F_q^* and ${}^*G_q$ are given by (2.13). Also, both of the expressions in (3.14) are given by the expression

$$\int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]}\exp\Big\{\frac{1}{\sqrt2}\langle D_tw_1+D_tw_2,y\rangle-\frac{i}{2q}(\|w_1\|^2_{C'_{a,b}}+\|w_2\|^2_{C'_{a,b}})$$
$$+\,2i\Big(\frac{i}{2q}\Big)^{\frac12}(w_1,a)_{C'_{a,b}}\Big\}df(w_1)dg(w_2). \tag{3.15}$$

Proof. By using (2.10), (2.13), the Fubini theorem, and (2.15), we have that for $\lambda>0$,

$$T_\lambda((F*G)_\lambda)(y)=T_\lambda(F_\lambda^*)(y)T_\lambda(G_\lambda^*)(y) \tag{3.16}$$

for s-a.e. $y \in C_{a,b}[0,T]$. But both of the expressions on the right-hand side of equation (3.16) are analytic functions of λ throughout $\mathbb{C}_+$, and are continuous functions of λ on $\tilde{\mathbb{C}}_+$ for all $y \in C_{a,b}[0,T]$. By using (3.7), $T_q^{(p)}((F * G)_q)$ exists for all real q with $|q| \geq |q_0|$ and is given by (3.14) for all desired values of p and q. □

THEOREM 3.4. *Let F, G, f, g, and q_0 be as in Theorem 3.3. Then*

$$\begin{aligned}
&\int_{C_{a,b}[0,T]}^{anf_{-q}} T_q^{(p)}((F*G)_q)(y)d\mu(y) \\
&= \int_{C_{a,b}[0,T]}^{anf_{-q}} T_q^{(p)}(F_q^*)(y)T_q^{(p)}({}^*G_q)(y)d\mu(y) \\
&= \int_{C_{a,b}[0,T]}^{anf_{q}} (F_{-q}^*)^*_{\frac{q}{2}}(\sqrt{2}y)(G_{-q}^*)^*_{\frac{q}{2}}(-\sqrt{2}y)d\mu(y).
\end{aligned} \tag{3.17}$$

Proof. Fix p and q. Then for $\lambda > 0$, using (3.9), the Fubini theorem and (3.3), we have

$$\begin{aligned}
&\int_{C_{a,b}[0,T]} T_q^{(p)}((F*G)_q)(y/\sqrt{\lambda})d\mu(y) \\
&= \int_{C_{a,b}[0,T]}\int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]} \exp\bigg\{\frac{1}{\sqrt{2\lambda}}\langle D_t w_1 + D_t w_2, y\rangle \\
&\qquad - \frac{i}{2q}(\|w_1\|^2_{C'_{a,b}} + \|w_2\|^2_{C'_{a,b}}) + 2i\Big(\frac{i}{2q}\Big)^{\frac{1}{2}}(w_1,a)_{C'_{a,b}}\bigg\}df(w_1)dg(w_2)d\mu(y) \\
&= \int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]} \exp\bigg\{-\frac{1}{4\lambda}(\|w_1\|^2_{C'_{a,b}} + \|w_2\|^2_{C'_{a,b}} + 2(w_1,w_2)_{C'_{a,b}}) \\
&\qquad \frac{i}{\sqrt{2\lambda}}((w_1,a)_{C'_{a,b}} + (w_2,a)_{C'_{a,b}}) - \frac{i}{2q}(\|w_1\|^2_{C'_{a,b}} + \|w_2\|^2_{C'_{a,b}}) \\
&\qquad + 2i\Big(\frac{i}{2q}\Big)^{\frac{1}{2}}(w_1,a)_{C'_{a,b}}\bigg\}df(w_1)dg(w_2).
\end{aligned} \tag{3.18}$$

But the last expression is an analytic function of λ throughout $\tilde{\mathbb{C}}_+$ and is continuous throughout $\tilde{\mathbb{C}}_+$, and so letting $\lambda = -i(-q) = iq$, we obtain that

