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THE ORDER AND SPEED OF CONVERGENCE FOR THE k-FOLD PSEUDO-OLVER'S METHOD LOCATING A SIMPLE REAL ZERO[1]

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Abstract. A convergence behavior is under investigation near a simple real zero for the k-fold pseudo-Olver's method defined by extending the classical Olver's method. The order of convergence is shown to be at least $k+3$. The asymptotic error constant is explicitly given in terms of k and the corresponding simple zero. Various numerical examples with a proposed zero-finding algorithm are successfully confirmed with the use of symbolic and computational ability of Mathematica.

1.. Introduction and preliminaries

Consider a function $f : \mathbb{R} \to \mathbb{R}$ which has a simple real zero α and is sufficiently smooth in a neighborhood of α. The aim of this analysis is to find α accurately and to establish the order and speed of convergence by constructing and analyzing an iterative method

$$x_{n+1} = g(x_n),\ n = 0,\ 1,\ 2, \cdots, \tag{1.1}$$

where $g : \mathbb{R} \to \mathbb{R}$ has a real fixed point α and x_0 is given. Suppose further that g is sufficiently smooth in a neighborhood of α. Given $p \in \mathbb{N}$, we assume

$$\begin{cases} \left|\frac{d^p}{dx^p} g(x)\right|_{x=\alpha} = |g^{(p)}(\alpha)| < 1, & \text{if } p = 1. \\ g^{(i)}(\alpha) = 0 \text{ for } 1 \le i \le p-1 \text{ and } g^{(p)}(\alpha) \neq 0, & \text{if } p \ge 2. \end{cases} \tag{1.2}$$

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Let x_n belong to a sufficiently small neighborhood of α for $n \in \mathbb{N} \cup \{0\}$. Then Taylor series[1,6] expansion about α immediately gives

$$x_{n+1} = g(x_n) = g(\alpha) + g^{(p)}(\xi)\ (x_n - \alpha)^p/p!, \tag{1.3}$$

where $\xi \in (a, b)$ with $a = \min(\alpha, x_n)$ and $b = \max(\alpha, x_n)$. The continuity of g at α ensures, for all given $\epsilon > 0$, the existence of a number $\delta > 0$ satisfying

$$|x_{n+1} - \alpha| = |g(x_n) - g(\alpha)| = |g^{(p)}(\xi)|\ \frac{|(x_n - \alpha)^{p-1}|}{p!}\ |x_n - \alpha| < \epsilon, \tag{1.4}$$

whenever $|x_n - \alpha| < \delta$. Let $\boldsymbol{J} = \{x : |x - \alpha| \leq \delta\}$. Owing to the continuity of $g^{(p)}$ on $\boldsymbol{J}$, there exists a number $M > 0$ satisfying $|g^{(p)}(x)| \leq M$ for all $x \in \boldsymbol{J}$. We now choose

$$\delta = \begin{cases} \min(\epsilon,\ 1/M), \text{ if } p = 1. \\ \{\min(\epsilon^{p-1},\ p!/M)\}^{1/(p-1)}, \text{ if } p \geq 2. \end{cases}$$

Then $|x_{n+1} - \alpha| = |g(x_n) - g(\alpha)| \leq |x_n - \alpha|$. Hence $g : \boldsymbol{J} \to \boldsymbol{J}$. Since $|x_n - \alpha| < \delta$, it follows from (1.4) that

$$|x_{n+1} - \alpha| = |g(x_n) - g(\alpha)| \leq K|x_n - \alpha|, \tag{1.5}$$

where $0 < K = \sup\{M\ |(x_n - \alpha)|^{p-1}/p!\ :\ n \in \mathbb{N} \cup \{0\}\} < M\ \delta^{p-1}/p!\ \leq 1$ for $p \geq 2$. If $p = 1$, then $K = M < 1$ can be chosen according to (1.2). Hence g is contractive on $\boldsymbol{J}$ for any $p \in \mathbb{N}$ and the sequence $\{x_n\}_{n=0}^{\infty}$ with $x_0 \in \boldsymbol{J}$ defined by (1.1) converges to a fixed point $\alpha \in \boldsymbol{J}$[7]. Now introducing $e_n = x_n - \alpha$ with the fact that $\lim_{n\to\infty} \xi = \alpha$, for the iterative method (1.1) we obtain the *asymptotic error constant (also called the speed of convergence)* η and *order of convergence* p[3,7] as follows:

$$\eta = \lim_{n\to\infty} \left|\frac{e_{n+1}}{{e_n}^p}\right| = |g^{(p)}(\alpha)|\ /\ p!. \tag{1.6}$$

