JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **19**, No.1, March 2006

ON THE STABILITY OF A MIXED TYPE FUNCTIONAL EQUATION

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Abstract. The generalized Hyers–Ulam stability problems of the mixed type functional equation

$$f\left(\sum_{i=1}^{4} x_i\right) + \sum_{1\le i<j\le 4} f(x_i+x_j) = \sum_{i=1}^{4} f(x_i) + \sum_{1\le i<j<k\le 4} f(x_i+x_j+x_k)$$

is treated under the approximately even(or odd) condition and the behavior of the quadratic mappings and the additive mappings is investigated.

1. Introduction

In 1940, Ulam proposed the general Ulam stability problem (see [6]):

"*When is it true that by slightly changing the hypothesis of a theorem one can still assert that the thesis of the theorem remains true or approximately true?*"

In 1941, this problem was solved by Hyers [2] in the case of Banach spaces. Thereafter, this type of stability is called the Hyers–Ulam stability. In 1978 Th. M. Rassias [4] provided a remarkable generalization of the Hyers–Ulam stability of mappings by considering variables. This fact rekindled interest in the field. Such type of stability is now called the Hyers–Ulam–Rassias stability of functional equations. For the function case, the reader is referred to Găvruta [1]. Throughout this paper, let X be a real normed space and Y be a real Banach space in the case of functional inequalities, as well as let X and Y be real linear spaces for the case of functional equations.

Received February 12, 2006.
2000 Mathematics Subject Classification: Primary 39B52, 39B72.
Key words and phrases: Hyers–Ulam stability, approximately even (or odd) mapping.

Definition. A mapping $f : X \to Y$ is called *additive* (respectively, *quadratic*) if f satisfies the equation $f(x+y) = f(x)+f(y)$ (respectively, $f(x + y) + f(x - y) = 2f(x) + 2f(y)$) for all $x, y \in X$.

We here introduce a theorem of Găvruta [1]:

Theorem. *Let G be an abelian group and E a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a function such that*

$$\Phi(x, y) := \sum_{i=1}^{\infty} 2^{-i}\varphi(2^{i-1}x, 2^{i-1}) < \infty$$

for all $x, y \in G$. If a function $f : G \to E$ satisfies the inequality $\|f(x + y) - f(x) - f(y)\| \le \varphi(x, y)$ for any $x, y \in G$, then there exists a unique additive function $A : G \to E$ such that

$$\|f(x) - A(x)\| \le \Phi(x, x)$$

for each $x \in G$.

For a mapping $f : X \to Y$, consider the following functional equation:

(1)
$$f\left(\sum_{i=1}^{4} x_i\right) + \sum_{1\le i<j\le 4} f(x_i + x_j) = \sum_{i=1}^{4} f(x_i) + \sum_{1\le i<j<k\le 4} f(x_i + x_j + x_k).$$

The general mixed stability problem had been posed for the first time by Th.M. Rassias and J. Tabor [5] in the year 1992.

In Section 2, the generalized Hyers–Ulam stability will be discussed in the spirit of Găvruta.

2. Stability for approximately even mappings

Let $\varphi : X^4 \to [0, \infty)$ and $\psi : X \to [0, \infty)$ be two functions such that

$$\begin{aligned}\Phi(x_1, x_2, x_3, x_4) &:= \frac{5}{6}\varphi(0, 0, 0, 0) \\ &+ \frac{1}{2}\sum_{i=1}^{\infty} 2^{-2i}\varphi(2^{i-1}x_1, 2^{i-1}x_2, -2^{i-1}x_3, -2^{i-1}x_4) < \infty,\end{aligned}$$

(2)
$$\Psi(x) := \frac{1}{2}\sum_{i=1}^{\infty} 2^{-2i}\big(4\psi(2^{i-1}x) + \psi(2^i x)\big) < \infty$$

for all $x, x_1, x_2, x_3, x_4 \in X$. The generalized Hyers–Ulam stability of the mixed type equation (1) is proved under a suitable condition by using ideas from the paper of Găvruta [1]. We first prove the following lemma.

