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THE DOBRAKOV INTEGRAL OVER PATHS

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Abstract. In 2002, the author introduced the definition and its properties of an analogue of Wiener measure over paths. In this article, using these concepts, we will derive an operator-valued measure over paths and will investigate the properties for integral with respect to the measure. Specially, we will prove the Wiener integral formula for our integral and give some example of it.

1. Introduction

The Brownian motion was found by the British botanist Robert Brown in 1827. The theory for this motion was developed extensively and deeply by many scientists including Cantoni, Oehl, Delsaux, Guoy and Einstein. In 1923, for the probabilitic approach of Brownian motion, Wiener suggested a new measure, the so called Wiener measure [6]. But, through the Wiener measure theory, we can treat theories for one small particle, merely.

Recently, the author and Im [6] introduced a new definition of an analogue of Wiener measure space and investigate some theories on many small particles moving along the law of diffusion. Since then, they presented some paper related to it [4, 5]. From this concept, we can derived an operator-valued measure V on $C[0,t]$ which is the space of all continuous functions on a closed interval $[0,t]$.

Since 1970, Dobrakov presented several papers associated with the theories of integral of vector-valued function with respect to the operator-valued measures. the so-called Dobrakov's integral [3].

In this paper, we will treat the Dobrakov integral with respect to an operator-valued measure on $C[0,t]$.

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2. Preliminaries: Definitions and notations

In this section, we introduce some notations, definitions and facts which are needed to understand the subsequent sections.

Let t be a positive real number and n be a non-negative integer. For $\vec{t} = (t_0, t_1, \cdots, t_n)$ with $0 = t_0 < t_1 < \cdots < t_n \le t$, let $J_{\vec{t}} : C[0,t] \to \mathbb{R}^{n+1}$ be a function with

$$J_{\vec{t}}(x) \;=\; (x(t_0), x(t_1), \cdots, x(t_n)) \; . \tag{2.1}$$

For B_j $(j = 0,1,2,\cdots,n)$ in $\mathcal{B}(\mathbb{R})$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)$ of $C[0,t]$ is called an interval and let $\mathcal{I}$ be the set of all intervals. For a non-negative finite Borel measure φ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, we let

$$m_\varphi(J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)) \tag{2.2}$$

$$= \int_{B_0} [\int_{\prod_{j=1}^n B_j} W(n+1; \vec{t}; u_0, u_1, \cdots, u_n) \; d\prod_{j=1}^n m_L(u_1, \cdots, u_n)] \; d\varphi(u_0)$$

where

$$\begin{aligned} &W(n+1; \vec{t}; u_0, u_1, \cdots, u_n) \\ &= \; (\prod_{j=1}^n \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}}) exp\{-\frac{1}{2}\sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}}\} \; . \end{aligned}$$

Then $\mathcal{B}(C[0,t])$, the set of all Borel subsets in $C[0,t]$, coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique positive measure ω_φ on $(C[0,t], \mathcal{B}(C[0,t]))$ such that $\omega_\varphi(I) = m_\varphi(I)$ for all I in $\mathcal{I}$.

For φ in $\mathcal{M}(\mathbb{R})$ with the Jordan decomposition $\varphi = \sum_{j=1}^4 \alpha_j \varphi_j$, let $\omega_\varphi = \sum_{j=1}^4 \alpha_j \omega_{\varphi_j}$. We say that ω_φ is the complex-valued analogue of Wiener measure on $(C[0,t], \mathcal{B}(C[0,t]))$, associated with φ. If φ is a Dirac measure δ_0 at the origin in $\mathbb{R}$ then ω_φ is the classical Wiener measure.

By the change of variables formula, we can easily prove the following theorem.

Theorem 2.1. (The Wiener integration formula) If $f : \mathbb{R}^{n+1} \to \mathbb{C}$ is a Borel measurable function then the following equality holds.

$$\int_{C[0,t]} f(x(t_0), x(t_1), \cdots, x(t_n))\, d\omega_\varphi(x) \tag{2.3}$$

$$\stackrel{*}{=} \int_{\mathbb{R}^{n+1}} f(u_0, u_1, \cdots, u_n) W(n+1; \vec{t}; u_0, u_1, \cdots, u_n)$$

$$d(\prod_{j=1}^{n} m_L \times \varphi)((u_1, u_2, \cdots, u_n), u_0)$$

where $\stackrel{*}{=}$ means that if one side exists then both sides exist and the two values are equal.