Now for an arbitrarily given $x \in \mathbb{R}$, we define a function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(w) = w - \frac{f(w)}{f'(x)} - \frac{1}{2}\frac{f(w)^2 f''(x)}{f'(x)^3} \tag{1.7}$$

with $f'(x) \neq 0$. Let $w_0(x) = F(x)$, and let for $k \in \mathbb{N}$

$$w_k(x) = F(w_{k-1}(x)) = w_{k-1}(x) - \frac{f(w_{k-1}(x))}{f'(x)} - \frac{1}{2}\frac{f(w_{k-1}(x))^2 f''(x)}{f'(x)^3}. \tag{1.8}$$

Hence $w_k(x) = F^k(w_0) = F^{k+1}(x)$ for $k \in \mathbb{N}$, where $F^k(w_0) = F(F(\cdots F(w_0)\cdots))$. As a result of the preceding analysis, we have constructed an iterative method with $x_0 \in \mathbb{R}$

$$x_{n+1} = F^{k+1}(x_n) = g(x_n) \tag{1.9}$$

which is called the *k-fold pseudo-Olver's method*. If $k = 0$, the method becomes the classical *Olver's method*[2] and has the cubic convergence as shown from Halley's method and other methods of Laguerre's type[4,5]. If $k = 1$, it is simply called the *pseudo-Olver's method*.

2.. Convergence of the k-fold pseudo-Olver's method

In view of the fact that $f'(\alpha) \neq 0$ and from (1.8), one can easily show that

$$w_k(\alpha) = \alpha, \text{ for all } k \in \mathbb{N} \cup \{0\}, \tag{2.1}$$

$$w_0{}'(\alpha) = \frac{d}{dx} w_0(x)\Big|_{x=\alpha} = 0, w_0{}''(\alpha) = 0, w_0{}'''(\alpha) = 3c^2 - f'''(\alpha)/f'(\alpha), \tag{2.2}$$

with $c = f''(\alpha)/f'(\alpha)$. We further wish to establish the following lemma.

LEMMA 2.1. Let $w_k^{(m)}(\alpha) = \frac{d^m}{dx^m} w_k(x)|_{x=\alpha}$ for any $k, m \in \mathbb{N} \cup \{0\}$. Then the following holds.

$$w_k^{(m)}(\alpha) = \begin{cases} \alpha, \ if \ m = 0. \\ 0, \ if \ 1 \leq m \leq k+1. \\ \frac{(k+3)!}{3!} c^k w'''(\alpha) \ with \ c = f''(\alpha)/f'(\alpha), \ if \ m = k+2. \end{cases} \tag{2.3}$$

Proof. If $k = 0$, then the assertion holds from (2.1) and (2.2). It suffices to consider $k \in \mathbb{N}$. If $m = 0$, then the assertion immediately holds from (2.1). The remaining proof will be given based on induction on $m \geq 1$. We rewrite (1.8) for $k \in \mathbb{N}$ with the abbreviation w_{k-1} for $w_{k-1}(x)$ to obtain

$$2f'(x)^3 \cdot (w_k - w_{k-1}) = -f(w_{k-1}) \cdot (2f'(x)^2 + f(w_{k-1}) f''(x)). \tag{2.4}$$

Differentiating both sides of (2.4) with respect to x and evaluating at $x = \alpha$ yields

$$\begin{aligned} 3f'(x)^2 \cdot f''(x)(w_k - w_{k-1})|_\alpha + f'(x)^3 \cdot (w_k{}' - w_{k-1}{}')|_\alpha \\ = -f'(w_{k-1}) \cdot w_{k-1}{}' \cdot f'(x)^2|_\alpha. \end{aligned} \tag{2.5}$$

