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ON GENERAL BEST PROXIMITY PAIRS AND EQUILIBRIUM PAIRS IN FREE GENERALIZED GAMES[1]

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Abstract. In this paper, using the fixed point theorem for acyclic factorizable multifunctions, we shall prove an existence theorem of general best proximity pairs and equilibrium pairs for free generalized games.

1. Introduction

In 1969, Ky Fan [4] proved the best approximation theorem which generalizes the previous known fixed point theorems, and his main result states that if K is a non-empty compact convex subset of a locally convex Hausdorff topological vector space E with a continuous seminorm p, and $f : K \to E$ is a single valued continuous function, then there exists an element $\bar{x} \in K$ such that $p(f(\bar{x}) - \bar{x}) = \inf\{p(f(\bar{x}) - y) \mid y \in K\}$. Since then, a number of generalizations of this theorem have been obtained in various directions by several authors (e.g., see [7-10]).

In a recent paper [5], Kim and Lee introduce a new concept of free generalized games which generalizes the previous concept of generalized games in [2, 3, 11], and using the fixed point theorem for Kakutani factorizable multifunctions, they prove the existence theorems of best proximity pairs and equilibrium pairs for free generalized games.

In this paper, using Park's fixed point theorem for acyclic factorizable multifunctions, we shall prove some generalizations of the existence theorems for best proximity pairs due to Srinivasan and Veeramani [10], and Kim and Lee [5]. As an application, we shall prove new existence

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theorem of general equilibrium pairs for free generalized games which generalize the corresponding results due to Srinivasan and Veeramani [10], and Kim and Lee [5] in several aspects.

2. Preliminaries

Recall the following definitions of proximity concepts in [1,7]. Let X and Y be any two non-empty subsets of a Hausdorff topological vector space E, and let p be a continuous seminorm on E. For $y \in E$, define $d_p(y, X) := \inf\{p(y-x) \,|\, x \in X\}$, and $d_p(X, Y) := \inf\{p(x-y) \,|\, x \in X,\, y \in Y\}$. If $X = \{x\}$ and $Y = \{y\}$, then $d_p(x, y)$ denotes $d_p(X, Y)$ which is precisely $p(x-y)$. Let I be a finite (or an infinite) index set. For each $i \in I$, let X and Y_i be non-empty subsets of a Hausdorff topological vector space E with a continuous seminorm p. Then we can use the following notations: for each $i \in I$,

$d_p(X, Y_i) := \inf\{d_p(x, y) \mid x \in X,\ y \in Y_i\};$

$X^o := \{x \in X \mid \text{for each } i \in I,\ \exists\, y_i \in Y_i \text{ such that } d_p(x, y_i) = d_p(X, Y_i)\,\};$

$Y_i^o := \{y \in Y_i \mid \text{there exists } x \in X \text{ such that } d_p(x, y) = d_p(X, Y_i)\,\}.$

Let X and Y be two non-empty subsets of a Hausdorff topological vector space E with a continuous seminorm p, and let $T : X \to 2^Y$ be a multifunction. Then the pair $(\bar{x}, T(\bar{x}))$ is called the *best proximity pair* [10] for T if $d_p(\bar{x}, T(\bar{x})) = d_p(\bar{x}, \bar{y}) = d_p(X, Y)$ for some $\bar{y} \in T(\bar{x})$. Then the best proximity pair theorem seeks an appropriate solution which is optimal.

We shall need more definitions on proximity concepts as follows: Let X be a non-empty subset of a locally convex Hausdorff topological vector space E with a continuous seminorm p. If X is a compact and convex subset of E, then the set $P_X(z)$ of all $p-$best approximations in X to any element $z \in E$, defined by

$$P_X(z) := \{x \in X \mid d_p(z, x) = d_p(z, X)\},$$

is a non-empty compact and convex subset of X, and every point in $P_X(z)$ is called a *best proximity point* of z in X. Also, any point $x \in X$ for which $d_p(x, Y) = d_p(X, Y)$ is called a best proximity point of Y in X, and points $x \in X, y \in Y$ satisfying $p(x-y) = d_p(X, Y)$ are called best proximity points of the pair (X, Y). By Proposition 2.1 in [15], we can see that the metric projection mapping $z \mapsto P_X(z)$ is an upper semicontinuous multifunction on E. For the properties of metric projection (see [7, 8]).

