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ON AP-HENSTOCK-STIELTJES INTEGRAL

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Abstract. In this paper, we define and study the vector-valued ap-Henstock-Stieltjes integral, we prove the Cauchy extension theorem and the dominated convergence theorems for the ap-Henstock-Stieltjes integral.

1. Introduction

In the late 1950s, R. Henstock and J. Kurzweil, independently, gave a Riemann-type integral called the Henstock integral. It is a kind of non-absolute integral and includes the Riemann, improper Riemann, Newton, and Lebesgue integral. Many authors have studied Henstock integral [2, 4, 6, 8].

It is well known that the Henstock integral is equivalent to the Denjoy-Perron integral that recovers a continuous function from its derivative. In 1967, R. Henstock [3] gave an Riemann definition of an integral which is equivalent to the Burkill integral that recovers a function from its approximate derivative. It is called approximate continuous Henstock integral (br.ap-Henstock integral). As an extension of the Henstock integral, ap-Henstock integral has been discussed in [1, 2, 7]. In this paper, we define and study the vector-valued ap-Henstock-Stieltjes integral, we prove the Cauchy extension theorem and the dominated convergence theorem for the ap-Henstock-Stieltjes integral.

2. Definitions and basic properties

Throughout this paper, $[a, b]$ is a compact interval in R. X will denote a real Banach space with norm $\|\cdot\|$ and its dual X^*. The point c is called

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a point of density of E if

$$d_cE = \lim_{h\to 0^+} \frac{\mu(E\bigcap(c-h,c+h))}{2h} = 1$$

whenever d_cE denote the density of E at c. A measurable set $S_x \subseteq [a,b]$ is called an approximate neighborhood (br.ap-nbd) of $x \in [a,b]$ if it containing x as a point of density. We choose an ap-nbd $S_x \subseteq [a,b]$ for each $x \in E \subseteq [a,b]$ and denote a choice on E by $\Delta = \{S_x : x \in E\}$. A tagged interval-point pair $([u,v],\xi)$ is called to be Δ - fine if $\xi \in [u,v]$ and $u, v \in S_\xi$.

A partition D is a finite collection of interval-point pairs $\{([u_i,v_i],\xi_i)\}_{i=1}^n$, where $\{[u_i,v_i]\}_{i=1}^n$ are non-overlapping subintervals of $[a,b]$. We say that $D = \{([u_i,v_i],\xi_i)\}_{i=1}^n$ is

(1) a partial partition of $[a,b]$ if $\bigcup_{i=1}^n [u_i,v_i] \subset [a,b]$,

(2) a partition of $[a,b]$ if $\bigcup_{i=1}^n [u_i,v_i] = [a,b]$,

(3) Δ - fine partition of $[a,b]$ if $\xi_i \in [u_i,v_i]$ and $([u_i,v_i],\xi_i)$ is Δ - fine for all i=1,2,$\cdots$,n.

Given a Δ - fine partition $D = \{([u_i,v_i],\xi_i)\}_{i=1}^n$ we write

$$S(f,D) = \sum_{i=1}^n f(\xi_i)|v_i - u_i|$$

for integral sums over D, whenever $f : [a,b] \to X$.

Definition 2.1. A function $f : [a,b] \to X$ is ap-Henstock integrable if there exists a vector $A \in X$ such that for each $\varepsilon > 0$ there is a choice Δ such that

$$\|S(f,D) - A\| < \epsilon$$

for each Δ - fine *partition* $D = \{([u_i,v_i],\xi_i)\}_{i=1}^n$ of $[a,b]$. A is called the *ap-Henstock integral* of f on $[a,b]$, and we write $A = \int_a^b f$.

The function f is ap-Henstock integrable on the set $E \subset [a,b]$ if the function $f\chi_E$ is ap-Henstock integrable on $[a,b]$. We write$\int_E f = \int_a^b f\chi_E$.