In this section, we use the notation $\varphi_1(x) := \varphi(x,x,-x,-x)$ for all $x \in X$.

LEMMA 1. *Let $f : X \to Y$ be a mapping satisfying the inequalities*

$$\left\| f\left(\sum_{i=1}^{4} x_i\right) + \sum_{1\le i<j\le 4} f(x_i+x_j) - \sum_{i=1}^{4} f(x_i) - \sum_{1\le i<j<k\le 4} f(x_i+x_j+x_k)\right\| \le \varphi(x_1,x_2,x_3,x_4) \tag{3}$$

and

$$\|f(x) - f(-x)\| \le \psi(x) \tag{4}$$

for all $x, x_1, x_2, x_3, x_4 \in X$. Then

$$\|f(x) - 2^{-2n} f(2^n x)\| \le \frac{1}{8}\sum_{i=1}^{n} 2^{-2(i-1)}\big(5\varphi_1(0) + \varphi_1(2^{i-1}x) + 4\psi(2^{i-1}x) + \psi(2^i x)\big) \tag{5}$$

for all $x \in X$ and $n \in \mathbb{N}$.

Proof. Put $x_i = 0 (i = 1,2,3,4)$ in (3), then $\|f(0)\| \le \varphi_1(0)$. Also putting $x_i = x (i = 1,2)$ and $x_j = -x (j = 3,4)$ in (3) yields

$$\|4f(x) + 4f(-x) - f(2x) - f(-2x)\| \le 5\varphi_1(0) + \varphi_1(x) \tag{6}$$

for all $x \in X$. We apply induction on n to prove Lemma 1. By (4) and (6), we have

$$\begin{aligned}\|f(x) - 2^{-2}f(2x)\| &\le \frac{1}{8}\Big(\|4f(x) + 4f(-x) - f(2x) - f(-2x)\| \\ &\quad + \big\| - 4\big(f(-x) - f(x)\big)\big\| + \|f(-2x) - f(2x)\|\Big) \\ &\le \frac{5}{8}\varphi_1(0) + \frac{1}{8}\varphi_1(x) + \frac{1}{2}\psi(x) + \frac{1}{8}\psi(2x),\end{aligned} \tag{7}$$

which proves the validity of the inequality (5) when $n = 1$. Now assume that the inequality (5) holds true for some $n \in \mathbb{N}$. By using (6) and (7), and the following relation:

$$\begin{aligned}&\|f(x) - 2^{-2(n+1)}f(2^{n+1}x)\| \\ &\quad \le \|f(x) - 2^{-2n}f(2^n x)\| + \|2^{-2n}f(2^n x) - 2^{-2(n+1)}f(2^{n+1}x)\| \\ &\quad \le \frac{1}{8}\sum_{i=1}^{n} 2^{-2(i-1)}\big(5\varphi_1(0) + \varphi_1(2^{i-1}x) + 4\psi(2^{i-1}x) + \psi(2^i x)\big)\end{aligned}$$

$$+\frac{1}{8}\cdot 2^{-2n}\big(5\varphi_1(0)+\varphi_1(2^n x)+4\psi(2^n x)+\psi(2^{n+1}x)\big)$$
$$\le \frac{1}{8}\sum_{i=1}^{n+1} 2^{-2(i-1)}\big(5\varphi_1(0)+\varphi_1(2^{i-1}x)+4\psi(2^{i-1}x)+\psi(2^i x)\big),$$

one can easily verify inequality (5) for $n+1$. This completes the proof. □

Theorem 2. *Let $f : X \to Y$ be a mapping satisfying inequalities* (3) *and* (4) *for all $x, x_1, x_2, x_3, x_4 \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies* (1) *as well as the inequality*

$$\|f(x)-Q(x)\| \le \Phi(x,x,x,x)+\Psi(x) \tag{8}$$

for all $x \in X$.

