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HAMILTONIAN INSERTED GRAPHS AND SQUARES

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Abstract. In this paper we characterize the graphs whose inserted graphs are Hamiltonian, and we study the relationship between Hamiltonian graphs and inserted graphs. Also we prove that if a connected graph G contains at least 3 vertices then inserted graph of the square of G is Hamiltonian and if G contains at least 3 edges then the square of inserted graph of G is Hamiltonian.

1. Introduction

In this paper, by a graph we mean an undirected, finite graph without loops and multiple edges. Let G be a graph with vertex set V_G and edge set E_G. Each member of $V_G \cup E_G$ will be called an element of G. A graph G is called trivial graph if it has a vertex set with single vertex and a null edge set. The degree of a vertex v of the graph G is denoted by $d_G(v)$. If e be an edge of a graph G with end vertices x and y, then we denote the edge e, by $e = xy$ and we define the degree of e by $d_G(x)+d_G(y)-2$. A vertex or an edge is called odd or even depending on whether it has odd or even degree. The graph G is called Hamiltonian graph if it has a Hamiltonian cycle i.e, a cycle containing all vertices of G.

We introduce the notions of box graph $B(G)$ and inserted graph $I(G)$ of a non-trivial graph G in [3].

In §2, we recall some definitions and results which will be used in this paper.

In §3, we give a necessary and sufficient condition for the Hamiltonian inserted graphs. More particularly for a connected graph G the inserted graph $I(G)$ is Hamiltonian if and only if there is a closed trail in G which contains at least one end-vertex of each edge of G. Also prove that if G is graph with $n \geq 4$ vertices and at least one edge and for each edge

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$e = xy$, $d_G(x) + d_G(y) \geq n$, then the inserted graph $I(G)$ is Hamiltonian and if $G \subseteq K_{n,n}$ be a bipartite graph with at least one edge where $n \geq 3$ and for each edge $e = xy$, $d_G(x) + d_G(y) \geq n+1$, then the inserted graph $I(G)$ is Hamiltonian unless G is isomorphic to the graph of Figure 1.

In §4, we prove that if G be a connected graph with $p \geq 3$ vertices, then $I(G^2)$ is Hamiltonian. Also prove that if G be a connected graph with $q \geq 3$ edges, then $[I(G)]^2$ is Hamiltonian.

2. Preliminaries

In this section we recall some definitions and results.

Definition 2.1. A walk in a graph G is an alternating sequence of vertices and edges as $v_1, e_1, v_2, e_2, ..., v_{m-1}, e_{m-1}, v_m$ such that e_i is the edge joining v_i and $v_{i+1} (1 \leq i \leq m-1)$. We shall call $v_1, v_2, ..., v_{m-1}, v_m$ the vertex-sequence of this walk. A spanning walk of G is a walk in which each vertex of G appears at least once. The walk is called path if the vertices $v_1, v_2, ..., v_m$ are distinct. The walk is called trail if the edges $e_1, e_2, ..., e_{m-1}$ are distinct. The trail is closed if $v_1 = v_m$.

Definition 2.2. ([6]) A cycle of a graph G is a closed walk $v_1, e_1, v_2, e_2, \cdots, v_m, e_m, v_1$ in which the vertices $v_1, v_2, \cdots, v_m$ are distinct and $m \geq 3$. A graph G is called Hamiltonian if G has a cycle containing all the vertices of G such a cycle is called Hamiltonian cycle.

Definition 2.3. ([7]) For a graph G, the square of G, i.e., G^2 is a graph with the property that there always exists a one-one correspondence between its vertices and the vertices of G such that two vertices of G^2 are adjacent if the corresponding vertices of G are joined by a path of length one or two.

Definition 2.4. ([3]) A new graph can be constructed by inserting a new vertex on each edge of a non-trivial graph G, the resulting graph is called Box graph of G, denoted by $B(G)$. For an edge e of G, $\overline{e}$ denotes the vertex of $B(G)$ corresponding to the edge e.