Definition 2.2. Let $g : [a,b] \to R$ be an increasing function. A function $f : [a,b] \to X$ is ap-Henstock-Stieltjes integrable with respect to g on $[a,b]$ if there exists a vector $A \in X$ such that for each $\varepsilon > 0$ there is a choice Δ such that

$$\|S(f,g,D) - A\| < \epsilon$$

for each Δ - fine *partition* $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[a, b]$, whenever

$$S(f, g, D) = \sum_{i=1}^{n} f(\xi_i)(g(v_i) - g(u_i)).$$

A is called the *ap-Henstock-Stieltjes integral* of f with respect to g on $[a, b]$, and we write $A = \int_a^b f dg$.

The function f is ap-Henstock-Stieltjes integrable with respect to g on the set $E \subset [a, b]$ if the function $f\chi_E$ is ap-Henstock-Stieltjes integrable with respect to g on $[a, b]$. We write$\int_E f dg = \int_a^b f\chi_E dg$.

From the above definition we know that if $g(x) = x$ for all $x \in [a, b]$ then the ap-Henstock-Stieltjes integral reduces to the ordinary ap-Henstock integral. We can easily get the following two theorems.

THEOREM 2.3. Let $g : [a, b] \to R$ be an increasing function. A function $f : [a, b] \to X$ is ap-Henstock-Stieltjes integrable with respect to g on $[a, b]$ if and only if for each $\varepsilon > 0$ there is a choice Δ such that

$$\|S(f, g, D_1) - S(f, g, D_2)\| < \epsilon$$

for arbitrary Δ - fine partition D_1 and D_2 of I_0.

THEOREM 2.4. Let $f : [a, b] \to X$, $g : [a, b] \to R$ be an increasing function.

(1) If f is ap-Henstock-Stieltjes integrable with respect to g on $[a, b]$, then f is ap-Henstock-Stieltjes integrable with respect to g on every subinterval of $[a, b]$.

(2) If f is ap-Henstock-Stieltjes integrable with respect to g on each of the intervals $[a, c]$ and $[c, b]$, then f is ap-Henstock-Stieltjes integrable with respect to g on $[a, b]$ and $\int_a^c f dg + \int_c^b f dg = \int_a^b f dg$.

(3) If f is ap-Henstock-Stieltjes integrable with respect to g on $[a, b]$ and α is a real number, then αf is ap-Henstock-Stieltjes integrable with respect to g on $[a, b]$ and $\int_a^b \alpha f dg = \alpha \int_a^b f dg$.

LEMMA 2.5. (Saks-Henstock) Let $f : [a, b] \to X$ is ap-Henstock-Stieltjes integrable with respect to g on $[a, b]$. Then for $\varepsilon > 0$ there is a choice Δ such that

$$\|S(f, g, D) - \int_a^b f dg\| < \epsilon$$

for each Δ - fine partition $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[a, b]$. Particulary, if $D' = \{([u_i, v_i], \xi_i)\}_{i=1}^m$ is an arbitray Δ - fine partial partition of $[a, b]$, we have

$$\|S(f,g,D') - \sum_{i=1}^{m} \int_a^b fdg\| \le \epsilon.$$

Proof. Assume $D' = \{(I_i, \xi_i)\}_{i=1}^{m}$ is an arbitrary $\Delta-$ fine partial partition of $[a,b]$, then the complement $[a,b]\backslash \bigcup_{i=1}^{m}[u_i, v_i]$ can be expressed as a fine collection of closed subintervals and we denote $[a,b]\backslash \bigcup_{i=1}^{m}[u_i, v_i] = \bigcup_{j=1}^{k}[u'_j, v'_j]$.

Let $\eta > 0$ is arbitrary. From Theorem 2.4 we know $\int_{u'_j}^{v'_j} fdg$ exists for each $j = 1, 2, \cdots k$, then there exists a choice Δ_j on $[u'_j, v'_j]$ such that if D_j is a Δ_j - fine partition of $[u'_j, v'_j]$, then

$$\|S(f,g,D_j) - \int_{u'_j}^{v'_j} fdg\| < \frac{\eta}{k}.$$

Assume that $\Delta_j(\xi) \subset \Delta(\xi)$ for all $\xi \in [a,b]$. Let $D_0 = D' + D_1 + D_2 + \cdots + D_k$, obviously, D_0 is Δ - fine partition of $[a,b]$, We have

$$\|S(f,g,D_0) - \int_a^b fdg\| = \|S(f,g,D') + \sum_{j=1}^{k} S(f,g,D_j) - \int_a^b fdg\| < \epsilon.$$

