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# EXAMPLES OF SMASH PRODUCT

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Abstract. Several examples of smash product are given.

The aim of this short note is to see that the skew group ring, skew polynomial ring and skew enveloping algebra are explained by smash product. Throughout the note, $k$ will be a field of arbitrary characteristic and all vector spaces will be over the field $k$.

Let $H = (H, m, \mu, \Delta, \epsilon)$ be a bialgebra and let $A$ be an $H$-module algebra. That is, $A$ is a left $H$-module with module structure $H \times A \longrightarrow A, (x, a) \mapsto x \cdot a$, satisfying

$$x \cdot (ab) = \sum (x' \cdot a)(x'' \cdot b), \quad x \cdot 1 = \epsilon(x)1,$$

where $\Delta(x) = \sum x' \otimes x''$. Here we write out and prove the following well-known theorem for completion.

**Theorem**

Let $H$ be a bialgebra and let $A$ be an $H$-module algebra. Then there exists a unique algebra structure on the vector space $A \otimes H$, with multiplicative identity $1 \otimes 1$, such that its product is given by

$$(a \otimes x)(b \otimes y) = \sum a(x' \cdot b) \otimes x'' y.$$

This algebra is called a smash product of $A$ and $H$ and denoted by $A\sharp H$.

*Proof.* We use Sweedler's notation in [3] to prove the associativity of the product. Note that

$$(1 \otimes \Delta)\Delta(x) = \sum x' \otimes x'' \otimes x''' = (\Delta \otimes 1)\Delta(x)$$

for all $x \in H$.

For $a, b, c \in A$ and $x, y, z \in H$,

$$((a \otimes x)(b \otimes y))(c \otimes z) \;=\; \sum ((a(x' \cdot b)) \otimes (x'' y))(c \otimes z)$$

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$$
\begin{aligned}
&= \sum (a(x' \cdot b)((x''y)' \cdot c)) \otimes ((x''y)''z) \\
&= \sum (a(x' \cdot b)(x'' \cdot (y' \cdot c))) \otimes (x'''y''z) \\
&= \sum (a(x' \cdot (b(y' \cdot c)))) \otimes (x''y''z) \\
&= (a \otimes x)(\sum (b(y' \cdot c)) \otimes (y''z)) \\
&= (a \otimes x)((b \otimes y)(c \otimes z)).
\end{aligned}
$$

Hence the product satisfies the associativity. For $a \in A$ and $x \in H$, since $\Delta(1) = 1 \otimes 1$ and $\sum \epsilon(x')x'' = x$, we have

$$
\begin{aligned}
(a \otimes x)(1 \otimes 1) &= \sum (a(x' \cdot 1)) \otimes x'' \\
&= \sum (a\epsilon(x')1) \otimes x'' \\
&= a \otimes (\sum \epsilon(x')x'') \\
&= a \otimes x \\
&= (1 \otimes 1)(a \otimes x),
\end{aligned}
$$

thus $1 \otimes 1$ is unity. It completes the proof. □

Clearly the linear maps

$$
\begin{aligned}
i : A \longrightarrow A\sharp H, &\quad i(a) = a \otimes 1, \\
j : H \longrightarrow A\sharp H, &\quad j(x) = 1 \otimes x
\end{aligned}
$$

are monomorphisms and thus it makes sense that, for $a \in A$ and $x \in H$, both elements $a \otimes 1$ and $1 \otimes x$ in $A\sharp H$ are denoted by $a$ and $x$. Henceforce we omit the notation $\otimes$ in $A\sharp H$ and each element $a \otimes x \in A\sharp H$ is denoted by $ax$. That is, the smash product $A\sharp H$ is an algebra generated by $A$ and $H$ subject to the relation $xa = \sum (x' \cdot a)x''$ for all $a \in A$ and $x \in H$.

**1. Skew group ring.** Let $G$ be a group. The group algebra $k[G]$ is a bialgebra with

$$\Delta(x) = x \otimes x, \quad \epsilon(x) = 1$$

for all $x \in G$. Let $A$ be a $k$-algebra such that there exists a group homomorphism $\phi : G \longrightarrow \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ is the algebra of all automorphisms on $A$. Then $A$ is a $k[G]$-module algebra with module structure $x \cdot a = \phi(x)(a)$ for $a \in A, x \in G$ and thus there exists the $k$-algebra $A\sharp k[G]$. Every element of $A\sharp k[G]$ is of the form $\sum_{x \in G} a_x x$, $a_x \in A$ and $xa = (x \cdot a)x$ for $a \in A$, $x \in G$. Thus $A\sharp k[G]$ is the skew group ring of A and $G$. This approach should be compared with [2, 1.5.4].

