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# GENERALIZED QUASI–BANACH SPACES AND QUASI–$(2;p)$–NORMED SPACES

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Abstract. In this paper, the notion of a generalized quasi-normed space is introduced and its completion is investigated.

We introduce quasi-2-normed spaces and quasi-$(2;p)$-normed spaces, and investigate the properties of quasi-2-normed spaces and quasi-$(2;p)$-normed spaces.

## 1. Introduction and preliminaries

It is well-known that the rational line $\mathbb{Q}$ is not complete but can be *enlarged* to the real line $\mathbb{R}$ which is complete. And this *completion* $\mathbb{R}$ of $\mathbb{Q}$ is such that $\mathbb{Q}$ is dense in $\mathbb{R}$. It is quite important that an arbitrary incomplete normed space can be *completed* in a similar fashion.

Banach spaces play an important role in many branches of mathematics and its applications ([4], [5], [6], [8], [9]).

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1. ([1, 7]) Let $X$ be a linear space. A *quasi-norm* is a real-valued function on $X$ satisfying the following:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.
(3) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

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The pair $(X, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on $X$.

A *quasi-Banach space* is a complete quasi-normed space.

A quasi-norm $\|\cdot\|$ is called a *p-norm* $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a *p-Banach space*.

Given a $p$-norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on $X$. By the Aoki–Rolewicz theorem [7] (see also [1]), each quasi-norm is equivalent to some $p$-norm. Since it is much easier to work with $p$-norms than quasi-norms, henceforth we restrict our attention mainly to $p$-norms.

In [2], Cho *et al.* defined linear 2-normed spaces and investigated the properties of linear 2-normed spaces.

Definition 2 [2]. Let $X$ be a real linear space with $\dim X \geq 2$ and $\|\cdot, \cdot\| : X^2 \to [0, \infty)$ a function. Then $(X, \|\cdot, \cdot\|)$ is called a *linear 2-normed space* if

$(2N_1)$ $\|x, y\| = 0 \iff x$ and $y$ are linearly dependent,

$(2N_2)$ $\|x, y\| = \|y, x\|$,

$(2N_3)$ $\|\alpha x, y\| = |\alpha| \|x, y\|$,

$(2N_4)$ $\|x + y, z\| \leq \|x, z\| + \|y, z\|$

for all $\alpha \in \mathbb{R}$ and all $x, y, z \in X$. The function $\|\cdot, \cdot\|$ is called a *2-norm on* $X$.

In [3], Chu *et al.* defined the notion of 2-isometry and proved the Rassias and Šemrl's theorem in linear 2-normed spaces.

In this paper, we introduce the notion of generalized quasi-normed spaces, quasi-2-normed spaces and quasi-$(2; p)$-normed spaces, and investigate the properties of generalized quasi-normed spaces, quasi-2-normed spaces and quasi-$(2; p)$-normed spaces.

## 2. Completion of generalized quasi-normed spaces

In this section, we generalize the concept of quasi-normed spaces and investigate the completion of the generalized quasi-normed space.

Definition 3. Let $X$ be a linear space. A *generalized quasi-norm* is a real-valued function on $X$ satisfying the following:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.
(3) There is a constant $K \geq 1$ such that $\|\sum_{j=1}^{\infty} x_j\| \leq \sum_{j=1}^{\infty} K\|x_j\|$ for all $x_1, x_2, \cdots \in X$ with $\sum_{j=1}^{\infty} x_j \in X$.

The pair $(X, \|\cdot\|)$ is called a *generalized quasi-normed space* if $\|\cdot\|$ is a generalized quasi-norm on $X$.

A *generalized quasi-Banach space* is a complete generalized quasi-normed space.

A generalized quasi-norm $\|\cdot\|$ is called a *generalized $p$-norm* $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a generalized quasi-Banach space is called a *generalized $p$-Banach space.*

Definition 4. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be generalized quasi-normed spaces.

(1) A mapping $L : X \to Y$ is said to be *isometric* or an *isometry* if for all $x, y \in X$

$$\|Lx - Ly\|_Y = \|x - y\|_X.$$

(2) The space $X$ is said to be *isometric* with the space $Y$ if there exists a bijective isometry of $X$ onto $Y$. The spaces $X$ and $Y$ are called *isometric spaces.*

Theorem 1. *Let $X = (X, \|\cdot\|_X)$ be a generalized quasi-normed space. Assume that the generalized quasi-norm $\|\cdot\|$ is a $p$-norm. Then there exist* *a generalized quasi-Banach space $\widehat{X}$ and an isometry $L$ from $X$ onto a subspace $Y$ of $\widehat{X}$ which is dense in $\widehat{X}$. The space $\widehat{X}$ is unique up to isometry.*

*Proof.* We divide the proof into four steps.