Consequently, we obtain

$$\begin{aligned} & \|S(f,g,D') - \sum_{i=1}^{m} \int_{u_i}^{v_i} fdg\| \\ = \ & \|S(f,g,D_0) - \sum_{j=1}^{k} S(f,g,D_j) - (\int_a^b fdg - \sum_{j=1}^{k} \int_{u'_j}^{v'_j} fdg)\| \\ \le \ & \|S(f,g,D_0) - \int_a^b fdg\| + \sum_{j=1}^{k} \|S(f,g,D_j) - \int_{u'_j}^{v'_j} fdg\| \\ < \ & \epsilon + \frac{k\eta}{k} = \epsilon + \eta. \end{aligned}$$

$\eta > 0$ is arbitrary, then we have

$$\|S(f,g,D') - \sum_{i=1}^{m} \int_{u_i}^{v_i} fdg\| \le \epsilon,$$

as desired. □

The proof of the following theorem is easy and will be omitted.

THEOREM 2.6. Let $f_1, f_2 : [a,b] \to X$, $g_1, g_2 : [a,b] \to R$ are increasing functions.

(1) If f_1 and f_2 are ap-Henstock-Stieltjes integrable with respect to g_1 on $[a,b]$ and if α and β are real numbers, then $\alpha f_1 + \beta f_2$ is ap-Henstock-Stieltjes integrable with respect to g_1 on $[a,b]$ and $\int_a^b (\alpha f_1 + \beta f_2) dg_1 = \alpha \int_a^b f_1 dg_1 + \beta \int_a^b f_2 dg_1$.

(2) If f_1 is ap-Henstock-Stieltjes integrable with respect to both g_1 and g_2 on $[a,b]$ and if α and β are real numbers, then f_1 is ap-Henstock-Stieltjes integrable with respect to $\alpha g_1 + \beta g_2$ on $[a,b]$ and $\int_a^b f_1 d(\alpha g_1 + \beta g_2) = \alpha \int_a^b f_1 dg_1 + \beta \int_a^b f_1 dg_2$.

THEOREM 2.7. Let $g : [a,b] \to R$ be an increasing function and $g \in C^1[a,b]$. If $f = \theta$ almost everywhere on $[a,b]$, then f is ap-Henstock-Stieltjes integrable with respect to g on $[a,b]$ and $\int_a^b f dg = \theta$.

Proof. Since $g \in C^1[a,b]$, there exists a number $M > 0$ such that $|g'(\xi)| \leq M$ for each $\xi \in [a,b]$. From the mean-valued theorem we know there exists $\xi_i^{'} \in [u_i, v_i]$ such that

$$g(v_i) - g(u_i) = g'(\xi_i^{'})(v_i - u_i).$$

Assume $E = \{\xi \in [a,b] : f(\xi) \neq \theta\}$ and $E = \bigcup E_n \subset [a,b]$ where $E_n = \{\xi \in [a,b] : n-1 \leq \|f(\xi)\| < n\}$. Obviously, $\mu(E) = 0$ and therefore $\mu(E_n) = 0$, then there are open sets $G_n \subset [a,b]$ such that $E_n \subset G_n$ and $\mu(G_n) < \frac{\epsilon}{n \cdot 2^n \cdot M}$. We choose a choice Δ such that

$$\begin{aligned} \|S(f,g,D)\| &= \|\sum_{n=1}^{\infty} \sum_{\xi_{n_i} \in E_n} f(\xi_{n_i})[g(v_{n_i}) - g(u_{n_i})]\| \\ &= \|\sum_{n=1}^{\infty} \sum_{\xi_{n_i} \in E_n} f(\xi_{n_i}) g'(\xi_{n_i}^{'})(v_{n_i} - u_{n_i})\| \\ &< \sum_{n=1}^{\infty} n \cdot M \cdot \frac{\epsilon}{n \cdot 2^n \cdot M} \\ &< \epsilon \end{aligned}$$

for each Δ - fine partition $D = \{(I, \xi)\}$ of $[a,b]$. Hence f is ap-Henstock-Stieltjes integrable with respect to g on $[a,b]$ and $\int_a^b f dg = \theta$. $\square$