$$\begin{aligned}
&\int_{C_{a,b}[0,T]}^{anf_{-q}} T_q^{(p)}((F*G)_q)(y)d\mu(y) \\
&= \int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]} \exp\bigg\{\frac{i}{4q}(\|w_1\|^2_{C'_{a,b}} + \|w_2\|^2_{C'_{a,b}} + 2(w_1,w_2)_{C'_{a,b}}) \\
&\qquad + i\bigg(\frac{-i}{2q}\bigg)^{\frac{1}{2}}((w_1,a)_{C'_{a,b}} + (w_2,a)_{C'_{a,b}}) \\
&\qquad - \frac{i}{2q}(\|w_1\|^2_{C'_{a,b}} + \|w_2\|^2_{C'_{a,b}}) + 2i\bigg(\frac{i}{2q}\bigg)^{\frac{1}{2}}(w_1,a)_{C'_{a,b}}\bigg\}df(w_1)dg(w_2) \\
&= \int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]} \exp\bigg\{-\frac{i}{4q}(\|w_1\|^2_{C'_{a,b}} + \|w_2\|^2_{C'_{a,b}} - 2(w_1,w_2)_{C'_{a,b}}) \\
&+ i\bigg(\frac{-i}{2q}\bigg)^{\frac{1}{2}}((w_1,a)_{C'_{a,b}} + (w_2,a)_{C'_{a,b}}) + 2i\bigg(\frac{i}{2q}\bigg)^{\frac{1}{2}}(w_1,a)_{C'_{a,b}}\bigg\}df(w_1)dg(w_2).
\end{aligned} \tag{3.19}$$

Clearly, the condition (3.7) will imply the existence of (3.19). On the other hand, using (2.14),(3.8), the Fubini theorem, and (3.2), we obtain that

$$\begin{aligned}
(F^*_{-q})^*_{\frac{q}{2}}(\sqrt{2}y) &= \int_{C'_{a,b}[0,T]} \exp\bigg\{\frac{i}{\sqrt{2}}\langle D_t w_1, y\rangle \\
&\quad + i\bigg(\frac{i}{2q}\bigg)^{\frac{1}{2}}(w_1,a)_{C'_{a,b}} + i\bigg(\frac{-i}{2q}\bigg)^{\frac{1}{2}}(w_1,a)_{C'_{a,b}}\bigg\}df(w_1)
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
(G^*_{-q})^*_{\frac{q}{2}}(-\sqrt{2}y) &= \int_{C'_{a,b}[0,T]} \exp\bigg\{-\frac{i}{\sqrt{2}}\langle D_t w_2, y\rangle \\
&\quad + i\bigg(\frac{i}{2q}\bigg)^{\frac{1}{2}}(w_2,a)_{C'_{a,b}} + i\bigg(\frac{-i}{2q}\bigg)^{\frac{1}{2}}(w_2,a)_{C'_{a,b}}\bigg\}dg(w_2)
\end{aligned} \tag{3.21}$$

s-a.e. $y \in C_{a,b}[0,T]$. By using (3.20) and (3.21), we have that for $\lambda > 0$

$$\int_{C_{a,b}[0,T]} (F^*_{-q})^*_{\frac{q}{2}}(\sqrt{2}y/\sqrt{\lambda})(G^*_{-q})^*_{\frac{q}{2}}(-\sqrt{2}y/\sqrt{\lambda})d\mu(y)$$

$$= \int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]} \exp\bigg\{ -\frac{1}{4\lambda}(\|w_1\|^2_{C'_{a,b}} + \|w_2\|^2_{C'_{a,b}} - 2(w_1,w_2)_{C'_{a,b}})$$

(3.22)

$$+\frac{i}{\sqrt{2\lambda}}((w_1,a)_{C'_{a,b}} - (w_2,a)_{C'_{a,b}}) + i\bigg(\frac{i}{2q}\bigg)^{\frac{1}{2}}((w_1,a)_{C'_{a,b}} - (w_2,a)_{C'_{a,b}})$$

$$+ i\bigg(\frac{-i}{2q}\bigg)^{\frac{1}{2}}((w_1,a)_{C'_{a,b}} + (w_2,a)_{C'_{a,b}})\bigg\}df(w_1)dg(w_2).$$

But the last expression above is an analytic function of λ throughout $\tilde{\mathbb{C}}_+$ and is continuous throughout on $\tilde{\mathbb{C}}_+$ and so letting $\lambda \to -iq$ we obtain that

(3.23)

$$\int^{anf_q}_{C_{a,b}[0,T]} (F^*_{-q})^*_{\frac{q}{2}}(\sqrt{2}y)(G^*_{-q})^*_{\frac{q}{2}}(-\sqrt{2}y)d\mu(y)$$

$$= \int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]} \exp\bigg\{ -\frac{i}{4q}(\|w_1\|^2_{C'_{a,b}} + \|w_2\|^2_{C'_{a,b}} - 2(w_1,w_2)_{C'_{a,b}})$$

$$+ 2i\bigg(\frac{i}{2q}\bigg)^{\frac{1}{2}}(w_1,a)_{C'_{a,b}} + i\bigg(\frac{-i}{2q}\bigg)^{\frac{1}{2}}((w_1,a)_{C'_{a,b}} + (w_2,a)_{C'_{a,b}})\bigg\}df(w_1)dg(w_2).$$

