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THE RIESZ THEOREM IN LINEAR n-NORMED SPACES

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Abstract. We prove the Riesz theorem in linear n-normed spaces

F. Riesz [7] obtained the following theorem in a normed space.

Theorem 1.1. *Let Y and Z be subspaces of a normed space X and Y a closed proper subset of Z. For each $\theta \in (0,1)$, there exists an element $z \in Z$ such that*

$$\|z\| = 1, \qquad \|z-y\| \geq \theta$$

for all $y \in Y$.

In this note, we extend this result to the case of linear n-normed space. To do this, we introduce the notion of n-compact set, which is an analogous notion of sequentially compact set in normed space.

Let X be a real linear space with $\dim X \geq n$ and $\|\cdot, \cdots, \cdot\| : X^n \to \mathbb{R}$ a function. Then $(X, \|\cdot, \cdots, \cdot\|)$ is called a *linear n-normed space* if

(nN_1) $\|x_1, \cdots, x_n\| = 0 \Leftrightarrow x_1, \cdots, x_n$ are linearly dependent,

(nN_2) $\|x_1, \cdots, x_n\| = \|x_{j_1}, \cdots, x_{j_n}\|$ for every permutation $(j_1, \cdots, j_n)$ of $(1, \cdots, n)$,

(nN_3) $\|\alpha x_1, \cdots, x_n\| = |\alpha| \ \|x_1, \cdots, x_n\|$,

(nN_4) $\|x+y, x_2, \cdots, x_n\| \leq \|x, x_2, \cdots, x_n\| + \|y, x_2, \cdots, x_n\|$

for all $\alpha \in \mathbb{R}$ and all $x, y, x_1, \cdots, x_n \in X$.

The function $\|\cdot, \cdots, \cdot\|$ is called an *n-norm on X* (see [1, 12.3]).

The properties of n-normed spaces and n-inner product spaces have been investigated by many authors (see [2], [3], [4], [5], [6]).

We define an *n-convergent sequence* in a linear n-normed space.

Definition 1.2. A sequence $\{x_k\}$ in a linear n-normed space X is *n-convergent* to x and denoted by $x_k \to x$ as $k \to \infty$ if

$$\lim_{k\to\infty} \|x_k - x, w_2, \cdots, w_n\| = 0 \quad \text{for all} \ \ w_2, \cdots, w_n \in X.$$

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Now we introduce the concept of n-compact in a linear n-normed space.

DEFINITION 1.3. A subset Y of a linear n-normed space X is called an *n-compact subset* if for every sequence $\{y_k\}$ in Y, there exists a subsequence of $\{y_k\}$ which converges to an element $y \in Y$.

LEMMA 1.4. *Let x_i be an element of a linear n-normed space X for each $i \in \{1, \cdots, n\}$ and γ a real number. Then*

$$\|x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n\| = \|x_1, \cdots, x_i, \cdots, x_j + \gamma x_i, \cdots, x_n\|.$$

for all $1 \le i \ne j \le n$.

Proof. It is obviously true. □

LEMMA 1.5. *Let X be a linear n-normed space and Y an n-compact subspace of X. For $v_1, \cdots, v_n \in X$, $\inf_{y \in Y} \|v_1 - y, \cdots, v_n - y\| = 0$, then there exists an element $y_0 \in Y$ such that $\|v_1 - y_0, \cdots, v_n - y_0\| = 0$.*

Proof. For each positive integer k, there exists an element $y_k \in Y$ such that

$$\|v_1 - y_k, \cdots, v_n - y_k\| < \frac{1}{k}.$$

Since $\{y_k\}$ is a sequence in an n-compact space Y, we can consider that $\{y_k\}$ is a convergent sequence in Y without loss of generality. Let $y_k \to y_0$ as $k \to \infty$ for some $y_0 \in Y$. For every $\varepsilon > 0$, there exists a positive integer K with $\frac{1}{K} < \frac{\varepsilon}{n+1}$ such that

$$k > K \text{ implies } \|y_k - y_0, w_2, \cdots, w_n\| < \frac{\varepsilon}{n+1} \text{ for all } w_i \in X \ (i = 2, \cdots, n).$$