Let C be a non-empty convex subset of E. A single valued function $g : C \to E$ is said to be *quasi p-affine* [8] if for each real number $r \geq 0$ and $x \in E$, the set $\{y \in C \,|\, p(g(y) - x) \leq r\}$ is convex. Let A be a non-empty subset of E. Then, g is said to be *p-continuous* [8] in A if $\{p\big(g(x_\alpha) - g(x)\big)\} \to 0$ for each $x \in A$ and every net x_α in C converging to $x \in A$. If g is continuous, then it is clear that g is p-continuous.

A non-empty topological space is called *acyclic* if all its reduced Čech homology groups over rationals vanish. Then, a multifunction $T : X \to 2^Y$ from a topological space X to another topological space Y is said to be an *acyclic* multifunction [6] if the following conditions are satisfied: (i) T is upper semicontinuous; (ii) $T(x)$ is a non-empty compact and acyclic subset of Y.

The collection of all acyclic multifunctions from X to Y is denoted by $\mathbb{V}(X, Y)$. A multifunction $T : X \to 2^Y$ from a topological space X to another topological space Y is said to be an *acyclic factorizable multifunction* [6] if it can be expressed as a composition of finitely many acyclic multifunctions. The collection of all acyclic factorizable multifunctions from X to Y is denoted by $\mathbb{V}_C(X, Y)$.

The following fixed point theorem, which is a special form of a fixed point theorem due to Park [6], is also essential in proving our results:

LEMMA 2.1. *If X is a non-empty compact and convex subset of a locally convex Hausdorff topological vector space, then any acyclic factorizable multifunction $T : X \to 2^X$ has a fixed point, i.e., if $T \in \mathbb{V}_C(X, X)$, then there exists a point $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$.*

Now we recall the following equilibrium pair concept in [5] which generalizes the previous concept of abstract economies in [2, 3, 11]. Let I be a finite or an infinite set of locations or agents. For each $i \in I$, let X_i be a non-empty set of manufacturing commodities, and Y_i be a non-empty set of selling commodities. *A free generalized game* (or *free abstract economy*) [5] $\Gamma = (X_i, Y_i, A_i, P_i)_{i \in I}$ is defined as a family of ordered quadruples (X_i, Y_i, A_i, P_i) where X_i and Y_i are non-empty subsets of a locally convex Hausdorff topological vector space E with a continuous seminorm p, and $A_i : X = \Pi_{j \in I} X_j \to 2^{Y_i}$ is a constraint correspondence, and $P_i : Y = \Pi_{j \in I} Y_j \to 2^{Y_i}$ is a preference correspondence. An *equilibrium pair* [5] for Γ is a pair of points $(\bar{x}, \bar{y}) = \big((\bar{x}_i)_{i \in I}, (\bar{y}_i)_{i \in I}\big) \in X \times Y$ such that for each $i \in I$, $\bar{y}_i \in A_i(\bar{x})$ with $d_p(\bar{x}_i, \bar{y}_i) = d_p(X_i, Y_i)$, and $A_i(\bar{x}) \cap P_i(\bar{y}) = \emptyset$.

In particular, when $I = \{1, \cdots, n\}$, we may call Γ a *free n-person game*.

When $X_i = Y_i$ for each $i \in I$, then the previous definitions can be reduced to the standard definitions of equilibrium theory in mathematical economics due to Yannelis-Prabhakar [11]. For more economic meanings and interpretations (see [5]).

3. General best proximity pairs and equilibrium pairs

Using Lemma 2.1, we begin with the following existence theorem for general best proximity pairs, which generalizes Theorem 2 in [5] to the acyclic-valued multifunctions:

THEOREM 3.1. *For each* $i \in I = \{1, 2, \cdots, n\}$, *let* X *and* Y_i *be non-empty compact and convex subsets of a locally convex Hausdorff topological vector space* E *with a continuous seminorm* p, *and* X^o *is a non-empty subset of* X. *Let* $T_i : X \to 2^{Y_i}$ *be an upper semicontinuous multifunction in* X^o *such that each* $T_i(x)$ *is non-empty compact, and* $T_i(x) \cap Y_i^o$ *is an acyclic subset of* Y_i, *and let* $g : X^o \to X^o$ *be a* p*-continuous, proper, quasi* p*-affine, and surjective mapping on* X^o. *Assume that for each* $x \in X^o$, *there exists* $(y_1, \ldots, y_n) \in \Pi_{i \in I} T_i(x)$ *such that*

$$\exists\, x_o \in X \text{ with } d_p(x_o, y_i) = d_p(X, Y_i) \text{ for each } i \in I, \qquad (*)$$

and $\cap_{i \in I} P_X(y_i)$ *is non-empty for each* $(y_1, \ldots, y_n) \in \Pi_{i \in I} Y_i^o$.