Hence we get $2f'(\alpha)^3 \cdot \Big(w_k{}'(\alpha) - w_{k-1}{}'(\alpha)\Big) = -2f'(\alpha)^3 \cdot w_{k-1}{}'(\alpha)$, which states

$$w_k{}'(\alpha) = 0 \text{ for all } k \ \in \mathbb{N}. \tag{2.6}$$

Suppose now (2.3) holds for $m \geq 1$. By differentiating $(m+1)$ times both sides of (2.4) with respect to x via Leibnitz Rule[6] and evaluating at $x = \alpha$ we obtain

$$2\sum_{r=0}^{m+1} {}_{m+1}C_r \cdot (f'^3)^{(m+1-r)} \cdot \Big[w_k^{(r)}(x) - w_{k-1}^{(r)}(x)\Big]\Big|_{x=\alpha}$$

$$= -\sum_{r=0}^{m+1} {}_{m+1}C_r \cdot \Big[f(w_{k-1})\Big]^{(m+1-r)} \cdot \Big[2f'^2 + f(w_{k-1}) \cdot f''\Big]^{(r)}\Big|_{x=\alpha}, \quad (2.7)$$

where ${}_mC_r = \frac{m!}{(m-r)!r!}$, $f' = f'(x)$ and $f'' = f''(x)$. Since $w_k^{(r)}(\alpha) - w_{k-1}^{(r)}(\alpha) = 0$ for $0 \leq r \leq m-1 \leq k$, the leftmost side of (2.7) has possible nonvanishing terms for $r = m$ and $r = m+1$ as follows.

$$2\Big[(m+1)(f'^3)'(\alpha)\cdot\Big(w_k^{(m)}(\alpha) - w_{k-1}^{(m)}(\alpha)\Big) + f'^3(\alpha)\cdot\Big(w_k^{(m+1)}(\alpha) - w_{k-1}^{(m+1)}(\alpha)\Big)\Big]$$

$$= -2(m+1)(f'^3)'(\alpha)\cdot w_{k-1}^{(m)}(\alpha) + 2f'^3(\alpha)\cdot\Big(w_k^{(m+1)}(\alpha) - w_{k-1}^{(m+1)}(\alpha)\Big) \quad (2.8)$$

in view of the induction hypothesis that $w_k^{(m)}(\alpha) = 0$ for $1 \leq m \leq k+1$. Similarly, owing to the fact that $\Big[f(w_{k-1})\Big]^{(m+1-r)}(\alpha) = 0$ for $2 \leq r \leq m+1 \leq k+1$, the rightmost side of (2.7) has possible nonvanishing terms for $r = 0$ and $r = 1$ as follows.

$$-2\Big[f(w_{k-1})\Big]^{(m+1)}(\alpha) \cdot f'(\alpha)^2 - 4(m+1)\Big[f(w_{k-1})\Big]^{(m)}(\alpha) \cdot f'(\alpha) \cdot f''(\alpha). \quad (2.9)$$

Hence it follows from the right side of (2.8) and (2.9) that

$$w_k^{(m+1)}(\alpha) = \begin{cases} 0, \text{ if } 2 \leq m \leq k+1. \\ c\ (m+1)\ w_{k-1}^{(m)}(\alpha), \text{ if } m = k+2. \end{cases} \quad (2.10)$$

We also find for $m+1 = k+3$ that

$$w_k^{(m+1)}(\alpha) = w_k^{(k+3)}(\alpha) = c(k+3)\ w_{k-1}^{(k+1)}(\alpha) = c^2\ (k+3)(k+2)\ w_{k-2}^{(k)}(\alpha)$$

$$= (k+3)(k+2)(k+1)\cdots 4 \cdots c^k \cdot {w_0}'''(\alpha) = \frac{(k+3)!}{3!} c^k \cdot {w_0}'''(\alpha) \quad (2.11)$$

Hence (2.3) also holds for $m+1$, completing the induction proof. □

The result of the preceding analysis immediately leads us to the following theorem.