Theorem 2.2. $|\omega_\varphi| = \omega_{|\varphi|}$ on $(C[0,t], \mathcal{B}(C[0,t]))$.

Let $X : C[0,t] \to \mathbb{R}$ be a function with $X(x) = x(t)$
For B in $\mathcal{B}(C[0,t])$, we let

$$[V_\varphi(B)](E) \;=\; \omega_\varphi(B \cap X^{-1}(E)) \tag{2.4}$$

for E in $\mathcal{B}(\mathbb{R})$.

From [3, Theorem 4.2, p4933], we obtain theorems as follows.

Theorem 2.3. For φ in $\mathcal{M}(\mathbb{R})$, V_φ is a measure-valued measure on $(C[0,t],\, \mathcal{B}(C[0,t]))$ in the total variation norm sense.

Theorem 2.4. Let φ be in $\mathcal{M}(\mathbb{R})$ and let B be in $\mathcal{B}(C[0,t])$ with $|\omega_\varphi|(B) \;=\; 0$. Then $V_\varphi(B)$ is a zero measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Theorem 2.5. Let φ be in $\mathcal{M}(\mathbb{R})$ and let $\vec{t} = (t_0, t_1, \cdots, t_n)$ be a vector in $\mathbb{R}^{n+1}$ with $0 = t_0 < \cdots < t_n = t$. Let $f : \mathbb{R}^{n+1} \to \mathbb{C}$ be a Borel measurable function such that $f(u_0, u_1, \cdots, u_n) W(n+1; \vec{t}; u_0, \cdots, u_n)$ is $|\varphi| \times \prod_{j=1}^{n} m_L$- integrable. Let $F : C[0,t] \to \mathbb{C}$ be a function with $F(x) = (f \circ J_{\vec{t}})(x) = f(x(t_0), x(t_1), \cdots, x(t_n))$. Then F is V_φ-Bartle integrable on $C[0,t]$ and for E in $\mathcal{B}(\mathbb{R})$,

$$[(Ba) - \int_{C[0,t]} F(x)\, dV_\varphi(x)](E) \tag{2.7}$$

$$= \int_E \{ \int_{\mathbb{R}^{n-1}} (\int_{\mathbb{R}} f(u_0, u_1, \cdots, u_n) W(n+1; \vec{t}; u_0, \cdots, u_n)$$

$$d\varphi(u_0))\ d(\prod_{j=1}^{n-1} m_L)(u_1, \cdots, u_{n-1})\}\ dm_L(u_n)$$

Letting $V(B) : \mathcal{M}(\mathbb{R}) \to \mathcal{M}(\mathbb{R})$ with $[V(B)](\varphi) = V_\varphi(B)$, $V(B)$ is a bounded linear operator on $\mathcal{M}(\mathbb{R})$.

3. The Dobrakov integral

This section is the main section in this note. Here, we will treat the theory of Dobrakov integral over $C[0, t]$.

First of all, we introduce the definition of Dobrakov integral [1].

Definition 3.1. Let $\mathbb{X}, \mathbb{Y}$ be two Banach spaces and let $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ be the space of bounded linear operators from $\mathbb{X}$ in $\mathbb{Y}$. Let m be a $\mathcal{L}(\mathbb{X}, \mathbb{Y})$-valued measure countably additive in the strong operator topology. For a mesurable set E, we define a non-negative set function $\hat{m}$, which will be called the semi-variation of m, by

$$\hat{m}(E) = \sup\{|\sum_{i=1}^{n} m(E_i)\mathbb{X}_i|\}$$

where $E_1, E_2, \cdots, E_n$ are dissection of E and $||\mathbb{X}_i|| \leq 1$ for $i = 1, 2, \cdots, n$. We say that a measurable set E is an *integrable set* if $\hat{m}(E)$ is finite. A strongly measurable function $f : C[0, t] \to \mathbb{X}$ is called *integrable* if there exists a sequence of simple integrable functions $\{f_n\}_{n=1}^{\infty}$ converging almost everywhere m to f for which the integrals $\int f_n dm$, $n = 1, 2, \cdots$ are uniformly countably additive. In that case the integral of the function f on a set E is defined by the equality $\int_E f dm = lim_{n\to\infty} \int_E f_n dm$.

Lemma 3.2. For u_0 in $\mathbb{R}$, let $P_{u_0} = \{x \in C[0, t] | x(0) = u_0\}$. Then $\hat{V}(P_{u_0}) = 1$.