Now (3.19) and (3.23) together yield (3.17). □

REMARK 3.4. *In Theorem 3.4 above, if $a(t) \equiv 0$, then for all $q \neq 0$,*

(3.24)

$$T^{(p)}_q(F^*_q)(F)(y) = T^{(p)}_q(F)(y/\sqrt{2}) \text{and} T^{(p)}_q({}^*G_q)(F)(y) = T^{(p)}_q(G)(y/\sqrt{2})$$

for s-a.e. $y \in C_{a,b}[0,T]$. Furthermore

$$(F^*_{-q})^*_{\frac{q}{2}}(\sqrt{2}y) = F(y/\sqrt{2}) \qquad \text{and} \qquad (G^*_{-q})^*_{\frac{q}{2}}(-\sqrt{2}y) = G(-y/\sqrt{2}).$$

Hence we have the following Parseval's identity

$$\begin{aligned}&\int_{C_{a,b}[0,T]}^{anf_{-q}} T_q^{(p)}((F*G)_q)(y)d\mu(y)\\&=\int_{C_{a,b}[0,T]}^{anf_{-q}} T_q^{(p)}(F)(y/\sqrt{2})T_q^{(p)}(G)(y/\sqrt{2})d\mu(y)\\&=\int_{C_{a,b}[0,T]}^{anf_{q}} F(y/\sqrt{2})G(-y/\sqrt{2})d\mu(y).\end{aligned}$$

4. Multiple L_p analytic GFFT and the GCP

In this section we establish some relationships between the multiple L_p analytic GFFT and the GCP of functionals in $\mathcal{F}(C_{a,b}[0,T])$.

Definition 4.1 Let F be a measurable functional defined on $C_{a,b}[0,T]$ and define a transform $(T_\gamma)^{(n)}(\gamma>0)$ of F by

$$(T_\gamma)^{(n)}(F)=(\underbrace{T_\gamma\circ\cdots\circ T_\gamma}_{n-times})(F) \tag{4.1}$$

that is, $(T_\gamma)^{(n)}$ means the n-times composition of (T_γ), where T_γ is given by (2.10) in Definition 2.2 and n is a nonnegative integer. When γ is in $\mathbb{C}_+$, the transform $(T_\gamma)^{(n)}(F)$ means the analytic extension of $(T_\gamma)^n(F)(\gamma>0)$ as the function of $\lambda\in\mathbb{C}_+$. Let $(T_\lambda)^{(n)}(F)$ be analytic extension of $(T_\gamma)^n(F)$ as a function of $\lambda\in\mathbb{C}_+$. In case that $1<p\le 2$, for each $q\in\mathbb{R}-\{0\}$, we define the multiple L_p analytic GFFT $(T_q)^{(n)}(F)$ of F by

$$(T_q^{(p)})^{(n)}(F)=\text{l.i.m.}_{\lambda\to -iq}(T_\lambda)^{(n)}(F), \tag{4.2}$$

where λ approaches $-iq$ through $\mathbb{C}_+$. In case that $p=1$, for each $q\in\mathbb{R}-\{0\}$, we define the multiple L_1 analytic GFFT $(T_q^{(1)})^{(n)}(F)$ of F by

$$(T_q^{(1)})^n(F)=\lim_{\lambda\to -iq}(T_\lambda)^{(n)}(F), \tag{4.3}$$

where λ approaches $-iq$ through $\mathbb{C}_+$.

Note that $(T_\lambda)^{(0)}(F)\equiv F\equiv (T_q^{(p)})^{(0)}$, $(T_\lambda)^{(1)}(F)\equiv T_\lambda(F)$, and $(T_q^{(p)})^{(1)}\equiv T_q^{(p)}(F)$.

We have already shown that for $F \in \mathcal{F}(C_{a,b}[0,T])$ with condition (3.1), the L_p GFFT $T_q^{(p)}(F)$ belongs to the Fresnel type class $\mathcal{F}(C_{a,b}[0,T])$.Hence by using the mathematical induction and proceeding as in the proof of Theorem 3.1, we can obtain the following theorem.