By Lemma 1.4, if $k > K$, then we have

$$\begin{aligned}
\|v_1 - y_0, v_2 - y_0, \cdots, v_n - y_0\| &\le \|y_k - y_0, v_2 - y_0, \cdots, v_n - y_0\| \\
&\quad + \|v_1 - y_k, v_2 - y_0, \cdots, v_n - y_0\| \\
&\le \|y_k - y_0, v_2 - y_0, \cdots, v_n - y_0\| \\
&\quad + \|v_1 - y_k, y_k - y_0, v_3 - y_0, \cdots, v_n - y_0\| \\
&\quad + \|v_1 - y_k, v_2 - y_k, v_3 - y_0, \cdots, v_n - y_0\| \\
&\le \|y_k - y_0, v_2 - y_0, \cdots, v_n - y_0\| \\
&\quad + \|v_1 - y_k, y_k - y_0, v_3 - y_0, \cdots, v_n - y_0\| \\
&\quad + \|v_1 - y_k, v_2 - y_k, y_k - y_0, \cdots, v_n - y_0\| \\
&\quad + \|v_1 - y_k, v_2 - y_k, v_3 - y_k, \cdots, v_n - y_0\| \\
&\qquad \vdots
\end{aligned}$$

$$
\begin{aligned}
&\leq \|y_k - y_0, v_2 - y_0, \cdots, v_n - y_0\| \\
&\quad + \|v_1 - y_k, y_k - y_0, v_3 - y_0, \cdots, v_n - y_0\| \\
&\quad + \|v_1 - y_k, v_2 - y_k, y_k - y_0, \cdots, v_n - y_0\| \\
&\quad + \cdots \\
&\quad + \|v_1 - y_k, v_2 - y_k, v_3 - y_k, \cdots, y_k - y_0, v_n - y_0\| \\
&\quad + \|v_1 - y_k, v_2 - y_k, v_3 - y_k, \cdots, v_{n-1} - y_k, v_n - y_0\| \\
&\leq \|y_k - y_0, v_2 - y_0, \cdots, v_n - y_0\| \\
&\quad + \|v_1 - y_k, y_k - y_0, v_3 - y_0, \cdots, v_n - y_0\| \\
&\quad + \|v_1 - y_k, v_2 - y_k, y_k - y_0, \cdots, v_n - y_0\| \\
&\quad + \cdots \\
&\quad + \|v_1 - y_k, v_2 - y_k, v_3 - y_k, \cdots, y_k - y_0, v_n - y_0\| \\
&\quad + \|v_1 - y_k, v_2 - y_k, v_3 - y_k, \cdots, v_{n-1} - y_k, y_k - y_0\| \\
&\quad + \|v_1 - y_k, v_2 - y_k, v_3 - y_k, \cdots, v_{n-1} - y_k, v_n - y_k\| \\
&= \|y_k - y_0, v_2 - y_0, \cdots, v_n - y_0\| \\
&\quad + \|v_1 - y_0, y_k - y_0, v_3 - y_0, \cdots, v_n - y_0\| \\
&\quad + \|v_1 - y_0, v_2 - y_0, y_k - y_0, \cdots, v_n - y_0\| \\
&\quad + \cdots \\
&\quad + \|v_1 - y_0, v_2 - y_0, v_3 - y_0, \cdots, y_k - y_0, v_n - y_0\| \\
&\quad + \|v_1 - y_0, v_2 - y_0, v_3 - y_0, \cdots, v_{n-1} - y_0, y_k - y_0\| \\
&\quad + \|v_1 - y_k, v_2 - y_k, v_3 - y_k, \cdots, v_{n-1} - y_k, v_n - y_k\| \\
&< n\frac{\varepsilon}{n+1} + \frac{1}{k} < n\frac{\varepsilon}{n+1} + \frac{1}{K} < n\frac{\varepsilon}{n+1} + \frac{\varepsilon}{n+1} = \varepsilon.
\end{aligned}
$$

Since ε is arbitrary, $\|v_1 - y_0, \cdots, v_n - y_0\| = 0$ □

THEOREM 1.6. *Let Y and Z be subspaces of a linear n-normed space X and Y an n-compact proper subset of Z with codimension greater than $n-1$. For each $\theta \in (0,1)$, there exists an element $(z_1, \cdots, z_n) \in Z^n$ such that*

$$\|z_1, \cdots, z_n\| = 1, \qquad \|z_1 - y, \cdots, z_n - y\| \geq \theta$$

for all $y \in Y$.