Then there exists a point $\bar{x} \in X$ *satisfying the system of best proximity pairs, i.e., for each* $i \in I$, $\{g(\bar{x})\} \times T_i(\bar{x}) \subseteq X \times Y_i$ *such that* $d_p\big(g(\bar{x}), T_i(\bar{x})\big) = d_p(X, Y_i)$.

Proof. As shown in the proof of Theorem 1 of [5], we can see that X^o and Y_i^o are non-empty compact and convex. Also, since X is non-empty compact and convex, it is known that the metric projection map $P_X : E \to 2^X$ is upper semicontinuous in E such that $P_X(z)$ is a non-empty compact and convex subset of X for each $z \in E$. For each $i \in I$, we now define a multifunction $T_i' : X^o \to 2^{Y_i^o}$ by

$$T_i'(x) := T_i(x) \cap Y_i^o \quad \text{for each } x \in X^o.$$

Then, by the assumption, T_i' is upper semicontinuous in X_o such that each $T_i'(x)$ is a non-empty compact and acyclic subset of Y_i^o. Also, we can see that $P_X(Y_i^o) \subseteq X^o$ as in the proof of Theorem 1 in [5].

Now we introduce the multifunctions $T' : X^o \to 2^{\Pi_{i \in I} Y_i^o}$, defined by

$$T'(x) := \Pi_{i \in I} T_i'(x) \quad \text{for each } x \in X^o;$$

and $P_X' : \Pi_{i \in I} Y_i^o \to 2^{X^o}$, defined by

$$P_X'(y_1, \ldots, y_n) := \bigcap_{i \in I} P_X(y_i) \quad \text{for each } (y_1, \ldots, y_n) \in \Pi_{i \in I} Y_i^o.$$

Then, T' is upper semicontinuous in X^o such that $T'(x)$ is non-empty compact and acyclic in $\Pi_{i\in I}Y_i^o$. By the assumption $(*)$, each $P'_X(y_1,\dots,y_n)$ is a non-empty subset in X^o so that each $P'_X(y_1,\dots,y_n)$ is non-empty closed convex. And we can see that the multifunction $g^{-1}\circ P'_X : \Pi_{i\in I}Y_i^o \to 2^{X^o}$ is an acyclic multifunction. Therefore, the composition map $(g^{-1}\circ P'_X)\circ T' : X^o \to 2^{X^o}$ is an acyclic factorizable multifunction in X^o. Therefore, by Lemma 2.1, there exists a fixed point $\bar{x} \in X^o$ such that $\bar{x} \in ((g^{-1}\circ P'_X)\circ T')(\bar{x})$, that is, $g(\bar{x}) \in (P'_X \circ T')(\bar{x})$. Then, $g(\bar{x}) \in P'_X\big(T'_1(\bar{x}),\dots,T'_n(\bar{x})\big)$ so that there exists $(\bar{y}_1,\dots,\bar{y}_n) \in \Pi_{i\in I}\big(T_i(\bar{x})\cap Y_i^o\big)$ such that $g(\bar{x}) \in P'_X(\bar{y}_1,\dots,\bar{y}_n) = \bigcap_{i\in I} P_X(\bar{y}_i)$. Since each $\bar{y}_i$ is an element in Y_i^o, there exists an $x'_i \in X$ such that $d_p(x'_i,\bar{y}_i) = d_p(X,Y_i)$ for each $i\in I$. Therefore, for each $i \in I$,

$$d_p(g(\bar{x}),T_i(\bar{x})) \le d_p(g(\bar{x}),\bar{y}_i) = d_p(\bar{y}_i,X) \le d_p(\bar{y}_i,x'_i) = d_p(X,Y_i)$$

so that $d_p(g(\bar{x}),T_i(\bar{x})) = d_p(X,Y_i)$, which completes the proof. □

Remark 1. (1) Theorem 1 further generalizes Theorem 2 in [5] in the following aspects:

(i) the underlying space is a locally convex Hausdorff topological vector space instead of a normed linear space;

(ii) each T_i is an acyclic multifunction instead of a Kakutani multifunction;

(2) When $g = id_{X^o}$ and E is a normed linear space in Theorem 1, then the conclusion is reduced to Theorem 2 in [5].