The graph $B(G)$ has the property that, there always exists a one-one correspondence between the vertices and the elements of G such that any two vertices of $B(G)$ are adjacent if and only if the corresponding elements of G are an edge and an incident vertex. Obviously $B(G)$ is a bipartite graph whose number of vertices is equal to the number of elements of G. Moreover if $V_G = \{v_1, v_2, ..., v_n\}$ and $E_G = \{e_1, e_2, ..., e_m\}$ then $V_{B(G)} = \{v_1, v_2, ..., v_n, \overline{e}_1, \overline{e}_2, ..., \overline{e}_m\}$.

DEFINITION 2.5. ([3]) Let I_G be the set of all inserted vertices in $B(G)$. A graph $I(G)$with vertex set I_G is called the inserted graph in which any two vertices are adjacent if they are joined by a path of length two in $B(G)$. Therefore if $E_G = \{e_1, e_2, ..., e_m\}$ then $I_G = V_{I(G)} = \{\overline{e}_1, \overline{e}_2, ..., \overline{e}_m\}$. Also the degree of $\overline{e}_i$ in $I(G)$ is equals to the degree of $e_i, i = 1, 2, ..., m$ in G.

3. Hamiltonian $I(G)$

THEOREM 3.1. *The graph $I(G)$ is Hamiltonian if and only if there is a closed trail in G which includes at least one end vertex of each edge of G.*

Proof. Let us suppose, that there is a closed trail $v_1, e_1, v_2,\ e_2, \cdots, v_n, e_n, v_1$ in G which includes at least one end vertex of each edge of G. Divide the edges of G not in this closed trail into n disjoint sets $S_1, ..., S_n$ such that the member of S_i are incident with v_i. If $S_i = \{x_i^1, ..., x_i^{r(i)}\}$, then $\overline{e}_n, \overline{x}_1^1, \cdots, \overline{x}_1^{r(1)}, \overline{e}_1, \overline{x}_2^1, \cdots, \overline{x}_2^{r(2)}, \overline{e}_2, \cdots, \overline{x}_n^1, \cdots, \overline{x}_n^{r(n)}, \overline{e}_n$ is the vertex-sequences of an Hamiltonian cycle of $I(G)$.

To prove the converse, let us assume that $I(G)$ has an Hamiltonian cycle with vertex-sequence $\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_k, \overline{e}_1$, where $e_1, e_2, \cdots, e_k$ are the distinct edges of G. Let v_i be the common end vertex of e_i and e_{i+1} in G for $i = 1, 2, \cdots, k-1$ and also let v_k be the common end vertex of e_k and e_1. If $v_1 = v_2 = ... = v_k$ the sequence with sole term v_1 is in a trivial sense, a closed trail which includes an end vertex of each edge of G. Otherwise, let $v_{j(1)}, v_{j(2)}, ..., v_{j(p)}$ be the subsequence of $v_1, v_2, ..., v_k$ consisting of the consecutively distinct vertices among $v_1, v_2, ..., v_k$; specifically

$$v_{j(r)} = v_i \neq v_{j(r+1)} \quad for \quad j(r) \leq i \leq j(r{+}1) \quad and \quad r = 1, 2, ..., p{-}1 ...(1)$$

$$and \quad v_{j(p)} = v_h \neq v_{j(1)} \quad for \quad h \geq j(p) \quad and \quad h < j(1) ....(2)$$

Then for $r = 2, ..., p$ the vertices $v_{j(r-1)} = v_{j(r)-1}$ and $v_{j(r)}$ are distinct vertices incident with $e_{j(r)}$ in G, they are joint by $e_{j(r)}$; and similarly $v_{j(p)}$ is joined by $e_{j(p)}$ and $e_{j(1)}$. Hence $v_{j(p)}, e_{j(1)}, v_{j(1)}, e_{j(2)}, v_{j(2)}, ..., e_{j(p)}, v_{j(p)}$ is a closed trail in G which includes each v_j and hence includes an end vertex of each edge of G, as desired. □