Proof. For any $x \in X$ and for every positive integer n, we define $Q_n(x) := 2^{-2n} f(2^n x)$. From (5), one has, for every positive integers $n \ge m$,

$$\begin{aligned}
&\|2^{-2n}f(2^n x)-2^{-2m}f(2^m x)\|\\
&= 2^{-2m}\|2^{-2(n-m)}f(2^{n-m}\cdot 2^m x)-f(2^m x)\| \qquad (9)\\
&\le 2^{-2m}\frac{1}{8}\sum_{i=1}^{n-m}2^{-2(i-1)}\big(5\varphi_1(0)+\varphi_1(2^{i-1}2^m x)+4\psi(2^{i-1}2^m x)\\
&\quad +\psi(2^i 2^m x)\big)
\end{aligned}$$

for all $x \in X$. By (2), since the right-hand side of the inequality (9) tends to zero as m tends to infinity, the sequence $\{Q_n(x)\}$ is a Cauchy sequence. Therefore, we may apply a direct method to the definition of Q. Define $Q(x) = \lim_{n\to\infty} Q_n(x)$ for all $x \in X$. The inequality (3) implies that

$$\Bigg\|Q_n\left(\sum_{i=1}^{4}x_i\right)+\sum_{1\le i<j\le 4}Q_n(x_i+x_j)-\sum_{i=1}^{4}Q_n(x_i)$$
$$-\sum_{1\le i<j<k\le 4}Q_n(x_i+x_j+x_k)\Bigg\| \le 2^{-2n}\varphi(2^n x_1, 2^n x_2, 2^n x_3, 2^n x_4)$$

for all $x_1, x_2, x_3, x_4 \in X$ and all $n \in \mathbb{N}$. Letting n tend to infinity in the last inequality, then by (2), Q satisfies (1). By (1) and (4), we can show that $Q(0) = 0$ (with $x_i = 0 (i = 1, 2, 3, 4)$ in (1)) and that Q is even. Putting $x_1 = -x_2 = x$ and $x_3 = -x_4 = y$ in (1), one gets that Q as an even solution of (1) is quadratic. According to (5), the inequality (8) holds. Now, let $R : X \to Y$ be another quadratic mapping which satisfies equation (1) and inequality (8). Obviously, we have $Q(2^n x) = 2^{2n}Q(x)$ and $R(2^n x) = 2^{2n}R(x)$ for all $x \in X$ and $n \in \mathbb{N}$. Hence, it follows from (8) that

$$\begin{aligned}\|Q(x) - R(x)\| &= 2^{-2n}\|Q(2^n x) - R(2^n x)\| \\ &\leq 2^{-2n}\big(\|Q(2^n x) - f(2^n x)\| + \|f(2^n x) - R(2^n x)\|\big) \\ &\leq 2^{-2n+1}\bigg(\Phi(2^n x, 2^n x, 2^n x, 2^n x) + \Psi(2^n x)\bigg)\end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. By letting $n \to \infty$ in the preceding inequality, one immediately obtains the uniqueness of Q. Therefore the proof is complete. □

3. Stability for approximately odd mappings

From now on, let $\varphi : X^4 \to [0, \infty)$ and $\psi : X \to [0, \infty)$ be two functions such that

$$\begin{aligned}\Phi(x_1, x_2, x_3, x_4) &:= \frac{11}{7}\varphi(0,0,0,0) \\ &+\frac{1}{12}\sum_{i=1}^{\infty}\left(\frac{1}{2^i} - \frac{1}{8^i}\right)\big(3\varphi(2^{i-1}x_1, 2^{i-1}x_2, 2^{i-1}x_3, -2^{i-1}x_4) \\ &+3\varphi(-2^{i-1}x_1, -2^{i-1}x_2, -2^{i-1}x_3, 2^{i-1}x_4) \\ &+2\varphi(2^{i-1}x_1, 2^{i-1}x_2, 2^i x_3, -2^{i-1}x_4)\big) < \infty, \\ \Psi(x) &:= \frac{1}{12}\sum_{i=1}^{\infty}\left(\frac{1}{2^i} - \frac{1}{8^i}\right)\big(17\psi(2^{i-1}x) + 12\psi(2^i x) \\ &+3\psi(3\cdot 2^{i-1}x)\big) < \infty\end{aligned} \tag{10}$$

for all $x, x_1, x_2, x_3, x_4 \in X$.

