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ON THE GENERALIZED HYERS–ULAM STABILITY OF A CUBIC FUNCTIONAL EQUATION

Kil-Woung Jun* and Sang-Baek Lee**

Abstract. The generalized Hyers–Ulam stability problems of the cubic functional equation

$$\begin{aligned} &f(x+y+z)+f(x+y-z)+2f(x-y)+4f(y) \\ &\quad = f(x-y+z)+f(x-y-z) \\ &\qquad +2f(x+y)+2f(y+z)+2f(y-z) \end{aligned}$$

shall be treated under the approximately odd condition and the behavior of the cubic mappings and the additive mappings shall be investigated. The generalized Hyers–Ulam stability problem for functional equations had been posed by Th.M. Rassias and J. Tabor [7] in 1992.

1. Introduction

In 1940, Ulam proposed the general Ulam stability problem (see [8]):
"*When is it true that by slightly changing the hypothesis of a theorem one can still assert that the thesis of the theorem remains true or approximately true?*"

In 1941, this problem was solved by Hyers [3] in the case of Banach spaces. Thereafter, this type of stability is called the Hyers–Ulam stability. In 1978 Th. M. Rassias [5] provided a remarkable generalization of the Hyers–Ulam stability of mappings by considering variables.

This fact rekindled interest in the field. Since then a number of papers have appeared in the subject([6], [1]). Such type of stability is now called the Hyers–Ulam–Rassias stability of functional equations. For the function case, the reader is referred to Găvruta [2]. Throughout this paper, let X be a real normed space and Y be a real Banach space

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in the case of functional inequalities, as well as let X and Y be real linear spaces for the case of functional equations.

We here introduce a theorem of Găvruta [2]:

THEOREM. *Let G be an abelian group and E a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a function such that*

$$\Phi(x, y) := \sum_{i=1}^{\infty} 2^{-i} \varphi(2^{i-1}x, 2^{i-1}) < \infty$$

for all $x, y \in G$. If a function $f : G \to E$ satisfies the inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y)$$

for any $x, y \in G$, then there exists a unique additive function $A : G \to E$ such that

$$\|f(x) - A(x)\| \leq \Phi(x, x)$$

for each $x \in G$.

For a mapping $f : X \to Y$, consider the following functional equations:

$$\begin{aligned} &f(x + y + z) + f(x + y - z) + 2f(x) + 2f(y) \\ &\quad = 2f(x + y) + f(x + z) + f(x - z) + f(y + z) + f(y - z) \end{aligned} \tag{1}$$

and

$$\begin{aligned} &f(x + y + z) + f(x + y - z) + 2f(x - y) + 4f(y) \\ &\quad = f(x - y + z) + f(x - y - z) + 2f(x + y) \\ &\quad\quad +2f(y + z) + 2f(y - z) \end{aligned} \tag{2}$$

for all $x, y, z \in X$. Recently, H. Kim [4] investigated the solution and the stability of the functional equation (1). The general Hyers–Ulam stability problem had been posed for the first time by Th.M. Rassias and J. Tabor [7] in the year 1992.

2. Solution of the functional equation (2)

In this section, we investigate the solution of the functional equation (2).

THEOREM 1. *A mapping $f : X \to Y$ satisfies the cubic functional equation (2) for all $x, y, z \in X$ if and only if there exist two mappings $B : X^3 \to Y$ and $A : X \to Y$ such that $f(x) = B(x, x, x) + A(x) + f(0)$ for all $x \in X$, where B is symmetric for each fixed one variable and is additive for each fixed two variables and A is additive.*

Proof. First we assume that f is a solution of (1). Setting $g(x) = f(x) - f(0)$, we get g is also a solution of (1) and $g(0) = 0$. Next we assume that f is a solution of (2). Setting $h(x) = f(x) - f(0)$, we get h is also a solution of (2) and $h(0) = 0$. Thus we may assume without loss of generality that $f(0) = 0$.

First we show that (1) implies (2). Replacing y by $-y$ in (1), we obtain that f satisfies

$$\begin{aligned} &f(x-y+z) + f(x-y-z) + 2f(x) + 2f(-y) \\ &\quad = 2f(x-y) + f(x+z) + f(x-z) + f(-y+z) + f(-y-z) \end{aligned} \tag{3}$$

for all $x, y, z \in X$. Putting $x = y = 0$ in (1), one can obtain that f is odd. By (1) and (3), we obtain that f satisfies (2) for all $x, y, z \in X$.