THEOREM 3.2. *Let G be a graph with $n \geq 4$ vertices and at least one edge and for each edge $e = xy$, $d_G(x) + d_G(y) \geq n$, then the inserted graph $I(G)$ is Hamiltonian.*

Proof. According to the Theorem 3.1, it it is sufficient to show that G has a closed trail which includes at least one vertex of each edge of G. First suppose that G has no cycle. Then there is an edge xy where $d_G(x) = 1$. But then $d_G(y) = n - 1$ and y is joined to all other vertices of G. It now follows that G is isomorphic to $K_{1,n-1}$ and hence $I(G)$ is isomorphic to K_{n-1}. Thus $I(G)$ has a Hamiltonian cycle. We now assume G has a cycle and hence a non-trivial closed trail.

Let γ be a closed trail which omits the fewest number $n-t$ of vertices, among all closed trails of G. Let A consist of those vertices which occur in γ, and $\overline{A}$ those that do not. So $\mid A \mid= t$ and $\mid \overline{A} \mid= n - t$. Since γ omits the fewest possible number of vertices, neither of the following can exist:

A cycle μ which has at least one vertex in A and at least one vertex not in A but none of whose edges belong to γ. ....(3)

A cycle μ exactly one of whose edges is an edge of γ and at least one of whose vertices is not in A. ....(4)

Suppose a cycle μ satisfying (3) existed. Let a be a vertex common to μ and γ. Then there is a closed trail which includes all vertices and edges of μ and γ. Suppose a cycle μ satisfying (4) existed. Let ab be the edge belonging to μ and γ. Then there is a closed trail which includes all vertices and edges of μ and γ except for the edge ab. In either circumstances, we distinguish three cases.

Case 1. There is an edge $e = xy$ such that $x, y \in \overline{A}$ and neither x nor y is joined to a vertex in A.

Let ab be any edge of γ. Let $\overline{A}_1$ consist of those vertices in $\overline{A}$ to which a is joined, let $\overline{A}_2$ consist of those vertices in $\overline{A}$ to which b is joined. Let $\mid \overline{A}_1 \mid= k$ and $\mid \overline{A}_2 \mid= l$. Then by the nonexistence of (4) $\overline{A}_1 \cap \overline{A}_2 = \phi$. By hypothesis,

$n \le d_G(a) + d_G(b) \le (t - 1 + k) + (t - 1 + l)$ or $n \le 2t + k + l - 2$..(5)

Since x and y are joined by an edge, it follows from the nonexistence of (3) and (4) that there is at most one edge joining x to $\overline{A}_1 \cup \overline{A}_2$ and at most one edge joining y to $\overline{A}_1 \cup \overline{A}_2$. Hence

$n \le d_G(x) + d_G(y) \le 2 + 2(n - t - k - l - 1)$ or $2t + 2k + 2l \le n$...(6)

Combining (5) and (6) we obtain $k + l \le -2$, a contradiction.

Case 2. There is an edge $e = xy$ such that $x, y \in \overline{A}$ and x but not y is joined to a vertex in A.

Let $d_G(y) = r$ and y be joined to the vertices $z_1, z_2, ..., z_{r-1}$ of $\overline{A}$ different from x. If any of the vertices $z_1, z_2, ..., z_{r-1}$ were not joined to a vertex to of A, then we would have an edge $z_i y$ to which Case 1 applies. Hence we may assume there are vertices $a_1, a_2, ..., a_{r-1} \in A$ such that z_i

is joined to a_i $(i = 1, 2, ..., r-1)$. It follows from the nonexistence of (3) that $a_i \neq a_j$ $(1 \leq i \leq j \leq r-1)$ and that x can not be joined to any of $a_1, a_2, ..., a_{r-1}$. Thus

$n \leq d_G(x)+d_G(y) \leq \{t-(r-1)+n-t-1\}+r$..................................(7)

Thus $d_G(x) + d_G(y) = n$, and x is joined to all the vertices of $A - \{a_1, a_2, ..., a_{r-1}\}$. If $r \geq 2$, we may pick an edge ua_i of γ where $u \neq a_1, a_2, ..., a_{r-1}$ and x is joined to u, contradicting the nonexistence of (4). Thus $d_G(y) = r = 1$ and $d_G(x) = n-1$. Since x is adjacent to all vertices of G and hence to all vertices in A, it follows from the nonexistence of (4) again that $t = 1$. But this means G has no non-trivial closed trails, contrary to our assumption.