COROLLARY 2.8. Let $g : [a,b] \to R$ be an increasing function and $g \in C^1[a,b]$. If f_1 is ap-Henstock-Stieltjes integrable with respect to g on $[a,b]$ and $f_1 = f_2$ almost everywhere on $[a,b]$, then f_2 is ap-Henstock-Stieltjes integrable with respect to g on $[a,b]$ and $\int_a^b f_1 dg = \int_a^b f_2 dg$.

THEOREM 2.9. Let $g : [a,b] \to R$ be an increasing function, f is ap-Henstock-Stieltjes integrable with respect to g on $[a,b]$.

(1) for each $x^* \in X^*$, the function x^*f is ap-Henstock-Stieltjes integrable with respect to g on $[a,b]$ and $\int_a^b x^* f dg = x^*(\int_a^b f dg)$.

(2) If $T : X \to Y$ is a continuous linear operator, then Tf is ap-Henstock-Stieltjes integrable with respect to g on $[a,b]$ and $\int_a^b Tf dg = T(\int_a^b f dg)$.

Proof. (1) For each $x^* \in X^*$, since $f : [a,b] \to X$ is ap-Henstock-Stieltjes integrable with respect to g on $[a,b]$, for each $\varepsilon > 0$ there is a choice Δ such that

$$\|S(f,g,D) - \int_a^b f dg\| < \frac{\epsilon}{\|x^*\|}$$

for each Δ - fine *partition* $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[a,b]$. Hence for each $x^* \in X^*$, we have

$$|S(x^*f,g,D) - x^*(\int_a^b f dg)| \leq \|x^*\| \cdot \|S(f,g,D) - \int_a^b f dg\| < \epsilon.$$

Hence x^*f is ap-Henstock-Stieltjes integrable with respect to g on $[a,b]$ and $\int_a^b x^* f dg = x^*(\int_a^b f dg)$.

(2) $T : X \to Y$ is a continuous linear operator, then there exists a number $M > 0$ such that $\|Tx\| \leq M\|x\|$ for each $x \in X$. Since $f : [a,b] \to X$ is ap-Henstock-Stieltjes integrable with respect to g on $[a,b]$, for each $\varepsilon > 0$ there is a choice Δ such that

$$\|S(f,g,D) - \int_a^b f dg\| < \frac{\epsilon}{M}$$

for each Δ - fine *partition* $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[a,b]$. Hence we have

$$\begin{aligned} \|S(Tf,g,D) - T(\int_a^b f dg)\| &= \|T(S(f,g,D) - \int_a^b f dg)\| \\ &\leq M \cdot \|S(f,D) - \int_a^b f\| \\ &< M \cdot \frac{\epsilon}{M} = \epsilon, \end{aligned}$$

as desired. □

3. Convergence theorems

DEFINITION 3.1. Let $g : [a, b] \to R$ be an increasing function, $\{f_n\}$ be a sequence of integrable function defined on $[a, b]$ and X valued. The sequence $\{f_n\}$ is said ap-Henstock-Stieltjes equi-integrable with respect to g on $[a, b]$ if $\{f_n\}$ is ap-Henstock-Stieltjes integrable with respect to g on $[a, b]$ and for each $\varepsilon > 0$ there is a choice Δ such that

$$\|S(f_n, g, D) - \int_a^b f_n dg\| < \epsilon$$

holds for each Δ - fine *partition* $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[a, b]$ and $n \in \mathbb{N}$.