**Step I**. We construct a generalized quasi-Banach space $(\widehat{X}, \|\cdot\|_{\widehat{X}})$.

Let $\{x_n\}$ and $\{x'_n\}$ be Cauchy sequences in $X$. Define $\{x_n\}$ to be equivalent to $\{x'_n\}$, written $\{x_n\} \sim \{x'_n\}$, if

$$lim_{n\to\infty}\|x_n - x'_n\|_X = 0. \tag{2.1}$$

Let $\widehat{X}$ be the set of all equivalence classes of Cauchy sequences. We write $\{x_n\} \in \widehat{x}$ to mean $\{x_n\}$ is a member of $\widehat{x}$ and a *representative* of the class $\widehat{x}$.

We must make $\widehat{X}$ into a vector space. To define on $\widehat{X}$ the two algebraic operations of a vector space, we consider any $\widehat{x}, \widehat{y} \in \widehat{X}$ and representatives $\{x_n\} \in \widehat{x}$ and $\{y_n\} \in \widehat{y}$. We set $z_n = x_n + y_n$. Then $\{z_n\}$ is Cauchy in $X$ since

$$\|z_n - z_m\|_X = \|x_n + y_n - (x_m + y_m)\|_X \le K\|x_n - x_m\|_X + K\|y_n - y_m\|_X.$$

We define the sum $\widehat{z} = \widehat{x} + \widehat{y}$ of $\widehat{x}$ and $\widehat{y}$ to be the equivalence class for which $\{z_n\}$ is a representative, i.e., $\{z_n\} \in \widehat{z}$. This definition is independent of the particular choice of Cauchy sequences belonging to $\widehat{x}$ and $\widehat{y}$. In fact, the equality (2.1) shows that if $\{x_n\} \sim \{x'_n\}$ and $\{y_n\} \sim \{y'_n\}$, then $\{x_n + y_n\} \sim \{x'_n + y'_n\}$ because

$$\|x_n + y_n - (x'_n + y'_n)\|_X \le K\|x_n - x'_n\|_X + K\|y_n - y'_n\|_X.$$

Similarly, we define the product $\alpha\widehat{x} \in \widehat{X}$ of a scalar $\alpha$ and $\widehat{x}$ to be the equivalence class for which $\{\alpha x_n\}$ is a representative. Again, this definition is independent of the particular choice of a representative of $\widehat{x}$. The zero element of $\widehat{X}$ is the equivalence class containing all Cauchy sequences which converge to zero. It is not difficult to see that those two algebraic operations have all the properties required by the definition, so that $\widehat{X}$ is a vector space.

We now set

$$\|\widehat{x}-\widehat{y}\|_{\widehat{X}} = lim_{n\to\infty}\|x_n - y_n\|_X, \tag{2.2}$$

where $\{x_n\} \in \widehat{x}$ and $\{y_n\} \in \widehat{y}$. We show that this limit exists. We have

$$\|x_n - y_n\|_X^p \leq \|x_n - x_m\|_X^p + \|x_m - y_m\|_X^p + \|y_m - y_n\|_X^p.$$

So

$$\|x_n - y_n\|_X^p - \|x_m - y_m\|_X^p \leq \|x_n - x_m\|_X^p + \|y_m - y_n\|_X^p,$$

and a similar inequality with $m$ and $n$ interchanged, i.e.,

$$\|x_m - y_m\|_X^p - \|x_n - y_n\|_X^p \leq \|x_n - x_m\|_X^p + \|y_m - y_n\|_X^p.$$

Hence

$$\big|\|x_n - y_n\|_X^p - \|x_m - y_m\|_X^p\big| \leq \|x_n - x_m\|_X^p + \|y_m - y_n\|_X^p. \tag{2.3}$$

Since $\{x_n\}$ and $\{y_n\}$ are Cauchy, we can make the right side as small as we please. This implies that the limit in (2.2) exists since $\mathbb{R}$ is complete.