THEOREM 4.1. *Let q_0 be a nonzero real number and let n be a nonnegative integer. Let $F \in \mathcal{F}(C_{a,b}[0,T])$ be given by (3.26) whose associated measure f satisfies the condition*

$$\int_{C'_{a,b}[0,T]} \exp\left\{n|2q_0|^{-1/2}\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\right\}|df(w)| < \infty. \tag{4.4}$$

Then for all $p \in [1,2]$ and all real q with $|q| \geq |q_0|$, the multiple L_p analytic GFFT $(T_q^{(p)})^{(n)}(F)$ exists and is given by

$$\begin{aligned}(T_q^{(p)})^{(n)}(F) \\ = \int_{C'_{a,b}[0,T]} \exp\left\{i\langle D_t w, y\rangle - \frac{in}{2q}\|w\|^2_{C'_{a,b}} + in\left(\frac{i}{q}\right)^{\frac{1}{2}}(w,a)_{C'_{a,b}}\right\}df(w)\end{aligned} \tag{4.5}$$

for s-a.e. $y \in C_{a,b}[0,T]$. And $(T_q^{(p)})^{(n)}(F)$ is an element of $\mathcal{F}(C_{a,b}[0,T])$ with associated measure

$$\phi_n(E) = \int_E \exp\left\{-\frac{in}{2q}\|w\|^2_{C'_{a,b}} + in\left(\frac{i}{q}\right)^{\frac{1}{2}}(w,a)_{C'_{a,b}}\right\}df(w)$$

for $E \in \mathcal{B}(C'_{a,b}[0,T])$. Note that (4.5) is reduced to (3.3), if we take $n = 1$ in (4.5).

Next, we obtain the GCP of the multiple L_p analytic GFFT's of functionals in $\mathcal{F}(C_{a,b}[0,T])$.

THEOREM 4.2. *Let q_0 be a nonzero real number and let n be nonnegative integer. Let F and G be elements of $\mathcal{F}(C_{a,b}[0,T])$ whose associated measures f and g satisfy the condition*

$$\int_{C'_{a,b}[0,T]} \exp\left\{n|2q_0|^{-1/2}\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\right\}[|df(w)| + |dg(w)|] < \infty. \tag{4.6}$$

Then for all $p \in [1,2]$, all q with $|q| \geq |q_0|$ and nonnegative integer m, the GCP $((T_q^{(p)})^{(n)}(F) * ((T_q^{(p)})^{(m)}(G))_q(y)$ exists and is given by (4.7) below. Furthermore $((T_q^{(p)})^{(n)}(F) * ((T_q^{(p)})^{(m)}(G))_q(y)$ is an elements of $\mathcal{F}(C_{a,b}[0,T])$.

Proof. By using (4.5) and (3.8) we observe that for all $p \in [1,2]$ and all q with $|q| \geq |q_0|$

$$
\begin{aligned}
&((T_q^{(p)})^{(n)}(F) * ((T_q^{(p)})^{(m)}(G))_q(y) \\
&= \int_{C'_{a,b}[0,T]} \int_{C'_{a,b}[0,T]} \exp\bigg\{ \frac{i}{\sqrt{2}}\langle D_t w_1 + D_t w_2, y\rangle - \frac{in}{2q}\|w_1\|^2_{C'_{a,b}} - \frac{im}{2q}\|w_2\|^2_{C'_{a,b}} \\
&\qquad + in\Big(\frac{i}{q}\Big)^{\frac{1}{2}}(w_1,a)_{C'_{a,b}} + im\Big(\frac{i}{q}\Big)^{\frac{1}{2}}(w_2,a)_{C'_{a,b}} \\
&\qquad - \frac{i}{4q}(\|w_1\|^2_{C'_{a,b}} + \|w_2\|^2_{C'_{a,b}} + 2(w_1,w_2)_{C'_{a,b}}) \\
&\qquad + i\Big(\frac{i}{2q}\Big)^{\frac{1}{2}}((w_1,a)_{C'_{a,b}} + (w_2,a)_{C'_{a,b}})\bigg\} df(w_1)dg(w_2)
\end{aligned} \tag{4.7}
$$

for s-a.e. $y \in C_{a,b}[0,T]$. Furthermore, proceeding as in the proof of Theorem 3.2 above and using (4.6), we see that $((T_q^{(p)})^{(n)}(F) * ((T_q^{(p)})^{(m)}(G))_q(y)$ is an element of $\mathcal{F}(C_{a,b}[0,T])$ □

Note that (4.7) is reduced to (3.8), if we take $m = n = 0$ in (3.8).

In our next theorem, we obtain the multiple L_p analytic GFFT of the convolution product for two functionals in $\mathcal{F}(C_{a,b}[0,T])$.