Before stating the existence of equilibrium pair for the free n-person game, we shall need the following lemma which is the special case of Theorem 3 due to Ding-Kim-Tan [3] in a compact convex setting:

Lemma 3.2. *Let $\Gamma = (Y_i,\Phi_i)_{i\in I}$ be a qualitative game where I is a (possibly infinite) set of agents such that for each $i\in I$,*

(1) Y_i is a non-empty compact and convex subset of a locally convex Hausdorff topological vector space;

(2) the correspondence $\Phi_i : Y = \Pi_{j\in I}Y_j \to 2^{Y_i}$ is L^*-majorized in Y;

(3) the set $W_i := \{y\in Y \mid \Phi_i(y)\neq\emptyset\}$ is (possibly empty) open in Y.

Then there exists $\bar{y}\in Y$ such that for each $i\in I$, $\Phi_i(\bar{y}) = \emptyset$.

Next, using Lemma 3.2, we shall prove the existence of equilibrium pairs for a free n-person game which generalizes Theorem 4 in [5] as follows:

THEOREM 3.3. *Let* $\Gamma = (X, Y_i, A_i, P_i)_{i\in I}$ *be a free* n*-person game such that for each* $i \in I = \{1, \ldots, n\}$,

(1) X *is a non-empty compact and convex subset of a locally convex Hausdorff topological vector space* E *with a continuous seminorm* p, X^o *a non-empty subset of* X, Y_i *a non-empty compact and convex subset of* E, *and let* $Y := \Pi_{j\in I} Y_j$;

(2) $A_i : X \to 2^{Y_i}$ *is an upper semicontinuous correspondence such that each* $A_i(x)$ *is a non-empty closed convex subset of* Y_i, *and satisfies the condition* $(*)$ *in Theorem 1;*

(3) $P_i : Y \to 2^{Y_i}$ *is a preference correspondence which is* L^**-majorized in* Y*;*

(4) $P_i(y)$ *is non-empty for each* $y = (y_i)_{i\in I} \in Y$ *with* $y_i \in Y \setminus \mathcal{A}_{i_x}$, *whenever* $\mathcal{A}_{i_x} := \{z \in Y_i \mid z \in A_i(x) \text{ and } d_p(x, z) = d_p(X, Y_i)\ \}$ *is non-empty;*

(5) the set $W_i := \{y \in Y \mid A_i(x) \cap P_i(y) \neq \emptyset\}$ *is open in* Y *whenever* $\mathcal{A}_{i_x}$ *is non-empty.*

Then there exists an equilibrium pair $(\bar{x}, \bar{y}) = (\bar{x}, (\bar{y}_i)_{i\in I}) \in X \times Y$ *for* Γ, *i.e., for each* $i \in I$, $\bar{y}_i \in A_i(\bar{x})$ *and* $d_p(\bar{x}, \bar{y}_i) = d_p(X, Y_i)$ *such that* $A_i(\bar{x}) \cap P_i(\bar{y}) = \emptyset$.

Proof. For each $i \in I$, since A_i satisfies the whole assumptions of Theorem 3.1 in case $g = id_{X^o}$, there exists a point $\bar{x} \in X$ satisfying the system of best proximity pairs, i.e., for each $i \in I$, $\{\bar{x}\} \times A_i(\bar{x}) \subseteq X \times Y_i$ such that $d_p(\bar{x}, A_i(\bar{x})) = d_p(X, Y_i)$. Now we may denote the non-empty best proximity set of the correspondence A_i at $\bar{x}$ simply by

$$\mathcal{A}_i := \{z \in Y_i \mid z \in A_i(\bar{x}) \text{ and } d_p(\bar{x}, z) = d_p(X, Y_i)\ \}.$$

Then, it is easy to see that each $\mathcal{A}_i$ is a closed and convex subset of a compact convex set $A_i(\bar{x})$. It remains to show that there exists a point $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that for each $i \in I$, $\bar{y}_i \in A_i(\bar{x})$ and $A_i(\bar{x}) \cap P_i(\bar{y}) = \emptyset$.