We must show that the limit in (2.2) is independent of the particular choice of representatives. If $\{x_n\} \sim \{x'_n\}$ and $\{y_n\} \sim \{y'_n\}$, then by (2.1)

$$\big|\|x_n - y_n\|_X^p - \|x'_n - y'_n\|_X^p\big| \leq \|x_n - x'_n\|_X^p + \|y_n - y'_n\|_X^p,$$

which tends to zero as $n \to \infty$. This implies the assertion

$$\lim_{n\to\infty}\|x_n - y_n\|_X = \lim_{n\to\infty}\|x'_n - y'_n\|_X.$$

Now we prove that $\|\cdot\|_{\widehat{X}}$ in (2.2) is a generalized quasi-norm on $\widehat{X}$.

Obviously, $\|\cdot\|_{\widehat{X}}$ satisfies Definition 3 (1) and (2). Furthermore, since $\|\cdot\|_X$ is a generalized quasi-norm on $X$, there is a constant $K \geq 1$ such that

$$\|\sum_{j=1}^{\infty} x_j\|_X \leq \sum_{j=1}^{\infty} K\|x_j\|_X$$

for all $x_1, x_2, \cdots \in X$. Thus

$$\|\sum_{j=1}^{\infty} \widehat{x_j}\|_{\widehat{X}} \leq \sum_{j=1}^{\infty} K\|\widehat{x_j}\|_{\widehat{X}}$$

for all $\widehat{x_1}, \widehat{x_2}, \cdots \in \widehat{X}$. So $\|\cdot\|_{\widehat{X}}$ is a generalized quasi-norm on $\widehat{X}$.

Note that the inequality (2.2) implies that the generalized quasi-norm $\|\cdot\|_{\widehat{X}}$ is equivalent to a generalized $p$-norm.

**Step II**. We construct an isometry $L : X \to Y \subset \widehat{X}$.

With each $b \in X$ we associate the class $\widehat{b} \in \widehat{X}$ which contains the constant Cauchy sequence $(b, b \cdots)$. This defines a mapping $L : X \to Y$ onto the subspace $Y = L(X) \subset \widehat{X}$. The mapping $L$ is given by $b \mapsto \widehat{b} = Lb$, where $(b, b, \cdots) \in \widehat{b}$. We see that $L$ is an isometry since (2.2) becomes simply

$$\|\widehat{b} - \widehat{c}\|_{\widehat{X}} = \|b - c\|_X.$$

Here $\widehat{c}$ is the class of $\{y_n\}$ where $y_n = c$ for all $n \in \mathbb{N}$. Any isometry is injective, and $L : X \to Y$ is surjective since $L(X) = Y$. Hence $Y$ and $X$ are isometric.

From the definition it follows that on $Y$ the operations of vector induced from $\widehat{X}$ agree with those induced from $X$ by means of $L$.

We show that $Y$ is dense in $\widehat{X}$. We consider any $\widehat{x} \in \widehat{X}$. Let $\{x_n\} \in \widehat{x}$. For every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$\|x_n - x_N\|_X < \frac{\epsilon}{2}$$

for all $n > N$. Let $(x_N, x_N, \cdots) \in \widehat{x_N}$. Then $\widehat{x_N} \in Y$. By (2.2),

$$\|\widehat{x} - \widehat{x_N}\|_{\widehat{X}} = \lim_{n\to\infty} \|x_n - x_N\|_X \leq \frac{\epsilon}{2} < \epsilon.$$

This shows that every $\epsilon$-neighborhood of the arbitrary $\widehat{x} \in \widehat{X}$ contains an element of $Y$. Hence $Y$ is dense in $\widehat{X}$.

**Step III**. We prove the completeness of $\widehat{X}$.

Let $\{\widehat{x_n}\}$ be any Cauchy sequence in $\widehat{X}$. Since $Y$ is dense in $\widehat{X}$, for every $\widehat{x_n}$ there is a $\widehat{z_n} \in Y$ such that

$$\|\widehat{x_n} - \widehat{z_n}\|_{\widehat{X}} < \frac{1}{n}. \tag{2.4}$$

Hence by Definition 3 (3),

$$\begin{aligned}\|\widehat{z_m}-\widehat{z_n}\|_{\widehat{X}} &\le K\|\widehat{z_m}-\widehat{x_m}\|_{\widehat{X}}+K\|\widehat{x_m}-\widehat{x_n}\|_{\widehat{X}}+K\|\widehat{x_m}-\widehat{z_n}\|_{\widehat{X}}\\ &< \frac{K}{m}+K\|\widehat{x_m}-\widehat{x_n}\|_{\widehat{X}}+\frac{K}{n}\end{aligned}$$