THEOREM 4.3. *Let F, G, f, g and q_0 be as in Theorem 4.2. Then for all $p \in [1,2]$ and all real q the following equation with $|q| \geq |q_0|$,*

$$
\begin{aligned}
&(T_q^{(p)})^{(n)}((F*G)_q)(y)\\
&=\int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]}\exp\bigg\{\frac{i}{\sqrt{2}}\langle D_t w_1+D_t w_2,y\rangle\\
&\qquad -\frac{i}{4q}(\|w_1\|^2_{C'_{a,b}}+\|w_2\|^2_{C'_{a,b}}-2(w_1,w_2)_{C'_{a,b}})\\
&\qquad +i\bigg(\frac{i}{2q}\bigg)^{\frac{1}{2}}((w_1,a)_{C'_{a,b}}-(w_2,a)_{C'_{a,b}})\\
&\qquad -\frac{in}{4q}(\|w_1\|^2_{C'_{a,b}}+\|w_2\|^2_{C'_{a,b}}+2(w_1,w_2)_{C'_{a,b}})\\
&\qquad +in\bigg(\frac{i}{2q}\bigg)^{\frac{1}{2}}((w_1,a)_{C'_{a,b}}+(w_2,a)_{C'_{a,b}})\bigg\}df(w_1)dg(w_2)
\end{aligned}
\tag{4.8}
$$

holds of s-a.e. $y\in C_{a,b}[0,T]$, where n is a nonnegative integer. Futhermore, $(T_q^{(p)})^{(n)}((F*G)_q)(y)$ is an element of $\mathcal{F}(C_{a,b}[0,T])$.

Proof. By using equations (3.8) and (4.5), we can easily obtain the equation (4.8) above. Moreover, the condition (4.6) will imply the existence of the equation (4.8). □

Finally, we show that the L_p analytic GFFT of the GCP of the multiple L_p analytic GFFT's of the transforms for functionals in $\mathcal{F}(C_{a,b}[0,T])$.

THEOREM 4.4. *Let F, G, f, g and q_0 be as in Theorem 4.2. Then for all $p\in[1,2]$ and all real q the following equation with $|q|\geq|q_0|$,*

$$
\begin{aligned}
&T_q^{(p)}(((T_q^{(p)})^{(n)}(F)*(T_q^{(p)})^{(m)}(G))_q)(y)\\
&=(T_{q/2}^{(p)})^{(n)}(T_q^{(p)}(F_q^*))(y)(T_{q/2}^{(p)})^{(m)}(T_q^{(p)}(^*G_q))(y)
\end{aligned}
\tag{4.9}
$$

holds for s-a.e. $y\in C_{a,b}[0,T]$, where F_q^* and $^*G_q$ are as in (2.14). Also, both expressions in (4.9) are given by the expression

$$\int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]}\exp\bigg\{\frac{i}{\sqrt{2}}\langle D_tw_1+D_tw_2,y\rangle-\frac{i(n+1)}{2q}\|w_1\|^2_{C'_{a,b}}$$
$$-\frac{i(m+1)}{2q}\|w_2\|^2_{C'_{a,b}}+in\Big(\frac{i}{q}\Big)^{\frac{1}{2}}(w_1,a)_{C'_{a,b}}$$
$$+im\Big(\frac{i}{q}\Big)^{\frac{1}{2}}(w_2,a)_{C'_{a,b}}+i\sqrt{2}\Big(\frac{i}{q}\Big)^{\frac{1}{2}}(w_1,a)_{C'_{a,b}}\bigg\}df(w_1)dg(w_2).$$

Furthermore, the transform $T_q^{(p)}((T_q^{(p)})^{(n)}(F)*((T_q^{(p)})^{(m)}(G))_q)$ *is an element of* $\mathcal{F}(C_{a,b}[0,T])$.

Proof. By using (4.7), (3.8) and (3.3), we can obtain the equation (4.9) above. □

REMARK 4.1. *In Theorem 4.4 above, if* $a(t)\equiv 0$, *then*

$$(T^{(p)}_{q/2})^{(n)}(T^{(p)}_q(F)(\cdot/\sqrt{2}))(y)=(T^{(p)}_q)^{(n+1)}(F)(y/\sqrt{2}) \tag{4.10}$$

and

$$(T^{(p)}_{q/2})^{(m)}(T^{(p)}_q(G))(\cdot/\sqrt{2})(y)=(T^{(p)}_q)^{(m+1)}(G)(y/\sqrt{2}) \tag{4.11}$$

Hence by using (3.19), (4.10) and (4.11) we obtain that

$$\begin{aligned}&T_q^{(p)}((T_q^{(p)})^{(n)}(F)*((T_q^{(p)})^{(m)}(G))_q)(y)\\&=(T_q^{(p)})^{(n+1)}(F)(y/\sqrt{2})(T_q^{(p)})^{(m+1)}(G)(y/\sqrt{2})\end{aligned}$$

for s-a.e. $y\in C_{a,b}[0,T]$.

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