Next we show that (2) implies (1). Interchanging x and y in (2), we get that

$$\begin{aligned} &f(y+x+z) + f(y+x-z) + 2f(y-x) + 4f(x) \\ &\quad = f(y-x+z) + f(y-x-z) + 2f(y+x) \\ &\qquad +2f(x+z) + 2f(x-z) \end{aligned} \tag{4}$$

for all $x, y, z \in X$. Putting $x = y = 0$ in (2), one can obtain that f is odd. By (2) and (4), we obtain that

$$\begin{aligned} &f(x-y+z) + f(x-y-z) + 2f(x) - 2f(y) \\ &\quad = 2f(x-y) + f(x+z) + f(x-z) - f(y-z) - f(y+z) \end{aligned} \tag{5}$$

for all $x, y, z \in X$. Replacing y by $-y$ in (5), we obtain that f satisfies (1) for all $x, y, z \in X$.

By Theorem 2.4 in [4], the proof is completed. $\square$

3. Stability of the cubic equation (2)

In this section, we investigate the generalized Hyers–Ulam stability of the functional equation (2) in the spirit of Găvruta.

Let $\varphi : X^3 \to [0, \infty)$ and $\psi : X \to [0, \infty)$ be two functions such that

$$\begin{aligned} \Phi(x,y,z) &:= \frac{1}{6}\sum_{i=1}^{\infty}\left(\frac{1}{2^i} - \frac{1}{8^i}\right)\big(2\varphi(2^{i-1}x, 2^{i-1}y, -2^{i-1}z) \\ &\quad +2\varphi(-2^{i-1}x, -2^{i-1}y, 2^{i-1}z) + \varphi(2^{i-1}x, 2^{i}y, , 2^{i-1}z)\big) < \infty, \end{aligned} \tag{6}$$

$$\Psi(x) := \frac{2}{7}\psi(0)$$

$$+ \ \frac{1}{6}\sum_{i=1}^{\infty}\left(\frac{1}{2^i}-\frac{1}{8^i}\right)\bigl(12\psi(2^{i-1}x)+9\psi(2^ix)+2\psi(3\cdot 2^{i-1}x)\bigr)<\infty$$

for all $x, y, z \in X$.

For simplicity of calculation in this section, we use the notation $\varphi_1(x) := \varphi(x, x, -x)$, $\varphi_2(x) := \varphi(x, 2x, x)$, $\psi_1(x) := \psi(3x) + 4\psi(2x) + 5\psi(x)$, $\psi_2(x) := \psi(2x)+2\psi(x)+\frac{1}{2}\psi(0)$, and $\chi(x) := \frac{1}{3}\varphi_1(x)+\frac{1}{3}\varphi_1(-x)+\frac{1}{6}\varphi_2(x) + \frac{1}{3}\psi_1(x) + \frac{1}{6}\psi_2(x)$ for all $x \in X$.

LEMMA 2. *Let $f : X \to Y$ be a mapping satisfying*

$$\begin{aligned}&\|f(x+y+z)+f(x+y-z)+2f(x-y)+4f(y)\\&\quad -f(x-y+z)-f(x-y-z)-2f(x+y)\\&\quad -2f(y+z)-2f(y-z)\|\le\varphi(x,y,z)\end{aligned}\tag{7}$$

and

$$\|f(x)+f(-x)\|\le\psi(x)\tag{8}$$

for all $x, y, z \in X$. Then

$$\begin{aligned}&\left\|f(x)-\frac{1}{6}\left(\frac{8}{2^n}-\frac{2}{8^n}\right)f(2^nx)+\frac{1}{6}\left(\frac{1}{2^n}-\frac{1}{8^n}\right)f(2^{n+1}x)\right\|\\&\qquad\le\sum_{i=1}^{n}\left(\frac{1}{2^i}-\frac{1}{8^i}\right)\chi(2^{i-1}x)\end{aligned}\tag{9}$$

for all $x \in X$ and $n \in \mathbb{N}$.