Case 3. There is an edge $e = xy$ such that $x, y \in \overline{A}$ and each of x and y is joined to at least one vertex of A.

Let x be joined to $k \geq 1$ vertices in A and to $l \geq 0$ vertices in $\overline{A}$. Let y be joined to $p \geq 1$ vertices in A and to $q \geq 0$ vertices in $\overline{A}$. Let x be joined to $a \in A$. Then it follows from the nonexistence of (3) and (40 that the p vertices of A joined to y can not be joined to a. By the nonexistence of (3) y and the q vertices different from x of $\overline{A}$ joined to it can not be joined to a. Thus

$d_G(a) \leq (t-1-p)+(n-t-q-1) = n-p-q-2$................................(8)

But then

$n \leq d_G(a)+d_G(x) \leq n-p-q-2+k+l+1$ or, $p+q+1 \leq k+1$..........(9)

Similar reasoning shows that if $b \in A$ is joined to y then

$d_G(b) \leq n-k-l-2$ and, $k+l+1 \leq p+q$.......................................(10)

Hence from (9) and (10) we have

$p+q+1 \leq k+l \leq p+q-1$, ...(11)

which is a contradiction. □

Remark 1. It may be of interest that a graph may satisfy the hypothesis of Theorem 3.2 and yet fail to have a Hamiltonian cycle. Such a graph is obtained for $n \geq 4$ by taking G to be the complete graph K_{n-1} with an additional vertex joined to one vertex of K_{n-1}.

We now state and prove a stronger theorem for bipartite graph. Parts of the proof are similar to those of Theorem 3.2.

THEOREM 3.3. *Let $G \subseteq K_{n,n}$ be a bipartite graph with at least one edge where $n \geq 3$ and for each edge $e = xy$, $d_G(x) + d_G(y) \geq n+1$, then the inserted graph $I(G)$ is Hamiltonian unless G is isomorphic to the graph of Figure 1.*

Proof. Let X and Y be disjoint sets of n vertices each such that every edge of G joins a vertex of X to a vertex of Y. First suppose that G has no cycle. Then there exists an edge $e = xy$ such that $d_G(x) = 1$ where $x \in X$ and $y \in Y$. Hence $n+1 \leq d_G(x) + d_G(y) = 1 + d_G(y)$, so that y is joined to each vertex in X. Since G has no cycle, every other vertex

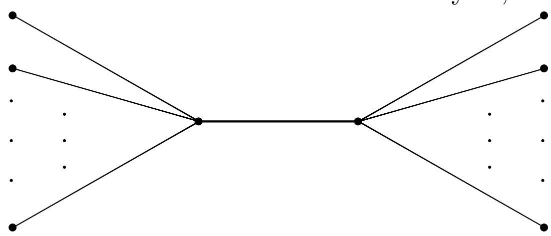

Figure 1

of Y is joined to at most one vertex in X. Suppose there were an edge $x^{'}y^{'}$ where $y^{'} \in Y$ and $y^{'} \neq y$. Then $d_G(y^{'}) = 1$ and it follows that $x^{'}$ is joined to all vertices in Y. Since G has no cycles, it now follows that either G is isomorphic to the graph consisting of $K_{n,1}$ and $n-1$ isolated vertices or the graph G_0 of Figure 1. Since $I(K_{n,1})$ is isomorphic to K_n and $n \geq 3$, $I(K_{n,1})$ has a Hamiltonian cycle. The graph $I(G_0)$ is isomorphic to the graph consisting two disjoint copies of $K_{n,1}$ along with a new vertex joined to each of the order $2(n-1)$ vertices and does not have a Hamiltonian cycle. Hence the theorem holds when G has no cycle.