THEOREM 3.2. Assume that $g : [a, b] \to R$ is an increasing function, $f_n : [a, b] \to X$ is ap-Henstock-Stieltjes equi-integrable with respect to g on $[a, b]$ such that

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Then the function $f : [a, b] \to X$ is ap-Henstock-Stieltjes integrable with respect to g on $[a, b]$ and we have

$$\lim_{n\to\infty} \int_a^b f_n dg = \int_a^b f dg.$$

Proof. From the definition of ap-Henstock-Stieltjes equi-integrability of $\{f_n\}$, for each $\varepsilon > 0$ there is a choice Δ such that

$$\|S(f_n, g, D) - \int_a^b f_n dg\| < \epsilon$$

for each Δ - fine *partition* $D = \{(I_i, \xi_i)\}_{i=1}^n$ of $[a, b]$ and $n \in \mathbb{N}$. Assume D is fixed.

Since $\lim_{n\to\infty} f_n(x) = f(x)$, then there is a $N \in \mathbb{N}$ such that

$$\|S(f_n, g, D) - S(f, g, D)\| < \epsilon$$

for all $n > N$. Then we have

$$\begin{aligned}
& \|\int_a^b f_n dg - \int_a^b f_m dg\| \\
\leq\ & \|S(f, g, D) - \int_a^b f_n dg\| + \|S(f, g, D) - \int_a^b f_m dg\| \\
\leq\ & \|S(f_n, g, D) - S(f, g, D)\| + \|S(f_n, g, D) - \int_a^b f_n dg\| + \\
& \|S(f_m, g, D) - S(f, g, D)\| + \|S(f_m, g, D) - \int_a^b f_m dg\|
\end{aligned}$$

$$< \ 4\epsilon$$

for all $n, m > N$. Hence the sequence $\int_a^b f_n dg$ of elements of X is Cauchy and therefore

$$\lim_{n\to\infty}\int_a^b f_n dg = A \in X \quad \textit{exists.}$$

In other words, there is a $M \in \mathbb{N}$ such that $\| \int_a^b f_n dg - A\| < \epsilon$ for all $n > M$.

Take any Δ - fine *partition* $D = \{(I,\xi)\}$ of $[a,b]$. Since $\lim_{n\to\infty} f_n(x) = f(x)$, then there is a $K > M$ such that

$$\|S(f_K, g, D) - S(f, g, D)\| < \epsilon.$$

Then we have

$$\begin{aligned}\|S(f,g,D) - A\| &\leq \|S(f,g,D) - S(f_K,g,D)\| + \\ &\quad \|S(f_K,g,D) - \int_a^b f_K dg\| + \|\int_a^b f_K dg - A\| \\ &< \ 3\epsilon.\end{aligned}$$

Hence f is ap-Henstock-Stieltjes integrable with respect to g on $[a,b]$ and $\lim_{n\to\infty}\int_a^b f_n dg = \int_a^b f dg$. □

THEOREM 3.3. Assume that $g : [a,b] \to R$ is an increasing function and continuous from the left side at point b, $f : [a,b] \to X$ is ap-Henstock-Stieltjes integrable with respect to g on interval $[a,c]$ for each $c \in (a,b)$ and $\lim_{c\to b^-}\int_a^c f dg$ exists, then f is ap-Henstock-Stieltjes integrable with respect to g on $[a,b]$ and

$$\lim_{c\to b^-}\int_a^c f dg = \int_a^b f dg.$$

Proof. Let

$$a = c_1 < c_2 < \cdots, \lim_{n\to\infty} c_n = b.$$

From Theorem 2.4, f is ap-Henstock-Stieltjes integrable with respect to g on interval $[c_{k-1}, c_k]$. For $\epsilon > 0$, there is a choice Δ_k such that

$$\|S(f,g,D_k) - \int_{c_{k-1}}^{c_k} f dg\| < \frac{\epsilon}{2^k}$$

for each Δ_k - fine *partition* $D = \{([u,v],\xi)\}$ of $[c_{k-1}, c_k]$. Let $\varepsilon > 0$. Since $\lim_{c\to b^-}\int_a^c f dg = A$, there exists $\eta > 0$ and a measurable set $E \subset [b-\eta, b]$ such that

$$\|\int_a^x f dg - A\| < \epsilon \quad \textit{and} \quad \|f(b)(g(b) - g(x))\| < \epsilon$$

whenever $x \in E$ and x is a point of density of E. We define a choice in such a way