For simplicity of calculation in this section, we use the notation $\varphi_1(x) := \varphi(x, x, x, -x)$, $\varphi_2(x) := \varphi(x, x, 2x, -x)$ and $\chi(x) := 3\varphi_1(x) + 3\varphi_1(-x) + 2\varphi_2(x) + 17\psi(x) + 12\psi(2x) + 3\psi(3x)$ for all $x \in X$.

LEMMA 4. *Let $f : X \to Y$ be a mapping satisfying* (3) *and*

$$\|f(x) + f(-x)\| \leq \psi(x) \tag{11}$$

for all $x, x_1, x_2, x_3, x_4 \in X$*. Then*

$$\left\| f(x) - \frac{1}{6}\left(\frac{8}{2^n} - \frac{2}{8^n}\right) f(2^n x) + \frac{1}{6}\left(\frac{1}{2^n} - \frac{1}{8^n}\right) f(2^{n+1}x) \right\|$$

$$\leq \frac{1}{12}\sum_{i=1}^{n}\left(\frac{1}{2^i} - \frac{1}{8^i}\right)\left(22\varphi_1(0) + \chi(2^{i-1}x)\right) \tag{12}$$

for all $x \in X$ *and* $n \in \mathbb{N}$*.*

Proof. Putting $x_i = x(i = 1, 2, 3)$ and $x_j = -x(j = 4)$ in (3) it follows that

$$\|6f(x) + f(-x) - 4f(2x) + f(3x)\| \leq 3\varphi_1(0) + \varphi_1(x) \tag{13}$$

for all $x \in X$. Substitute $-x$ for x in (13), then

$$\| - 6f(-x) - f(x) + 4f(-2x) - f(-3x)\| \leq 3\varphi_1(0) + \varphi_1(-x) \tag{14}$$

for all $x \in X$. From (13) and (14), one has

$$\begin{aligned}&\|5f(x) - 5f(-x) - 4f(2x) + 4f(-2x) + f(3x) - f(-3x)\|\\ &\leq 6\varphi_1(0) + \varphi_1(x) + \varphi_1(-x)\end{aligned}$$

for all $x \in X$. By (11),

$$\begin{aligned}&2\|5f(x) - 4f(2x) + f(3x)\|\\ &\leq \|5f(x) - 5f(-x) - 4f(2x) + 4f(-2x) + f(3x) - f(-3x)\|\\ &\quad + \|5f(x) + 5f(-x)\| + \| - 4f(2x) - 4f(-2x)\| + \|f(3x) + f(-3x)\|\\ &\leq 6\varphi_1(0) + \varphi_1(x) + \varphi_1(-x) + 5\psi(x) + 4\psi(2x) + \psi(3x)\end{aligned}$$

for all $x \in X$. Hence

$$\begin{aligned}&\|5f(x) - 4f(2x) + f(3x)\|\\ &\leq 3\varphi_1(0) + \frac{1}{2}\varphi_1(x) + \frac{1}{2}\varphi_1(-x) + \frac{5}{2}\psi(x) + 2\psi(2x) + \frac{1}{2}\psi(3x)\end{aligned} \tag{15}$$

for all $x \in X$. Putting $x_i = x(i = 1, 2)$, $x_3 = 2x$ and $x_4 = -x$ in (3), it yields

$$\|2f(x) + f(-x) + 2f(2x) - 3f(3x) + f(4x)\| \leq 2\varphi_1(0) + \varphi_2(x) \tag{16}$$