THEOREM 2.1. Let $k \in \mathbb{N} \cup \{0\}$ be given and α be a simple real zero of the smooth function f described in Section 1. Then the k-fold pseudo-Olver's method defined by (1.9) is at least of order $k+3$ and its asymptotic error constant η is given by $|c^k \cdot d|/6$, where $c = f''(\alpha)/f'(\alpha)$ and $d = 3c^2 - f'''(\alpha/f'(\alpha)$.

Proof. Let $g(x) = w_k(x) = F^{k+1}(x)$ as described in (1.8) and (1.9). Define the iteration $x_{n+1} = g(x_n)$ with $x_0 \in \boldsymbol{J}$ and the error $e_n = x_n - \alpha$ for $n \in \mathbb{N} \cup \{0\}$. Then Lemma 2.1 yields the asymptotic error constant η and the order of convergence p in view of (1.6)

$$\eta = \lim_{n\to\infty}\left|\frac{e_{n+1}}{{e_n}^{k+3}}\right| = \frac{1}{(k+3)!}\,|g^{(k+3)}(\alpha)| = \frac{1}{(k+3)!}\,|w_k^{(k+3)}(\alpha)| = \frac{|c^k \cdot d|}{6},$$

completing the proof. $\square$

3.. Algorithm, numerical results and discussions

Based on the discussion in Sections 1 and 2, we construct a zero-finding algorithm with the aid of symbolic and computational ability of *Mathematica*[8] as follows.

Claim 3.2. Algorithm 3.1 (Zero-Finding Algorithm)

Step 1. For $k \in \mathbb{N} \cup \{0\}$, construct the iteration function $g = F^{k+1}$ with the given function f having a simple zero α, according to the description in Section 1.
Step 2. Set the minimum number of precision digits. With exact zero α or most accurate zero, supply the theoretical asymptotic error constant η. Set the error range ϵ, the maximum iteration number n_{max} and the initial value x_0. Compute $f(x_0)$ and $|x_0 - \alpha\ |$.
Step 3. Compute $x_{n+1} = g(x_n)$ for $0 \le n \le n_{max}$ and display the computed values of $n,\ x_n,$
$f(x_n), |x_n - \alpha|$, $|e_{n+1}/{e_n}^{k+3}|$ and η.

We illustrate the order of convergence and asymptotic error constant for a function

$$f(x) = x^3 - e^x \cos(\frac{\pi x}{2}) + \ln(\frac{2}{x^2+1}) + 1$$

having a simple real zero $\alpha = -1$. The symbolic computation of $f'(x)$ and $f''(x)$in (1.8) has been easily done with the aid of Mathematica. Table 1 lists the numerical results for approximated zeros of $f(x)$ computed with the aid of Mathematica programming. To obtain sufficient accuracy and keep track of the asymptotic error constant requiring highly accurate arithmetic, the minimum number of precision digits was chosen as 300 by assigning *$MinPrecision=300* in Mathematica. The error bound

ϵ for $\mid x_n - \alpha \mid < \epsilon$ was chosen as 0.5×10^{-250} for the current experiment. As can be seen in Table 1, the number of computation gets smaller due to high-order convergence as k increases. For each $0 \leq k \leq 5$, the order of convergence has been confirmed to be of at least $k+2$. As a second numerical example, we take

$$f(x) = e^{-x}(1 + x^2 - x^3) - x^2 \ln x + 1$$

with a simple zero

$$\begin{aligned}\alpha = 1.&5168172265969339164987227503248485468964080154954276162894902011640301542947543256\\&21293918926263676073375817561810310805778775998164929807624089391720851322462813460187\\&508241194796520149664659675927792304679148311701968690090738671227458264485912605,\end{aligned}$$

which is accurate up to 250 significant decimal digits. Table 2 also shows a good agreement with the theory presented in this paper. The computed asymptotic error constants have shown to be in good agreement with the theoretical asymptotic error constants η up to 10 significant digits. Even though the computed root was rounded to be accurate up to the 250 significant digits, the limited space allowed us to list it only up to 15 significant digits. Although not shown here, other examples for $f(x) = \frac{x^3}{x^2+1} - x\cos(\frac{\pi x^2}{4}) - \frac{2\sqrt{2}}{3}$ with $\alpha = \sqrt{2}$ and $f(x) = x^3 + x^4 \cos \pi x - \frac{1}{8}$ with $\alpha = 1/2$ have shown similar convergence behavior with high accuracy.