Proof. Putting $y = x$ and $z = -x$ in (7) it follows that

$$\|4f(x)-f(-x)-4f(2x)+f(3x)\|\le\varphi_1(x)\tag{10}$$

for all $x \in X$. Substitute $-x$ for x in (10), then

$$\|-4f(-x)+f(x)+4f(-2x)-f(-3x)\|\le\varphi_1(-x)\tag{11}$$

for all $x \in X$. From (10) and (11), one can obtain that

$$\begin{aligned}&\|5f(x)-5f(-x)-4f(2x)+4f(-2x)+f(3x)-f(-3x)\|\\&\quad\le\varphi_1(x)+\varphi_1(-x)\end{aligned}$$

for all $x \in X$. By (8),

$$\begin{aligned}&2\|5f(x)-4f(2x)+f(3x)\|\\&\quad\le\|5f(x)-5f(-x)-4f(2x)+4f(-2x)+f(3x)-f(-3x)\|\\&\quad+\|5f(x)+5f(-x)\|+\|-4f(2x)-4f(-2x)\|+\|f(3x)+f(-3x)\|\\&\quad\le\varphi_1(x)+\varphi_1(-x)+5\psi(x)+4\psi(2x)+\psi(3x)\\&\quad=\varphi_1(x)+\varphi_1(-x)+\psi_1(x)\end{aligned}\tag{12}$$

for all $x \in X$. Putting $y = 2x$ and $z = x$ in (7) it follows that

$$\begin{aligned}
&\|f(4x) + f(2x) + 2f(-x) + 4f(2x) - f(0) \\
&\quad -f(-2x) - 2f(3x) - 2f(3x) - 2f(x)\| \\
(13) &= \|f(4x) - 4f(3x) + 5f(2x) - f(-2x) - 2f(x) + 2f(-x) - f(0)\| \\
&\le \varphi_2(x)
\end{aligned}$$

for all $x \in X$. By (8), we have

$$\|f(0)\| \le \frac{1}{2}\psi(0). \tag{14}$$

By (8), (13) and (14), we have

$$\|f(4x) - 4f(3x) + 6f(2x) - 4f(x)\| \le \varphi_2(x) + \psi_2(x) \tag{15}$$

By (12) and (15), we have

$$\begin{aligned}
&\|16f(x) - 10f(2x) + f(4x)\| \\
&\le 2\varphi_1(x) + 2\varphi_1(-x) + \varphi_2(x) + 2\psi_1(x) + \psi_2(x) = 6\chi(x)
\end{aligned} \tag{16}$$

for all $x \in X$. By (16), we obtain that

$$\begin{aligned}
&\|f(x) - \frac{1}{6}\left(\frac{8}{2^{n+1}} - \frac{2}{8^{n+1}}\right) f(2^{n+1}x) + \frac{1}{6}\left(\frac{1}{2^{n+1}} - \frac{1}{8^{n+1}}\right) f(2^{n+2}x)\| \\
&\le \|f(x) - \frac{1}{6}\left(\frac{8}{2^{n}} - \frac{2}{8^{n}}\right) f(2^{n}x) + \frac{1}{6}\left(\frac{1}{2^{n}} - \frac{1}{8^{n}}\right) f(2^{n+1}x)\| \\
&+\frac{1}{6}\left(\frac{1}{2^{n+1}} - \frac{1}{8^{n+1}}\right) \|16f(2^n x) - 10f(2^{n+1}x) + f(2^{n+2}x)\| \\
&\le \sum_{i=1}^{n} \left(\frac{1}{2^i} - \frac{1}{8^i}\right) \chi(2^{i-1}x) + \left(\frac{1}{2^{n+1}} - \frac{1}{8^{n+1}}\right) \chi(x) \\
&= \sum_{i=1}^{n+1} \left(\frac{1}{2^i} - \frac{1}{8^i}\right) \chi(2^{i-1}x)
\end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. □

THEOREM 3. *Let $f : X \to Y$ be a mapping satisfying* (7) *and* (8) *for all $x, y, z \in X$. Then there exist two mappings $B : X^3 \to Y$ and $A : X \to Y$ which satisfy the inequality*

$$\|f(x) - B(x, x, x) - A(x)\| \le \Phi(x, x, x) + \Psi(x) \tag{17}$$

for all $x \in X$,where B is symmetric for each fixed one variable and is additive for each fixed two variable and A is additive.