for all $x \in X$. By (11),

$$\|f(x) + 2f(2x) - 3f(3x) + f(4x)\| \leq 2\varphi_1(0) + \varphi_2(x) + \psi(x) \tag{17}$$

for all $x \in X$. We use induction on n to prove the lemma. From (15) and (17), it follows

$$\|16f(x) - 10f(2x) + f(4x)\| \leq 11\varphi_1(0) + \frac{1}{2}\chi(x) \tag{18}$$

for all $x \in X$. So

$$\left\| f(x) - \frac{5}{8} f(2x) + \frac{1}{16} f(4x) \right\| \leq \frac{11}{16} \varphi_1(0) + \frac{1}{32} \chi(x) \tag{19}$$

which proves the validity of the inequality (12) for the case $n = 1$. Now assume that inequality (12) holds true for some $n \in \mathbb{N}$. By using (18) and the following relation:

$$\begin{aligned}
&\left\| f(x) - \frac{1}{6}\left(\frac{8}{2^{n+1}} - \frac{2}{8^{n+1}}\right) f(2^{n+1}x) + \frac{1}{6}\left(\frac{1}{2^{n+1}} - \frac{1}{8^{n+1}}\right) f(2^{n+2}x) \right\| \\
&\leq \left\| f(x) - \frac{1}{6}\left(\frac{8}{2^{n}} - \frac{2}{8^{n}}\right) f(2^{n}x) + \frac{1}{6}\left(\frac{1}{2^{n}} - \frac{1}{8^{n}}\right) f(2^{n+1}x) \right\| \\
&\quad + \left\| \frac{1}{6}\left(\frac{8}{2^{n}} - \frac{2}{8^{n}}\right) f(2^{n}x) - \frac{1}{6}\left(\frac{1}{2^{n}} - \frac{1}{8^{n}}\right) f(2^{n+1}x) \right. \\
&\quad \left. - \frac{1}{6}\left(\frac{8}{2^{n+1}} - \frac{2}{8^{n+1}}\right) f(2^{n+1}x) + \frac{1}{6}\left(\frac{1}{2^{n+1}} - \frac{1}{8^{n+1}}\right) f(2^{n+2}x) \right\| \\
&= \left\| f(x) - \frac{1}{6}\left(\frac{8}{2^{n}} - \frac{2}{8^{n}}\right) f(2^{n}x) + \frac{1}{6}\left(\frac{1}{2^{n}} - \frac{1}{8^{n}}\right) f(2^{n+1}x) \right\| \\
&\quad + \frac{1}{6}\left(\frac{1}{2^{n+1}} - \frac{1}{8^{n+1}}\right) \| 16 f(2^n x) - 10 f(2^{n+1}x) + f(2^{n+2}x) \| \\
&\leq \frac{1}{12} \sum_{i=1}^{n} \left(\frac{1}{2^i} - \frac{1}{8^i}\right) \big(22\varphi_1(0) + \chi(2^{i-1}x)\big) \\
&\quad + \frac{1}{6}\left(\frac{1}{2^{n+1}} - \frac{1}{8^{n+1}}\right) \big(11\varphi_1(0) + \frac{1}{2}\chi(2^n x)\big) \\
&\leq \frac{1}{12} \sum_{i=1}^{n+1} \left(\frac{1}{2^i} - \frac{1}{8^i}\right) \big(22\varphi_1(0) + \chi(2^{i-1}x)\big),
\end{aligned}$$

one can can easily verify inequality (12) for $n+1$. This completes the proof. □

THEOREM 5. *Let $f : X \to Y$ be a mapping satisfying* (3) *and* (11) *for all $x, x_1, x_2, x_3, x_4 \in X$. Then there exist two mappings $A : X \to Y$ and $B : X^3 \to Y$ which satisfy the inequality*

$$\| f(x) - A(x) - B(x, x, x) \| \leq \Phi(x, x, x, x) + \Psi(x) \tag{20}$$

for all $x \in X$. Furthermore, $g(x) := A(x) + B(x, x, x)$ satisfies (1) *for all $x \in X$.*