The high-order convergence established in Theorem 1 has been confirmed through many additional numerical experiments. This new development will play a role in the highly accurate computation of zeros for the nonlinear equation. The current study will be extended to the case when zeros are not simple.

Convergence of k-fold pseudo-Olver's Method for
$f(x) = x^3 - e^x \cos(\frac{\pi x}{2}) + \ln(\frac{2}{x^2+1}) + 1$

k	n	x_n	$f(x_n)$	$\mid x_n - \alpha \mid$	$e_{n+1}/{e_n}^{k+3}$	η
	0	-1.50000000000000	-2.70273	0.500000		1.957641668
	1	-1.06408632741116	-0.234212	0.0640863	0.5126906193	
	2	-1.00040971378055	-0.00140270	0.000409714	1.556626497	
0	3	-1.00000000013443	-4.60044×10^{-10}	1.34432×10^{-10}	1.954611612	
	4	-1.00000000000000	-1.62756×10^{-29}	4.75598×10^{-30}	1.957641667	
	5	-1.00000000000000	-7.20692×10^{-88}	2.10597×10^{-88}	1.957641668	
	6	1.00000000000000	$-6.25731 \times 10^{-263}$	1.82848×10^{-263}		
	0	-1.50000000000000	-2.70273	0.500000		4.093451779
	1	-1.03260294780328	-0.115402	0.0326029	0.5216471649	
	2	-1.00000387276317	-0.0000132532	3.87276×10^{-6}	3.427626289	
1	3	-1.00000000000000	-3.15109×10^{-21}	9.20797×10^{-22}	4.093363871	
	4	-1.00000000000000	-1.00703×10^{-83}	2.94270×10^{-84}	4.093451779	
	5	-1.00000000000000	1.06319×10^{-320}	0		
	0	-1.50000000000000	-2.70273	0.500000		8.559455868
	1	-1.01723955086851	-0.0600634	0.0172396	0.5516656278	
	2	-1.00000001150590	-3.93748×10^{-8}	1.15059×10^{-8}	7.555983644	
2	3	-1.00000000000000	-5.90673×10^{-39}	1.72604×10^{-39}	8.559455148	
	4	-1.00000000000000	$-4.48738 \times 10^{-193}$	1.31128×10^{-193}	8.559455868	
	5	-1.00000000000000	1.06319×10^{-320}	0		
	0	-1.50000000000000	-2.70273	0.500000		17.897923
	1	-1.00927950711051	-0.0320645	0.00927951	0.593888455	
3	2	-1.00000000001051	-3.59672×10^{-11}	1.05102×10^{-11}	16.46115442	
	3	-1.00000000000000	-8.25579×10^{-65}	2.41247×10^{-65}	17.89792300	
	4	-1.00000000000000	1.06319×10^{-320}	0		
	0	-1.50000000000000	-2.70273	0.500000		37.42476774
	1	-1.00503986366130	-0.0173381	0.00503986	0.6451025486	
4	2	-1.00000000000000	-1.00173×10^{-14}	2.92722×10^{-15}	35.44246738	
	3	-1.00000000000000	$-2.35855 \times 10^{-100}$	6.89203×10^{-101}	37.42476774	
	4	-1.00000000000000	1.06319×10^{-320}	0		
	0	-1.50000000000000	-2.70273	0.500000		78.25563
	1	-1.00275015721112	-0.00943849	0.00275016	0.7040402460	
5	2	-1.00000000000000	-8.46554×10^{-19}	2.47376×10^{-19}	75.59576211	
	3	-1.00000000000000	$-3.75555 \times 10^{-147}$	1.09743×10^{-147}	78.25563000	
	4	-1.00000000000000	1.06319×10^{-320}	0		

References

[1] R. G. Bartle, *The Elements of Real Analysis*, 2nd ed., John Wiley & Sons., New York, 1976.