Proof. We define

$$g_n(x) := \frac{1}{6}\left(\frac{8}{2^n} - \frac{2}{8^n}\right) f(2^n x) - \frac{1}{6}\left(\frac{1}{2^n} - \frac{1}{8^n}\right) f(2^{n+1}x)$$

for all $x \in X$ and all $n \in \mathbb{N}$. By (15), we obtain that

$$\begin{aligned}
&\|g_{n+1}(x) - g_n(x)\| \\
&\quad = \left\| \frac{1}{6}\left(\frac{8}{2^{n+1}} - \frac{2}{8^{n+1}}\right) f(2^{n+1}x) - \frac{1}{6}\left(\frac{1}{2^{n+1}} - \frac{1}{8^{n+1}}\right) f(2^{n+2}x) \right. \\
&\qquad \left. -\frac{1}{6}\left(\frac{8}{2^n} - \frac{2}{8^n}\right) f(2^n x) + \frac{1}{6}\left(\frac{1}{2^n} - \frac{1}{8^n}\right) f(2^{n+1}x) \right\| \\
&\quad = \frac{1}{6\cdot 8^{n+1}} \big\|(8\cdot 4^{n+1} - 2) f(2^{n+1}x) - (4^{n+1} - 1) f(2^{n+2}x) \\
&\qquad -(64\cdot 4^n - 16) f(2^n x) + (8\cdot 4^n - 8) f(2^{n+1}x)\big\| \\
&\quad \le \frac{4^{n+1}}{6\cdot 8^{n+1}} \big\|8f(2^{n+1}x) - f(2^{n+2}x) - 16f(2^n x) + 2f(2^{n+1}x)\big\| \\
&\qquad + \frac{1}{6\cdot 8^{n+1}} \big\|2f(2^{n+1}x) - f(2^{n+2}x) - 16f(2^n x) + 8f(2^{n+1}x)\big\| \\
&\quad = \frac{4^{n+1}+1}{6\cdot 8^{n+1}} \big\|16f(2^n x) - 10f(2^{n+1}x) + f(2^{n+2}x)\big\| \\
&\quad \le \frac{4^{n+1}+1}{8^{n+1}} \chi(2^n x)
\end{aligned}$$

for all $n \in \mathbb{N}$. For $n \ge m$,

$$\begin{aligned}
\|g_n(x) - g_m(x)\| &\le \sum_{i=m}^{n-1} \|g_{i+1}(x) - g_i(x)\| \\
&\le \sum_{i=m}^{n-1} \frac{4^{i+1}+1}{8^{i+1}} \chi(2^i x) \qquad (18)
\end{aligned}$$

for all $x \in X$. By (5), since the right-hand side of the inequality (18) tends to zero as m tends to infinity, the sequence $\{g_n(x)\}$ is a Cauchy sequence. Therefore, we may apply a direct method to the definition of g. Define

$$g(x) := \lim_{n\to\infty} g_n(x)$$

for all $x \in X$. The inequality (6) implies that

$$\begin{aligned}
&\|g_n(x+y+z) + g_n(x+y-z) + 2g_n(x-y) + 4g_n(y) - g_n(x-y+z) \\
&\quad -g_n(x-y-z) - 2g_n(x+y) - 2g_n(y+z) - 2g_n(y-z)\|
\end{aligned}$$

$$\leq \frac{1}{6}\left(\frac{8}{2^n} - \frac{2}{8^n}\right)\varphi(2^n x, 2^n y, 2^n z)$$
$$+\frac{1}{6}\left(\frac{1}{2^n} - \frac{1}{8^n}\right)\varphi(2^{n+1} x, 2^{n+1} y, 2^{n+1} z)$$

for all $x, y, z \in X$ and all $n \in \mathbb{N}$. Letting n tend to infinity in the last inequality, by (3), g satisfies (2). By Theorem 1, there exist two mappings $B : X^3 \to Y$ and $A : X \to Y$such that $g(x) = B(x, x, x) + A(x) + g(0)$ for all $x \in X$, where B is symmetric for each fixed one variable and is additive for each fixed two variables and A is additive. By the definition of g, one can easily see $g(0) = 0$. The validity of inequality (17) follows directly from Lemma 2 and the definition of g. Hence, the proof is completed. $\Box$

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Department of Mathematics
Chungnam National University
305–764 Daejeon
Republic of Korea
E-mail: kwjun@math.cnu.ac.kr

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Department of Mathematics
Chungnam National University
305–764 Daejeon
Republic of Korea
E-mail: mcsquarelsb@hanmail.net