Now assume that G has a cycle and hence a closed trail containing at least two vertices of X and at lest two vertices of Y. Among all closed trails of G, let γ be one which omit the fewest number of vertices. In addition, we choose such a γ so that there is a minimum number of edges having no vertex as a vertex of γ. Let γ omit $n-r$ vertices of X and $n-s$ vertices of Y. Let A_1 and A_2 consist of those vertices which occur in γ of X and Y respectively. Also let $\overline{A}_1$ and $\overline{A}_2$ consist of those vertices which not occur in γ of X and Y respectively. Then $\mid A_1 \mid= r$ and $\mid A_2 \mid= s$. Let $A = A_1 \cap A_2$ and $\overline{A} = \overline{A}_1 \cap \overline{A}_2$. Then neither (3) nor (4) can exist where (3) and (4) are defined in the proof of the Theorem 3.2.

Suppose there is an edge neither of whose vertices belong to A.

Case 1. There is an edge $e = xy$ such that $x \in \overline{A}_1$, $y \in \overline{A}_2$ and neither x nor y is joined to a vertex in A.

Let ab be any edge of γ where $a \in A_1$ and $b \in A_2$. Suppose a is joined to k vertices of $\overline{A}_2$ and b is joined to l vertices of $\overline{A}_1$. Then

$n+1 \leq d_G(a)+d_G(b) \leq s+k+r+l$..(12)

Since (3) and (4) can not occur, it follows that

$n+1 \leq d_G(x)+d_G(y) \leq (n-r-l)+(n-s-k)+1$............................(13)

From (12) and (13) we obtain the contradiction $n+1 \leq n$.
Case 2. There is an edge $e = xy$ such that $x \in \overline{A}_1$, $y \in \overline{A}_2$ and x, but not y is joined to a vertex in A.

Let $d_G(y) = t$ and y be joined to the vertices $x_1, x_2, ..., x_{t-1}$ of $\overline{A}$ different from x. If any of the vertices $x_1, x_2, ..., x_{t-1}$ were not joined to a vertex of A_2, then we would have an edge $x_i y$ to which Case 1 applied. Hence we may assume there are vertices $y_1, y_2, ..., y_{t-1} \in A_2$ such that x_i is joined to y_i $(i = 1, 2, ..., t-1)$. It follows from the nonexistence of (3) that the vertices $y_1, y_2, ..., y_{t-1}$ are distinct and that x can not be joined to any of $y_1, y_2, ..., y_{t-1}$. Thus

$n+1 \leq d_G(x)+d_G(y) \leq \{s-(t-1)+n-s\}+t = n+1$.....................(14)

We conclude that $d_G(x) = n-t+1$ and that each vertex of A_2 is joined to at least one of the vertices $x, x_1, x_2, ..., x_{t-1}$. Since γ is a non-trivial closed trail, $s \geq 2$. Suppose $t \geq 2$. Consider a portion of γ of the form u, uv, v, vw, w, where $u, w \in A_2$ and u is joined to x and w is joined to x_j. Then we may replace this portion of γ by $u, ux, x, e, y, yx_j, x_j, x_jw, w$ to obtain a closed trail which includes more vertices of G than γ. It follows that $t = 1$, and that y is joined only to x but x is joined to all vertices in Y. Consider an edge ab of γ where $a \in A_1$ and $b \in A_2$. Since (3) and (4) can not exist, it follows that a can not joined to a vertex in $\overline{A}_2$. Now consider a portion of γ of the form b, ba, a, ac, c, where $b, c \in A_2$. Then x is joined to both b and c and we may replace the above portion of γ by b, bx, x, xc, c. The resulting closed trail $\gamma^{'}$ omits no more vertices than γ does. Moreover, the each edge with a vertex in γ also has a vertex in $\gamma^{'}$. In addition the edge e has a vertex in $\gamma^{'}$. This contradicts our choice of γ.
Case 3. There is an edge $e = xy$ such that $x \in \overline{A}_1$, $y \in \overline{A}_2$ such that x and y are joined to at least one vertex in A.