and this is less than any given $\epsilon > 0$ for sufficiently large $m$ and $n$ because $\{\widehat{x_n}\}$ is a Cauchy sequence. Hence $\{\widehat{z_n}\}$ is a Cauchy sequence. Since $L : X \to Y$ is isometric and $\widehat{z_n} \in Y$, the sequence $\{z_n\}$, where $z_n = L^{-1}(\widehat{z_n})$, is a Cauchy sequence in $X$. Let $\widehat{x} \in \widehat{X}$ be the class to which $\{z_n\}$ belongs. We show that $\widehat{x}$ is the limit of $\{\widehat{x_n}\}$. By Definition 3 (3) and (2.4),

$$\begin{aligned}\|\widehat{x_n}-\widehat{x}\|_{\widehat{X}} &\le K\|\widehat{x_n}-\widehat{z_n}\|_{\widehat{X}}+K\|\widehat{z_n}-\widehat{x}\|_{\widehat{X}}\\ &< \frac{K}{n}+K\|\widehat{z_n}-\widehat{x}\|_{\widehat{X}}.\end{aligned} \tag{2.5}$$

Since $\{z_n\} \in \widehat{x}$ and $\widehat{z_m} \in Y$, so that $(z_m, z_m, z_m, \cdots) \in \widehat{z_m}$, the inequality (2.5) becomes

$$\|\widehat{x_n}-\widehat{x}\|_{\widehat{X}} < \frac{K}{n}+K\lim_{m\to\infty}\|z_n-z_m\|_X$$

and the right side is smaller than any given $\epsilon > 0$ for sufficiently large $n$. Hence the arbitrary Cauchy sequence $\{\widehat{x_n}\}$ in $\widehat{X}$ has the limit $\widehat{x} \in \widehat{X}$, and $\widehat{X}$ is complete.

**Step IV**. We show the uniqueness of $\widehat{X}$ up to isometry.

If $(\widetilde{X}, \|\cdot\|_{\widetilde{X}})$ is another complete metric space with a subspace $Z$ dense in $\widetilde{X}$ and isometric with $X$, then for any $\widetilde{x}, \widetilde{y} \in \widetilde{X}$ we have sequences $\{\widetilde{x_n}\}, \{\widetilde{y_n}\}$ in $Z$ such that $\widetilde{x_n} \to \widetilde{x}$ and $\widetilde{y_n} \to \widetilde{y}$. So

$$\|\widetilde{x}-\widetilde{y}\|_{\widetilde{X}} = \lim_{n\to\infty}\|\widetilde{x_n}-\widetilde{y_n}\|_{\widetilde{X}}$$

follows from

$$\big|\|\widetilde{x}-\widetilde{y}\|^p_{\widetilde{X}}-\|\widetilde{x_n}-\widetilde{y_n}\|^p_{\widetilde{X}}\big| \le \|\widetilde{x}-\widetilde{x_n}\|^p_{\widetilde{X}}+\|\widetilde{y}-\widetilde{y_n}\|^p_{\widetilde{X}} \to 0.$$

Here the inequality is similar to (2.3). Since $Z$ is isometric with $Y \subset \widehat{X}$ and $\overline{Y} = \widehat{X}$, the norms on $\widetilde{X}$ and $\widehat{X}$ must be the same. Hence $\widetilde{X}$ and $\widehat{X}$ are isometric. □

Corollary 2. *Let* $X = (X, \|\cdot\|_X)$ *be a quasi-normed space. Then there exist a quasi-Banach space* $\widehat{X}$ *and an isometry* $L$ *from* $X$ *onto a subspace* $Y$ *of* $\widehat{X}$ *which is dense in* $\widehat{X}$. *The space* $\widehat{X}$ *is unique up to isometry.*

## 3. Quasi-$(2;p)$-normed spaces

In this section, we introduce the concept of quasi-2-normed spaces and quasi-$(2;p)$-normed spaces. and investigate the properties of quasi-2-normed spaces and quasi-$(2;p)$-normed spaces.

Definition 5. Let $X$ be a linear space. A *quasi-2-norm* is a real-valued function on $X \times X$ satisfying Definition 2 $(2N_1)$, $(2N_2)$, $(2N_3)$ and the following: There is a constant $K \geq 1$ such that $\|x+y, z\| \leq K\|x, z\| + K\|y, z\|$ for all $x, y, z \in X$. The pair $(X, \|\cdot,\cdot\|)$ is called a *quasi-2-normed space* if $\|\cdot,\cdot\|$ is a quasi-2-norm on $X$. The smallest possible $K$ is called the *modulus of concavity* of $\|\cdot,\cdot\|$.