[2] Ward Cheney and David Kincaid, *Numerical Mathematics and Computing*, Brooks/Cole Publishing Company, Monterey, California 1980

[3] S. D. Conte and Carl de Boor, *Elementary Numerical Analysis*, McGraw-Hill Inc., 1980

[4] Qiang Du, Ming Jin, T. Y. Li and Z. Zeng, *The Quasi-Laguerre Iteration*, Mathematics of Computation, Vol. 66, No. 217(1997), pp.345-361.

Convergence of k-fold pseudo-Olver's Method for $f(x) = x^3 + x^4 \cos \pi x - 1/8$

k	n	x_n	$f(x_n)$	$\mid x_n - \alpha \mid$	$\lvert e_{n+1}/e_n{}^{k+2} \rvert$	η
0	0	1.50000000000000	0.0598122	0.0168172		0.5202029052
	1	1.51681465963352	9.21023×10^{-6}	2.56696×10^{-6}	0.5397051357	
	2	1.51681722659693	3.15708×10^{-17}	8.79902×10^{-18}	0.5202058035	
	3	1.51681722659693	1.27153×10^{-51}	3.54385×10^{-52}	0.5202029052	
	4	1.51681722659693	8.30712×10^{-155}	2.31526×10^{-155}	0.5202029052	
	5	1.51681722659693	6.14555×10^{-315}	1.08470×10^{-264}		
1	0	1.50000000000000	0.0598122	0.0168172		0.541533548
	1	1.51681727228101	-1.63914×10^{-7}	4.56841×10^{-8}	0.5711463101	
	2	1.51681722659693	-8.46323×10^{-30}	2.35877×10^{-30}	0.5415334701	
	3	1.51681722659693	$-6.01473 \times 10^{-119}$	1.67635×10^{-119}	0.5415335480	
	4	1.51681722659693	7.51195×10^{-317}	1.08470×10^{-264}		
2	0	1.50000000000000	0.0598122	0.0168172		0.5637388424
	1	1.51681722578377	2.91763×10^{-9}	8.13165×10^{-10}	0.6045147271	
	2	1.51681722659693	7.19156×10^{-46}	2.00434×10^{-46}	0.5637388443	
	3	1.51681722659693	6.54317×10^{-229}	1.82363×10^{-229}	0.5637388424	
	4	1.51681722659693	3.88549×10^{-318}	1.08470×10^{-264}		
3	0	1.50000000000000	0.0598122	0.0168172		0.586854653
	1	1.51681722661141	-5.19329×10^{-11}	1.44741×10^{-11}	0.6398308459	
	2	1.51681722659693	-1.93611×10^{-65}	5.39609×10^{-66}	0.5868546530	
	3	1.51681722659693	1.81323×10^{-317}	1.08470×10^{-264}		
4	0	1.50000000000000	0.0598122	0.0168172		0.6109183151
	1	1.51681722659668	9.24390×10^{-13}	2.57635×10^{-13}	0.6772101879	
	2	1.51681722659693	1.65144×10^{-88}	4.60269×10^{-89}	0.6109183151	
	3	1.51681722659693	$-1.29516 \times 10^{-318}$	1.08470×10^{-264}		
5	0	1.50000000000000	0.0598122	0.0168172		0.6359686949
	1	1.51681722659694	-1.64539×10^{-14}	4.58582×10^{-15}	0.7167732551	
	2	1.51681722659693	$-4.46294 \times 10^{-115}$	1.24386×10^{-115}	0.6359686949	
	3	1.51681722659693	$-1.29516 \times 10^{-318}$	1.08470×10^{-264}		

[5] L. D. Petkovic, M. S. Petkovic and D. Zivkovic, *Hansen-Patrick's Family Is of Laguerre's Type*, Novi Sad J. Math., Vol. 33, No. 1(2003), pp.109-115.

[6] Kenneth A. Ross, *Elementary Analysis*, Springer-Verlag New York Inc., 1980.

[7] J. Stoer and R. Bulirsh, *Introduction to Numerical Analysis*, pp.244-313, Springer-Verlag New York Inc., 1980.

[8] Stephen Wolfram, *The Mathematica Book, 4th ed.*, Cambridge University Press, 1999.

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