Proof. The complex Borel measure on $\{u_0\}$ has of the form $\alpha\delta_{u_0}$ where α is a complex number. So, for a dissection $E_1, E_2, \cdots, E_n$ of P_{u_0} and for complex numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ with $|\alpha_1| \leq 1, |\alpha_2| \leq 1, \cdots, |\alpha_n| \leq 1$,

$$\big|\sum_{i=1}^{n} V(E_i)(\alpha_i\delta_{u_0})\big|$$

$$= \Big|\sum_{i=1}^{n} \alpha_i V_{\delta_{u_0}}(E_i)\Big|$$
$$\leq \sum_{i=1}^{n} ||V_{\delta_{u_0}}(E_i)||$$
$$= \sum_{i=1}^{n} \omega_{\delta_{u_0}}(E_i)$$
$$= \ 1.$$

Hence, $\hat{V}(P_{u_0}) \leq 1$. Trivially, $\hat{V}(P_{u_0}) \geq V_{\delta_{u_0}}(P_{u_0}) = 1$, so our result proved.

Theorem 3.3. Let F be a subset of $\mathbb{R}$ and let $P(F) = \{x \in C[0,t] | x(0)$ belongs to $F\}$. Then F is finite if and only if $\hat{V}(P(F))$ is finite if and only if $P(F)$ is integrable.

Proof. If F is finite, say $F = \{u_1, u_2, \cdots, u_n\}$ then by the subadditivity of the semivariation

$$\hat{V}(P(F))$$
$$= \hat{V}(\cup_{i=1}^{n} P(u_i))$$
$$\leq \sum_{i=1}^{n} \hat{V}(P(u_i))$$
$$= \sum_{i=1}^{n} \omega_{\delta_{u_0}}(P(u_i))$$
$$= \ n.$$

Hence, if F is finite then $\hat{V}(P(F))$ is finite.
Now, we suppose that F is infinite. Let m be a given natural number. Let F_m be a subset of F such that a cardinal numbers of F_m is m, say $F_m = \{u_1, u_2, \cdots, u_m\}$. Then by the basic properties of the semivariation,

$$\hat{V}(P(F))$$
$$\geq \hat{V}(P(F_m))$$
$$\geq \sum_{i=1}^{m} ||V_{\delta_{u_i}}(P(F_m))||$$

$$= \sum_{i=1}^{m} \omega_{\delta_{u_i}}(P(u_i))$$
$$= m.$$

Since m was arbitrary, $\hat{V}(P(F))$ is infinite. So, if $\hat{V}(P(F))$ is finite then F is finite, as desired.

Theorem 3.4. V is an operator-valued measure countably additive in the strong operator topology but is not an operator-valued measure countably additive in the uniform operator topology.

Proof. From Theorem 2.3 in the above section, we know that V is an operator-valued measure countably additive in the strong operator topology. We suppose that V is an operator-valued measure countably additive in the uniform operator topology. Then it is true that $lim_{n\to\infty}|V(F_n)| = 0$ for a decreasing sequence $\langle E_n\rangle$ in $\mathcal{B}(C[0,t])$ with $\cap_{n=1}^{\infty} E_n = \emptyset$. But, letting $E_n = P((0, \frac{1}{n}))$ for a natural number n, $|V(F_n)| = 1$ for all $n \in \mathbb{N}$ and $\cap_{n=1}^{\infty} E_n = \emptyset$, a contradiction.

From [1], we have *-Theorem as follows. Let Y contain no subspace isomorphic to the space c_0, for example let Y be a weakly complete Banach space. Then the semivariation $\hat{m}$ is continuous on an integrable set, i.e., if $\langle E_n\rangle$ is a decreasing sequence of integrable sets with $E_n \downarrow \emptyset$, then $lim_{n\to\infty}\hat{m}(E_n) = 0$.

By the *-Theorem and the contrapositive of Theorem 3.3, we obtain the following theorems.

Theorem 3.6. $\mathcal{M}(\mathbb{R})$ is not a weakly complete Banach space i.e., there is a subspace of $\mathcal{M}(\mathbb{R})$ which is isomorphic to the space c_0.

Theorem 3.7. If F is a finite subset of $\mathbb{R}$, then V is continuous on $P(F)$.

From [1], we have following lemma.

Lemma 3.8. Let the semivariation $\hat{m}$ be continuous on an integrable set, let A be an integrable set and let f be a bounded strongly measurable function. Then the function $f \cdot \chi_A$ integrable.

The next theorem is the main theorem in this article.