We may assume Case 1 and Case 2 do not occur. Let B_2 consist of those vertices in A_2 joined to x and $\overline{B}_2$ consist of those vertices in $\overline{A}_2$ joined to x. Let B_1 consist of those vertices in A_1 joined to y and $\overline{B}_1$ consist of those vertices in $\overline{A}_1$ joined to y. Let $\mid B_2 \mid = k \geq 1$, $\mid \overline{B}_2 \mid = l \geq 0$, $\mid B_1 \mid = p \geq 1$, $\mid \overline{B}_1 \mid = q \geq 0$. Let $a \in B_2$ and $b \in B_1$. Then it follows from the nonexistence of (3) and (4) that a can not be joined to any vertex in B_1 and x is the only vertex of $\overline{B}_1$ joined to a. Thus

$n+1 \leq d_G(a) + d_G(x) \leq \{(r-p) + (n-r) - q\} + k + l + 1$ or, $p + q \leq k + l$......(15)
Similarly,

$n + 1 \leq d_G(b) + d_G(y) \leq \{(s-k) + (n-s) - l\} + p + q + 1$ or, $k + 1 \leq p + q$......(16)
Hence $p+q = k+l$. Thus $d_G(a) = n-p-q$ and $d_G(b) = n-k-l$. Thus a and hence each vertex of B_2 is joined to x, all vertices in $A_1 - B_1$ and all vertices in $\overline{A}_1 - \overline{B}_1$. Likewise each vertex of B_1 is joined to y all vertices in $A_2 - B_2$ and all vertices $\overline{A}_2 - \overline{B}_2$. Let $z \in \overline{A}_1 - \overline{B}_1$, if such a vertex exists. Suppose z were joined to a vertex $u \in A_2 - B_2$. Then z, u, b, y, x, a, z is the vertex-sequence of a cycle which satisfies (3) or (4). Thus z is joined to no vertex in $A_2 - B_2$. Suppose z were joined to a vertex $u \in B_2$ with $u \neq a$. Then z, u, x, a, z is the vertex-sequence of a cycle satisfying (3). Suppose z were joined to a vertex $u \in \overline{A}_2 - \overline{B}_2$. Then z, u, b, y, x, a, z is a vertex-sequence of a cycle satisfying (3). Finally suppose z were joined to a vertex $u \in \overline{B}_2$. Then z, u, x, a, z is a vertex-sequence of a cycle satisfying (3). Thus z can be joined only to a. Since az is an edge, it follows that $d_G(z) = 1$ and that $d_G(a) = n$. Thus $p = q = 0$, a contradiction. It follows that $\overline{A}_1 = \overline{B}_1$ and likewise that $\overline{A}_2 = \overline{B}_2$. Thus $\overline{A}_1$ contains exactly $q+1$ vertices, all joined to y and $\overline{A}_2$ contains exactly $l + 1$ vertices, all joined to x. By repeating the preceding argument on each of the edges joining x to a vertex of $\overline{A}_2$, we conclude that each vertex of $\overline{A}_1$ is joined to each vertex of $\overline{A}_2$.

Suppose there were a vertex $x' \neq x$ in $\overline{A}_1$. Then x' is joined to some vertex $u \in A_2 - B_2$. But then x', u, b, y, x' is the vertex-sequence of a cycle which contradicts the nonexistence of a cycle satisfying (4). Thus $\overline{A}_1 = \{x\}$ and similarly, $\overline{A}_2 = \{y\}$. It follows that $q = l = 0$ and since $p + q = k + l$, $p = k$.