Proof. For any $x \in X$ and for every positive integer n, we define

$$g_n(x) := \frac{1}{6}\left(\frac{8}{2^n} - \frac{2}{8^n}\right) f(2^n x) - \frac{1}{6}\left(\frac{1}{2^n} - \frac{1}{8^n}\right) f(2^{n+1}x).$$

By (19), we obtain that

$$\begin{aligned}
&\|g_{n+1}(x) - g_n(x)\| \\
&= \Big\| \frac{1}{6}\left(\frac{8}{2^{n+1}} - \frac{2}{8^{n+1}}\right) f(2^{n+1}x) - \frac{1}{6}\left(\frac{1}{2^{n+1}} - \frac{1}{8^{n+1}}\right) f(2^{n+2}x) \\
&\quad -\frac{1}{6}\left(\frac{8}{2^n} - \frac{2}{8^n}\right) f(2^n x) + \frac{1}{6}\left(\frac{1}{2^n} - \frac{1}{8^n}\right) f(2^{n+1}x)\Big\| \\
&= \frac{1}{6}\cdot\frac{1}{8^{n+1}}\big\|(8\cdot 4^{n+1} - 2) f(2^{n+1}x) - (4^{n+1} - 1) f(2^{n+2}x) \\
&\quad -(64\cdot 4^n - 16) f(2^n x) + (8\cdot 4^n - 8) f(2^{n+1}x)\big\| \\
&\le \frac{4^{n+1}}{6\cdot 8^{n+1}}\big\|8f(2^{n+1}x) - f(2^{n+2}x) - 16f(2^n x) + 2f(2^{n+1}x)\big\| \\
&\quad +\frac{1}{6\cdot 8^{n+1}}\big\|2f(2^{n+1}x) - f(2^{n+2}x) - 16f(2^n x) + 8f(2^{n+1}x)\big\| \\
&= \frac{4^{n+1}+1}{6\cdot 8^{n+1}}\big\|16f(2^n x) - 10f(2^{n+1}x) + f(2^{n+2}x)\big\| \\
&= \frac{4^{n+1}+1}{3\cdot 8^n}\left\|f(2^n x) - \frac{5}{8}f(2^{n+1}x) + \frac{1}{16}f(2^{n+2}x)\right\| \\
&\le \frac{4^{n+1}+1}{3\cdot 8^n}\left(\frac{11}{16}\varphi_1(0) + \frac{1}{32}\chi(2^n x)\right)
\end{aligned}$$

for all positive integers n. For $n \ge m$,

$$\begin{aligned}
\|g_n(x) - g_m(x)\| &\le \sum_{i=m}^{n-1} \|g_{i+1}(x) - g_i(x)\| \\
&\le \frac{4^{i+1}+1}{3\cdot 8^i}\left(\frac{11}{16}\varphi_1(0) + \frac{1}{32}\chi(2^i x)\right) \qquad (21)
\end{aligned}$$

for all $x \in X$. By (10), since the right-hand side of the inequality (21) tends to zero as m tends to infinity, the sequence $\{g_n(x)\}$ is a Cauchy sequence. Therefore, we may apply a direct method to the definition of g. Define $g(x) = \lim_{n\to\infty} g_n(x)$ for all $x \in X$. Similarly, as in the proof of Theorem 3, due to (11), the mapping g satisfies (1) and is odd. By putting $x_1 = y$, $x_2 = 2x$ and $x_3 = x_4 = -x$ in (1) and considering the oddness of A, we get

$$4g(x) - 2g(x+y) - 2g(x-y) = 2g(2x) - g(2x+y) - g(2x-y) \qquad (22)$$

for all $x, y \in X$. According to the proof of Theorem 2.1 in [3], there exist two mappings $A : X \to Y$ and $B : X^3 \to Y$ such that $g(x) = A(x) + B(x, x, x)$ for all $x \in X$. The validity of inequality (20) follows directly from Lemma 4 and the definition of g. Hence, the proof is complete. □

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