A quasi-2-norm $\|\cdot,\cdot\|$ is called a *quasi-p-norm* $(0 < p \leq 1)$ if

$$\|x + y, z\|^p \leq \|x, z\|^p + \|y, z\|^p$$

for all $x, y, z \in X$.

Theorem 3. *Let* $(X, \|\cdot,\cdot\|)$ *be a quasi-2-normed space. There is a* $p$ $(0 < p \leq 1)$ *and an equivalent quasi-2-norm* $|||\cdot,\cdot|||$ *on* $X$ *satisfying*

$$|||x + y, z|||^p \leq |||x, z|||^p + |||y, z|||^p$$

*for all* $x, y, z \in X$.

*Proof.* Let $K$ be the modulus of concavity of $\|\cdot,\cdot\|$ and $r$ a real number such that $2^{\frac{1}{r}} = 2K$. Let $z \in X$ be fixed and define a new quasi-2-norm by

$$|||x, z||| = \inf\{(\sum \|x_j, z\|^r)^{\frac{1}{r}} \mid x = \sum x_j\}.$$

It is obvious that $|||\alpha x, z||| = |\alpha| \cdot |||x, z|||$, that $|||x, z||| \leq \|x, z\|$, and that $|||\cdot,\cdot|||$ satisfies the required inequality. We are going to show that

$$\|\sum_{j=1}^{n} x_j, z\| \leq 4^{\frac{1}{r}}(\sum_{j=1}^{n} \|x_j, z\|^r)^{\frac{1}{r}} \tag{3.1}$$

for all $\{x_j\}_{j=1}^n$. This implies that $\|x,z\| \leq 4^{\frac{1}{r}}|||x,z|||$ and concludes the proof. For every $x \in X$ put $N_z(x) = 2^{\frac{m}{r}}$, where the integer $m$ is chosen so that $2^{\frac{m-1}{r}} < \|x,z\| \leq 2^{\frac{m}{r}}$. With this notation, the inequality (3.1) follows once we show that

$$\|\sum_{j=1}^n x_j, z\| \leq 2^{\frac{1}{r}}(\sum_{j=1}^n N_z(x_j)^r)^{\frac{1}{r}}. \tag{3.2}$$

If there are two $x_j$'s, say $x_1$ and $x_2$, such that $N_z(x_1) = N_z(x_2) = 2^{\frac{m}{r}}$, then we can replace the pair $x_1, x_2$ in the proof of (3.2) by their sum. Indeed,

$$\|x_1 + x_2, z\| \leq K\|x_1, z\| + K\|x_2, z\| \leq 2K \cdot 2^{\frac{m}{r}} = 2^{\frac{m+1}{r}}$$

by the choice of $r$. Thus

$$N_z(x_1 + x_2)^r \leq 2^{m+1} = N_z(x_1)^r + N_z(x_2)^r.$$

Using this remark and rearranging the $x_j$'s, we may assume that the sequence $\{N_z(x_j)\}_{j=1}^n$ is strictly decreasing. So

$$\|x_j, z\| \leq N_z(x_j) \leq 2^{-\frac{j-1}{r}} N_z(x_1)$$

for all $j$ $(1 \leq j \leq n)$.

Repeated use of the definition of the concavity modulus and the definition of $r$ give

$$\begin{aligned}\|\sum_{j=1}^n x_j, z\| &\leq K\|x_1, z\| + K^2\|x_2, z\| + \cdots + K^n\|x_n, z\| \\ &\leq \sum_{j=1}^n N_z(x_1)K^j 2^{-\frac{j-1}{r}} = 2^{\frac{1}{r}} N_z(x_1) \sum_{j=1}^n 2^{-j} \\ &\leq 2^{\frac{1}{r}} N_z(x_1) \leq 2^{\frac{1}{r}} (\sum_{j=1}^n N_z(x_j)^r)^{\frac{1}{r}},\end{aligned}$$

as desired. □

DEFINITION 6. Let $X$ be a quasi-2-normed space.

(1) A sequence $\{x_n\}$ in $X$ is called a *Cauchy sequence* if $\lim_{m,n\to\infty} \|x_m - x_n, z\| = 0$ for all $z \in X$.

(2) A sequence $\{x_n\}$ in $X$ is called a *convergent sequence* if there is an $x \in X$ such that $\lim_{n\to\infty} \|x_n - x, z\| = 0$ for all $z \in X$.

(3) A quasi-2-normed space in which every Cauchy sequence converges is called *complete*.

*Problem.* Construct a completion of a quasi-2-normed space.

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