$$\Delta = \bigcup_k \Delta_k \bigcup \bigcup_{x \in E} [x, b].$$

Take any Δ - fine *partition* $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ *of* $[a, b]$*, then we have*

$$\begin{aligned} & \|S(f, g, D) - A\| \\ = \ & \|\sum_{i=1}^{n-1}[f(\xi_i)(g(v_i) - g(u_i)) - \int_{u_i}^{v_i} f dg]\| + \\ & \|\sum_{i=1}^{n-1} \int_{u_i}^{v_i} f dg - A\| + \|f(b)(g(b) - g(u_n))\| \\ < \ & \epsilon + \epsilon + \epsilon = 3\epsilon. \end{aligned}$$

Hence f is ap-Henstock-Stieltjes integrable with respect to g on $[a, b]$ and

$$\lim_{c \to b^-} \int_a^c f dg = \int_a^b f dg,$$

as desired. □

Corollary 3.4. Assume that $g : [a, b] \to R$ is an increasing function, $f : [a, b] \to X$ is ap-Henstock-Stieltjes integrable with respect to g on each interval $[c, d] \subseteq (a, b)$. If $\lim_{\substack{c \to a^+ \\ d \to b^-}} \int_c^d f dg$ exists, then f is ap-Henstock-Stieltjes integrable with respect to g on $[a, b]$ and

$$\lim_{\substack{c \to a^+ \\ d \to b^-}} \int_c^d f dg = \int_a^b f dg.$$

Definition 3.5. Let $F : [a, b] \to R$ and let E be a subset of $[a, b]$.

(a)F is said to be AC_Δ on E if for each $\varepsilon > 0$ there is a constant $\eta > 0$ and a choice Δ such that $\sum_i |F(I_i)| < \epsilon$ for each Δ - fine partial partition $D = \{(I_i, \xi_i)\}$ of $[a, b]$ satisfying $\sum_i |I_i| < \eta$.

(b)F is said to be ACG_Δ on E if E can be expressed as a countable union of sets on each of which F is AC_Δ.

Theorem 3.6. Assume that $g : [a, b] \to R$ is an increasing function and $g \in C^1[a, b]$, $f_n : [a, b] \to X$ is ap-Henstock-Stieltjes integrable with respect to g on $[a, b]$ such that

1)$f_n(x) \to f(x)$ for all $x \in [a, b]$, 2)there exists a real-valued function h that is ap-Henstock-Stieltjes integrable with respect to g on $[a,b]$ and such that $\|f_n - f_m\| \leq h$ for each n, m.

Then f is ap-Henstock-Stieltjes integrable with respect to g on $[a,b]$ and

$$\lim_{n\to\infty}\int_a^b f_n dg = \int_a^b f dg.$$

Proof. Let $\epsilon > 0$ and $H(x) = \int_a^x hdg$. We claim that $H(x)$ is ACG_Δ on $[a,b]$.

Assume $E_n = \{\xi \in [a,b], n-1 \leq |h(\xi)| < n\}$ for each natural number n. then $[a,b] = \bigcup_n E_n$. By Saks-Henstock lemma, give $\epsilon > 0$, there is a choice Δ such that

$$\sum |h(\xi_i)(g(v_i) - g(u_i)) - H(u_i, v_i)| < \frac{\epsilon}{2}$$

for each $\Delta-$ fine partial partition $D = \{([u_i, v_i], \xi_i)\}$ of $[a,b]$ whenever $H(u_i, v_i) = \int_{u_i}^{v_i} hdg$. Assume $\xi_i \in E_n$, $i = 1, 2, \cdots$. Let M be a bound for the function g' on $[a,b]$. By the Mean Value Theorem, for each i, there exists $x_i \in (u_i, v_i)$ such that

$$g(v_i) - g(u_i) = g'(x_i)(v_i - u_i) \leq M(v_i - u_i).$$

Choose $\eta < \frac{\epsilon}{2Mn(b-a)}$ and let $\sum_i (v_i - u_i) < \eta$, then we have

$$\begin{aligned} & |\sum_i H(u_i, v_i)| \\ \leq\ & \sum_i |h(\xi_i)(g(v_i) - g(u_i)) - H(u_i, v_i)| + \sum_i |h(\xi_i)| g'(x_i)(v_i - u_i) \\ <\ & \frac{\epsilon}{2} + Mn \sum_i (v_i - u_i) < \epsilon \end{aligned}$$