**2. Semi-direct product.** Let $G$ and $H$ be groups such that there exists a group homomorphism $\theta : H \longrightarrow \mathrm{Aut}(G)$. Then the group algebra $k[H]$ is a bialgebra with

$$\Delta(x) = x \otimes x, \quad \epsilon(x) = 1$$

for all $x \in H$ and the group algebra $k[G]$ is a $k[H]$-module algebra with module structure $x \cdot g = \theta(x)(g)$ for all $x \in H$ and $g \in G$. It follows that there exists $k[G]\sharp k[H]$. All elements of the form $gh$, $g \in G, h \in H$, are unit and the set $\{gh \mid g \in G, h \in H\}$ forms a group under the multiplication on $k[G]\sharp k[H]$, called the semi-direct product of $G$ and $H$ and denoted by $G \times_\theta H$. Note that $k[G]\sharp k[H]$ is isomorphic to the group algebra $k[G \times_\theta H]$.

**3. Skew polynomial ring $A[x;\delta]$.** The polynomial ring $k[x]$ is a bialgebra with

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \epsilon(x) = 0.$$

Let $A$ be a $k$-algebra with a derivation $\delta$. Then $A$ is a $k[x]$-module algebra with module structure

$$x \cdot a = \delta(a), \quad a \in A.$$

Hence there exists the algebra $A\sharp k[x]$. Every element of $A\sharp k[x]$ is of the form $\sum_i a_i x^i$, $a_i \in A$ and $xa = ax + \delta(a)$ for all $a \in A$ and thus $A\sharp k[x]$ is the skew polynomial ring $A[x;\delta]$. This approach should be compared with [2, 1.2.3].

**4. Skew polynomial ring $A[x;\alpha,\delta]$.** Given a $k$-algebra $A$, let $\alpha$ be an endomorphism in $A$ and let $\delta$ be an $\alpha$-derivation. That is, $\delta$ is a $k$-linear map from $A$ into itself such that $\delta(ab) = \alpha(a)\delta(b) + \delta(a)b$ for all $a, b \in A$. The polynomial ring $k[\alpha][x]$ is a bialgebra with

$$\begin{aligned} \Delta(\alpha) &= \alpha \otimes \alpha \qquad & \Delta(x) &= \alpha \otimes x + x \otimes 1 \\ \epsilon(\alpha) &= 1 & \epsilon(x) &= 0 \end{aligned}$$

The $k$-algebra $A$ is a $k[\alpha][x]$-module algebra with module structure

$$\alpha \cdot a = \alpha(a), \quad x \cdot a = \delta(a)$$

for $a \in A$. Therefore there exists the $k$-algebra $A\sharp k[\alpha][x]$. Let $S$ be the subalgebra of $A\sharp k[\alpha][x]$ generated by $x$ and all elements of $A$. Then all elements of $S$ are of the form $\sum_i a_i x^i$, $a_i \in A$ and we have the relation $xa = \alpha(a)x + \delta(a)$ for all $a \in A$. Hence $S$ is just the skew polynomial ring $A[x;\alpha,\delta]$. This approach is more convenient than that given in [1, 1.10] because it is messy to prove the fact that the skew polynomial ring $A[x;\alpha,\delta]$ satisfies the associative law.

**5. Skew enveloping algebra.** Let $A$ be a $k$-algebra and let $\mathfrak{g}$ be a $k$-Lie algebra such that there is a Lie homomorphism $\phi : \mathfrak{g} \longrightarrow \mathrm{Der}_k(A)$, where $\mathrm{Der}_k(A)$ is the set of all $k$-derivations in $A$. Denote the universal enveloping algebra of $\mathfrak{g}$ by $U(\mathfrak{g})$. Then $A$ is a $U(\mathfrak{g})$-module algebra with module structure $x \cdot a = \phi(x)(a)$ for $x \in \mathfrak{g}$ and $a \in A$ since $U(\mathfrak{g})$ is a bialgebra such that the comultiplication $\Delta$ and the counit $\epsilon$ are given by

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \epsilon(x) = 0$$

for all $x \in \mathfrak{g}$. Hence there exists the $k$-algebra $A\sharp U(\mathfrak{g})$ satisfying the relation $xa = x \cdot a + ax$ for all $a \in A$ and $x \in \mathfrak{g}$. The smash product $A\sharp U(\mathfrak{g})$ is just the skew enveloping algebra of $A$ and $\mathfrak{g}$. (See [2, 1.7.10])

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