Theorem 3.9. (The Wiener integral integral formula) Let $F = \{u_1, u_2, \cdots, u_n\}$ and let $0 = t_0 < t_1 < \cdots < t_n = t$ be given. Suppose H is a function from $P(F)$ into $\mathcal{M}(\mathbb{R})$ such that $H(x) = \delta_{x(0)} h_{x(0)}(x(t_1), x(t_2), \cdots, x(t_n))$ and h_{u_k} $(k = 1, 2, \cdots, n)$ are bounded measurable functions on $\mathbb{R}^n$. Then the Dobrakov integral $\int_{P(F)} H(x)dV(x)$ exists and the following equality holds.

$$
\begin{aligned}
&[\int_{P(F)} H(x)dV(x)](E) \\
&= \frac{1}{\prod_{i=1}^n \sqrt{2\pi(t_i - t_{i-1})}} \sum_{k=1}^n \int_E [\int_{\mathbb{R}^{n-1}} h_{u_k}(v_1, v_2, \cdots, v_n) \\
&\quad exp\{-\frac{1}{2}\sum_{i=2}^n \frac{(v_i - v_{i-1})^2}{t_i - t_{i-1}}\} exp\{-\frac{1}{2}\frac{(v_1 - u_k)^2}{t_1}\} \\
&\quad dm_L(v_1) \cdots dm_L(v_{n-1})] dm_L(v_n)
\end{aligned}
$$

for all Borel subset E of $\mathbb{R}$.

Proof. By Lemma 3.8, H is integrable. Suppose that $H(x) = \delta_{x(0)} \chi_{B_{x(0)}}(x(t_1), x(t_2), \cdots, x(t_n))$, here B_{u_k} $(k = 1, 2, \cdots, n)$ are all Borel subsets of $\mathbb{R}^n$. Then for any Borel subset E of $\mathbb{R}$,

$$
\begin{aligned}
&[\int_{P(F)} H(x)dV(x)](E) \\
&\overset{(1)}{=} [\sum_{k=1}^n \int_{P_{u_k}} H(x)dV(x)](E) \\
&\overset{(2)}{=} \sum_{k=1}^n V_{\delta_{u_k}}(J^{-1}(B_{u_k}))(E) \\
&\overset{(3)}{=} \sum_{k=1}^n \omega_{\delta_{u_k}}(J^{-1}(B_{u_k}) \cap X^{-1}(E)) \\
&\overset{(4)}{=} \frac{1}{\prod_{i=1}^n \sqrt{2\pi(t_i - t_{i-1})}} \sum_{k=1}^n \int_E [\int_{\mathbb{R}^{n-1}} \chi_{B_{u_k}}(v_1, v_2, \cdots, v_n) \\
&\quad exp\{-\frac{1}{2}\sum_{i=2}^n \frac{(v_i - v_{i-1})^2}{t_i - t_{i-1}}\} exp\{-\frac{1}{2}\frac{(v_1 - u_k)^2}{t_1}\}
\end{aligned}
$$

$$dm_L(v_1)\cdots dm_L(v_{n-1})\big]dm_L(v_n)$$

By the basic properties of Dobrakov integral, Step (1) holds. Step (2) follows from the definition of V. Step (3) results from the definition of V_φ. From (2.7), we have Step (4).
Using the bounded convergence theorem for Dobrakov integral [2, Theorem 1, p260], we can prove our result in general case.

Example 3.10. Let $t = 2$ and let $F = \{3, 5\}$. Let $H(x) = \delta_{x(0)} exp\{-x(1)^2\}$. Then for $E \in \mathcal{B}(\mathbb{R})$,

$$\big[\int_{P(F)} H(x)dV(x)\big](E)$$
$$= \int_E \big[\int_{\mathbb{R}} \frac{1}{\sqrt{(2\pi)^2}} exp\{-u_1^2 - \frac{1}{2}(u_2-u_1)^2 - \frac{1}{2}(u_1-3)^2\}dm_L(u_1)\big]dm_L(u_2)$$
$$+ \int_E \big[\int_{\mathbb{R}} \frac{1}{\sqrt{(2\pi)^2}} exp\{-u_1^2 - \frac{1}{2}(u_2-u_1)^2 - \frac{1}{2}(u_1-5)^2\}dm_L(u_1)\big]dm_L(u_2)$$
$$= \frac{e^{-3}}{2\sqrt{2\pi}} \int_E exp\{-\frac{3}{8}(u_2-1)^2\}du_2 + \frac{e^{-\frac{25}{3}}}{2\sqrt{2\pi}} \int_E exp\{-\frac{3}{8}(u_2-\frac{5}{3})^2\}du_2$$

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