Hence $H(x)$ is ACG_Δ on $[a,b]$, i,e. there is a sequence of closed sets $\{E_i\}$ such that $\bigcup_i E_i = [a,b]$ and $H(x)$ is AC_Δ on E_i for each i. Then there exists $\eta_i > 0$ such that $\sum_i |H(v_i, u_i)| < \epsilon \cdot 2^{-i}$ whenever $\{[u_i, v_i]\}$ is a finite collection of non-overlapping intervals in $[a,b]$ satisfying $\sum_i |v_i - u_i| < \eta_i$ and $u_i, v_i \in E_i$.

$h(x)$ is ap-Henstock-Stieltjes integrable with respect to g on $[a,b]$, there is a choice Δ_h such that

$$|\sum [h(\xi)(g(v) - g(u)) - \int_u^v hdg]| < \epsilon$$

for each Δ_h- fine partition $D_h = \{([u,v], \xi)\}$ of $[a,b]$. Let $D_0 = \{([u,v], \xi)\}$ be a Δ_h- fine partial partition of $[a,b]$. Assume $u, v \in E_i$

and $\sum_{\xi\in E_i}|v-u|<\eta_i$, then for each n,m, we have

$$\begin{aligned}\|\sum\int_u^v f_n dg-\sum\int_u^v f_m dg\| &\le \sum\int_u^v \|f_n-f_m\|dg\\ &\le \sum\int_u^v hdg\\ &= \sum_{i=1}^{\infty}\sum_{\xi\in E_i}\int_u^v hdg<\epsilon.\end{aligned}$$

Since $\{f_n\}$ is ap-Henstock-Stieltjes integrable with respect to g on $[a,b]$, for $\epsilon>0$, there exists Δ_n and $\Delta_{n+1}\subset\Delta_n$ such that

$$\|\sum f_n(g(v)-g(u))-\sum\int_u^v f_n dg\|<\epsilon\cdot 2^{-n}$$

for each Δ_n - fine partition $D_n=\{([u,v],\xi)\}$ of $[a,b]$. For each $\xi\in E_i$, choose $m(\xi)\in\mathbb{N}$ such that $\|f_n(\xi)-f_m(\xi)\|<\epsilon$ for all $n,m>m(\xi)$.

Let $\Delta(\xi)=\Delta_{m(\xi)}(\xi)\bigcap\Delta_h(\xi),\xi\in E_i,i=1,2\cdots$. Take any $\Delta-$ fine partition $D=\{([u,v],\xi)\}$ of $[a,b]$, splitting the sum $\sum$ over D into two partial sums with $m(\xi)\ge n$ and $m(\xi)<n$ respectively, we have

$$\begin{aligned}&\|\sum f_n(g(v)-g(u))-\sum\int_u^v f_n dg\|\\ \le\ &\|\sum_{m(\xi)<n}[f_n(g(v)-g(u))-\int_u^v f_n dg]\|\\ +\ &\|\sum_{m(\xi)\ge n}[f_n(g(v)-g(u))-\int_u^v f_n dg]\|\\ <\ &\|\sum_{m(\xi)<n}(f_n-f_{m(\xi)})(g(v)-g(u))\|\\ +\ &\|\sum_{m(\xi)<n}[f_{m(\xi)}(g(v)-g(u))-\int_u^v f_{m(\xi)}dg]\|\\ &+\|\sum_{m(\xi)<n}[\int_u^v f_{m(\xi)}dg-\int_u^v f_n dg]\|+\epsilon\\ <\ &\epsilon+\epsilon(b-a)+\epsilon+\epsilon\\ =\ &\epsilon(b-a+3)\end{aligned}$$

Hence f is ap-Henstock-Stieltjes integrable with respect to g on $[a, b]$ and

$$\lim_{n\to\infty} \int_a^b f_n dg = \int_a^b f dg,$$

as desired. □

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