Now let $a' \in B_2$ and $b' \in A_1 - B_1$ so that there is an edge joining a' and b'. Then

$n+1 \leq d_G(a')+d_G(b') = 1+(n-p-l)+d_G(b')$...............................(17)
so that $p+1 \leq d_G(b')$. It follows that each vertex of $A_1 - B_1$ is joined to at least one vertex of $A_2 - B_2$. Similarly, each vertex of $A_2 - B_2$ is joined to at least one vertex of $A_1 - B_1$. Since γ is a closed trail and no vertex of B_2 is joined to a vertex of B_1, it follows that $n - 1 - p \geq 2$. Moreover, since $n + 1 \leq d_G(x) + d_G(y) = 2(1 + p)$, $p \geq n - 1 - p \geq 2$. Since each vertex of B_1 is joined to each vertex of $A_2 - B_2$, it follows by induction that there is trail which joins the vertex b of B_1 to any specified vertex u of $A_2 - B_2$ and which includes all vertices of $B_1 \cup (A_2 - B_2)$. Likewise, there is a trail joining the vertex a of B_2 to any specified vertex v of $A_1 - B_1$ and which includes all vertices of $B_2 \cup (A_1 - B_1)$. Choose u and v so that uv an edge. Since ax, e and yb are edges of G, there is a closed trail which includes all vertices of G, contradicting the choice of γ.

Hence the theorem follows. □

4. Hamiltonian $I(G^2)$ and $[I(G)]^2$

LEMMA 4.1. *Let G be a connected graph with $p \geq 3$ vertices and such that it contains a vertex u with $d_G(u) = 1$ and a vertex w with $d_G(w) = p - 1$. If v is a vertex of G such that $u \neq v \neq w$, then there exists a spanning path in $I(G)$ joining the vertices $\overline{uw}$ and $\overline{vw}$ of $I(G)$.*

Proof. The case when $p = 3$ is obvious. Assume that $p = n \geq 4$ and that for $p = n - 1$ the lemma is proved. The case when G is a star is simple. Assume G is not a star. Then there is a vertex x of G such that $d_G(x) \geq 2$ and $u \neq x \neq w$. By $v_1, .., v_k$ we denote the vertices of G different from w and adjacent to x. Obviously, there is a spanning path S in $I(G-x)$ joining the vertices $\overline{uw}$ and $\overline{vw}$. There is vertex $\overline{rs}$ of $I(G-x)$ such that $\overline{rs}\ \ \overline{v_1w}$ is an edge in S. It is evident that either $v_1 \in \{r, s\}$ or $w \in \{r, s\}$. If $v_1 \in \{r, s\}$, then $\overline{rs}, \overline{xv_1}, ..., \overline{xv_k}, \overline{xw}, \overline{v_1w}$ be the vertex-sequence of a path P. If $w \in \{r, s\}$, then $\overline{rs}, \overline{xw}, \overline{xv_k}, ..., \overline{xv_1}, \overline{v_1w}$ be the vertex-sequence of the path P. If in S we replace the edge $\overline{rs}\ \ \overline{v_1w}$ by the path P, we obtain a spanning path in $I(G)$ joining the vertices $\overline{uw}$ and $\overline{vw}$. □

THEOREM 4.2. *If G be a connected graph with $p \geq 3$ vertices, then $I(G^2)$ is Hamiltonian.*