For each $i \in I$, we now define a multifunction $\phi_i : Y \to 2^{Y_i}$ by for each $y = (y_1, \ldots, y_n) \in Y$,

$$\phi_i(y) := \begin{cases} P_i(y), & \text{if } \ y_i \notin \mathcal{A}_i, \\ A_i(\bar{x}) \cap P_i(y), & \text{if } \ y_i \in \mathcal{A}_i. \end{cases}$$

In order to apply Lemma 3.2 to ϕ_i for each $i \in I$, we should check the assumptions (2) and (3) of Lemma 3.2. We first show that the set $\{y \in Y \mid \phi_i(y) \neq \emptyset\}$ is open in Y for each $i \in I$. By the assumption (5), the set $W_i := \{y \in Y \mid A_i(\bar{x}) \cap P_i(y) \neq \emptyset\}$ is open in Y. Note that the projection map $\pi_i : Y \to Y_i$, defined by $\pi_i(y_1, \ldots, y_n) = y_i$, is an open map in Y. Then we have

$$\begin{aligned}&\{y\in Y \mid \phi_i(y)\neq\emptyset\}\\ &=\{y\in Y\setminus\pi_i^{-1}(\mathcal{A}_i)\mid\phi_i(y)\neq\emptyset\}\cup\{y\in\pi_i^{-1}(\mathcal{A}_i)\mid\phi_i(y)\neq\emptyset\}\\ &=\{y\in Y\setminus\pi_i^{-1}(\mathcal{A}_i)\mid P_i(y)\neq\emptyset\}\cup\{y\in\pi_i^{-1}(\mathcal{A}_i)\mid A_i(\bar{x})\cap P_i(y)\neq\emptyset\}\\ &=\big(Y\setminus\pi_i^{-1}(\mathcal{A}_i)\big)\cup\big(W_i\cap\pi_i^{-1}(\mathcal{A}_i)\big)=\big(Y\setminus\pi_i^{-1}(\mathcal{A}_i)\big)\cup W_i.\end{aligned}$$

Since the projection mapping π_i is open and $\mathcal{A}_i$ is compact, we have $\pi_i^{-1}(\mathcal{A}_i)$ is closed so that the set $\{y \in Y \mid \phi_i(y) \neq \emptyset\}$ is open in Y by the assumption (5).

Next, we shall show that ϕ_i is L^*-majorized in Y. By the assumption (4), for each $y \in Y$ with $y_i \notin \mathcal{A}_i$, then $\phi_i(y) = P_i(y)$ is non-empty so that there exists an L^*-majorant ψ_i of ϕ_i in Y by the assumption (3). For each $y \in Y$ with $y_i \in \mathcal{A}_i$, then $\phi_i(y) = A_i(\bar{x}) \cap P_i(y)$. If $\phi_i(y) = A_i(\bar{x}) \cap P_i(y)$ is non-empty, then $P_i(y)$ is non-empty. By the assumption (3) again, there exists an L^*-majorant ψ_i of P_i in Y. Since $\phi_i(y) = A_i(\bar{x}) \cap P_i(y) \subset P_i(y)$ for each $y \in Y$ with $y_i \in \mathcal{A}_i$, ψ_i is also an L^*-majorant of ϕ_i in Y. Therefore, ϕ_i is L^*-majorized in Y for each $i \in I$, and hence the whole hypotheses of Lemma 3.2 are satisfied so that there exists a point $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that $\phi_i(\bar{y}) = \emptyset$ for each $i \in I$. If $\bar{y}_i \notin \mathcal{A}_i$ for some $i \in I$, then by the assumption (4), $\phi_i(\bar{y}) = P_i(\bar{y})$ is a non-empty subset of Y_i, which is a contradiction. Therefore, for each $i \in I$, we must have $\bar{y}_i \in \mathcal{A}_i$ and $\phi_i(\bar{y}) = A_i(\bar{x}) \cap P_i(\bar{y}) = \emptyset$. This completes the proof. □

Remark 2. Theorem 3.3 further generalizes Theorem 4 in [5] in the following aspects:

(i) X need not be a compact set in a normed linear space;

(ii) P_i need not be of class L but L^*-majorized.

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