Proof. Let $v_1, \cdots, v_n \in Z \cap Y^{\perp}$ be linearly independent. Let

$$a = \inf_{y \in Y} \|v_1 - y, \cdots, v_n - y\|.$$

Assume that $a = 0$. By Lemma 1.5, there is an element $y_0 \in Y$ such that

$$\|v_1 - y_0, \cdots, v_n - y_0\| = 0. \tag{1}$$

If y_0 is zero, then $v_1, \cdots, v_n$ are linearly dependent, which is a contradiction. So y_0 is nonzero. Hence $v_1, \cdots, v_n, y_0$ are linear independent. On the other hand, it follows from the definition and (1) that $v_1 - y_0, \cdots, v_n - y_0$ are linearly dependent. Thus there exist real numbers $\alpha_1, \cdots, \alpha_n$ not all of which are zero such that

$$\alpha_1(v_1 - y_0) + \cdots + \alpha_n(v_n - y_0) = 0.$$

Thus we have

$$\alpha_1 v_1 + \cdots + \alpha_n v_n + (-1)(\alpha_1 + \cdots + \alpha_n)y_0 = 0.$$

Then $v_1, \cdots, v_n, y_0$ are linearly dependent, which is a contradiction. Hence $a > 0$.

For each $\theta \in (0, 1)$, there exists an element $y_0 \in Y$ such that

$$a \le \|v_1 - y_0, \cdots, v_n - y_0\| \le \frac{a}{\theta}.$$

For each $j = 1, \cdots, n$, let

$$z_j = \frac{v_j - y_0}{\|v_1 - y_0, \cdots, v_n - y_0\|^{\frac{1}{n}}}.$$

Then it is obvious that $\|z_1, \cdots, z_n\| = 1$.

$$\begin{aligned}
&\|z_1 - y, \cdots, z_n - y\| \\
&\quad = \|\frac{v_1 - y_0}{\|v_1 - y_0, \cdots, v_n - y_0\|^{\frac{1}{n}}} - y, \cdots, \frac{v_n - y_0}{\|v_1 - y_0, \cdots, v_n - y_0\|^{\frac{1}{n}}} - y\| \\
&\quad = \frac{1}{\|v_1 - y_0, \cdots, v_n - y_0\|}\|v_1 - (y_0 + y\|v_1 - y_0, \cdots, v_n - y_0\|^{\frac{1}{n}}), \cdots, \\
&\qquad\qquad v_n - (y_0 + y\|v_1 - y_0, \cdots, v_n - y_0\|^{\frac{1}{n}})\| \\
&\quad \ge \frac{1}{\|v_1 - y_0, \cdots, v_n - y_0\|}\, a \ge \frac{a}{\frac{a}{\theta}} = \theta
\end{aligned}$$

for all $y \in Y$.

This completes the proof. □

Now we introduce the concept of partially n-closed in linear n-normed spaces.

Definition 1.7. A subset Y of a linear n-normed space X is called a *partially n-closed subset* if for linear independent elements $x_1, \cdots, x_n$ in X there exists a sequence $\{y_k\}$ in Y such that $\|x_1 - y_k, \cdots, x_n - y_k\| \to 0$ as $k \to \infty$ then $x_j \in Y$ for some j.

Notice that a subset Y of a linear 1-normed space X is partially 1-closed if and only if the subset Y is closed.

Theorem 1.8. *Let Y and Z be subspaces of a linear n-normed space X and Y a partially n-closed proper subset of Z. Assume that* $\dim Z \geq n$. *For each $\theta \in (0,1)$, there exists an element $(z_1, \cdots, z_n) \in Z^n$ such that*

$$\|z_1, \cdots, z_n\| = 1, \qquad \|z_1 - y, \cdots, z_n - y\| \geq \theta$$

for all $y \in Y$.

Proof. Let $v_1, \cdots, v_n \in Z - Y$ be linearly independent. Let

$$a = \inf_{y \in Y} \|v_1 - y, \cdots, v_n - y\|.$$

Assume that $a = 0$. Then there is a sequence $\{y_k\}$ in Y such that $\|v_1 - y_k, \cdots, v_n - y_k\| \to 0$ as $k \to \infty$. Since Y is partially n-closed, $v_j \in Y$ for some j, which is a contradiction. Hence $a > 0$.

The rest of the proof is the same as in the proof of Theorem 1.6. □

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