Proof. The case when $p = 3$ is obvious. Assume that $p = n \geq 4$ and that for $p = n - 1$ the theorem is proved. The case when $G = K_p$ is simple. Assume that $G \neq K_p$. Then there is vertex w of G with degree not exceeding $p - 2$ and such that $G - w$ is connected. By F we denote the graph with the vertex t of G such that $d_G(t, w) \leq 2$, and with the edges t_1t_2 such that either $w \in \{t_1, t_2\}$ and $1 \leq d_G(t_1, t_2) \leq 2$, or $t_1 \neq w \neq t_2$ and $d_G(t_1, t_2) = 2 < d_{(G-w)}(t_1, t_2)$. Notice that the graphs $(G - w)^2$ and F are edge-disjoint and that x is an edge in G^2 if and only if it is an edge either in $(G - w)^2$ or in F. There are vertices u and v of G such that v is adjacent to w in G, u is adjacent to v in G and $d_G(u, w) = 2$. Obviously, u and v vertices both in $(G - w)^2$ and in F, and u has degree 1 in F. By Lemma 4.1, there is a spanning path S_0 in $I(F)$ joining $\overline{uw}$ with $\overline{vw}$. Similarly, there is a spanning path S_1 in $I(F)$ joining $\overline{vw}$ with $\overline{uw}$. By the induction hypothesis, there exits a a Hamiltonian cycle H in $I((G-w)^2)$. Consider a vertex $\overline{rs}$ of $I((G-w)^2)$ such that $\overline{rs}\,\overline{uv}$ is an edge in H. If $u \in \{r, s\}$, then by P we denote the path $\overline{rs}\,S_0\,\overline{uv}$; if $v \in \{r, s\}$, then by P we denote the path $\overline{rs}\,S_1\,\overline{uv}$. It is easy to see that if in H we replace the edge $\overline{rs}\;\overline{uv}$ by P we obtain a Hamiltonian cycle in $I(G^2)$. □

LEMMA 4.3. *Let T be any tree with $q \geq 3$ edges. Then $[I(T)]^2$ is Hamiltonian.*

Proof. The case when $q = 3$ is obvious. Let $q = n \geq 4$ and assume that for any q, $3 \leq q < n$ the lemma is proved. The case when T is a path is simple. We shall assume that T is not a path. Then T contains distinct vertices $v_0, ..., v_k$ such that $1 \leq k \leq q-2$ where v_0 is adjacent to v_1, . . . , v_{k-1} is adjacent to v_k, v_0 has degree at least 3, v_k has degree 1 and if $0 < j < k$, then v_j has degree 2. By T_0 we denote the tree which we obtain from T by deleting the vertices $v_1, ..., v_k$. By $u_1, ..., u_i$ we denote the vertices which are adjacent to v_0 in T_0; obviously $i \geq 2$. There is a Hamiltonian cycle H in $[I(T_0)]^2$. It is easy to verify that H contains such an edge $\overline{x}\,\overline{y}$ of $[I(T_0)]^2$ that the edge x is incident with one of the vertices $u_1, ..., u_i$ and y is incident with v_0. By P we denote the path in $[I(T)]^2$ such that if $k = 1$, then $\overline{x}, \overline{v_0v_1}, \overline{y}$ is the vertex-sequence of P, and if $k \geq 2$, then $\overline{x}, \overline{v_0v_1}, \overline{v_2v_3}, ..., \overline{v_{g-3}v_{g-2}}, \overline{v_{g-1}v_g}, \overline{v_hv_{h-1}}, ..., \overline{v_2v_1}, \overline{y}$ is the vertex-sequence of P, where g is the greatest odd integer not exceeding k and h is the the greatest even integer not exceeding k. If in H we replace $\overline{x}\,\overline{y}$ by P, we obtain a Hamiltonian cycle in $[I(T)]^2$. □

THEOREM 4.4. *If G be a connected graph with $q \geq 3$ edges, then $[I(G)]^2$ is Hamiltonian.*

Proof. Consider a spanning tree T_1 of G. Colour the edges of T_1 in red. Subdivide each uncoloured edges of G (if any) into two new edges and colour one of them in red and other of them in blue (the choice is arbitrary). By T_2 we denote the graph consisting of the red edges. Obviously T_2 is a tree with at least 3 edges. It is easy to see that $I(T_2)$ is isomorphic to a spanning subgraph of $I(G)$. This implies that $[I(T_2)]^2$ is isomorphic to a spanning subgraph of $[I(G)]^2$. By Lemma 4.3, $[I(T_2)]^2$ is Hamiltonian. Hence the theorem follows. □

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