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HYERS–ULAM–RASSIAS STABILITY OF ISOMORPHISMS IN C^*-ALGEBRAS

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Abstract. This paper is a survey on the Hyers–Ulam–Rassias stability of the Jensen functional equation in C^*-algebras. The concept of Hyers–Ulam–Rassias stability originated from the Th.M. Rassias' stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297–300.

Its content is divided into the following sections:
1. Introduction and preliminaries.
2. Approximate isomorphisms in C^*-algebras.
3. Approximate isomorphisms in Lie C^*-algebras.
4. Approximate isomorphisms in JC^*-algebras.
5. Stability of derivations on a C^*-algebra.
6. Stability of derivations on a Lie C^*-algebra.
7. Stability of derivations on a JC^*-algebra.

1. Introduction and preliminaries

The present paper is devoted only to the Jensen functional equation in C^*-algebras and its aim is to present in a more or less organic form the great number of results on the subject published in the recent years.

In 1940, S.M. Ulam [39] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

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By now an affirmative answer has been given in several cases, and some interesting variations of the problem have also been investigated. We shall call such an $f : G \to G'$ an *approximate homomorphism.*

In 1941, D.H. Hyers [7] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \le \epsilon.$$

No continuity conditions are required for this result, but if $f(tx)$ is continuous in the real variable t for each fixed $x \in E$, then L is linear, and if f is continuous at a single point of E then $L : E \to E'$ is also continuous.

In 1978, Th.M. Rassias [32] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

THEOREM 1.1. (Th.M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \le \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) *holds for $x, y \neq 0$ and* (1.2) *for $x \neq 0$.*

In 1990, Th.M. Rassias [33] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \ge 1$. In 1991, Z. Gajda [5] following the same approach as in Th.M. Rassias [32], gave an affirmative solution to this question for $p > 1$. It was shown by Z. Gajda [5] that one cannot prove a Th.M. Rassias' type Theorem when $p = 1$. The counterexamples of Z. Gajda [5] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, cf. P. Găvruta [6], S. Czerwik [3], S. Jung [13], who among others studied the Hyers–Ulam stability of functional equations. The inequality (1.1) that was introduced for the first time by Th.M. Rassias [32] provided a lot of influence in the development of a generalization of the Hyers–Ulam stability concept. This new concept is known as *Hyers–Ulam–Rassias stability* of functional equations (cf. the books of D.H. Hyers, G. Isac and Th.M. Rassias [8], S. Jung [14], P. Czerwik [4]).

Beginning around the year 1980 the topic of approximate homomorphisms and their stability theory in the field of functional equations and inequalities was taken up by several mathematicians (cf. D.H. Hyers and Th.M. Rassias [10], Th.M. Rassias [36] and the references therein).

J.M. Rassias [28] following the spirit of the innovative approach of Th.M. Rassias [32] for the unbounded Cauchy difference proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$ (The reader is referred to [29], [30] and [31] for essential work in the subject).

P. Găvruta [6] provided a further generalization of Th.M. Rassias' Theorem. In 1996, G. Isac and Th.M. Rassias [11] applied the Hyers–Ulam–Rassias stability theory to prove fixed point theorems and study some new applications in Nonlinear Analysis. In [9], D.H. Hyers, G. Isac and Th.M. Rassias studied the asymptoticity aspect of Hyers–Ulam–Rassias stability of mappings. During the several papers have been published on various generalizations and applications of Hyers–Ulam–Rassias stability and Hyers–Ulam–Rassias stability to a number of functional equations and mappings, for example: quadratic functional equation, invariant means, multiplicative mappings - superstability, bounded nth differences, convex functions, generalized orthogonality functional equation, Euler–Lagrange functional equation, Navier–Stokes equations. Several mathematician have contributed works on these subjects; we mention a few: C. Baak and M.S. Moslehian [1], D. Boo, S. Oh, C. Park and J. Park [2], C. Park [16, 18], F. Skof [38].

The main purpose of this paper is to prove the Hyers–Ulam–Rassias stability of the Jensen functional equation in unital C^*-algebras, Lie C^*-algebras and JC^*-algebras, and to prove the Hyers–Ulam–Rassias stability of derivations on a C^*-algebra, a Lie C^*-algebra and a JC^*-algebra. The results are applied to investigate isomorphisms in C^*-algebras, Lie C^*-algebras and JC^*-algebras.

2. Approximate isomorphisms in C^*-algebras

Throughout this section, assume that A is a unital C^*-algebra with unit e and norm $\|\cdot\|_A$ and that B is a unital C^*-algebra with unit e' and norm $\|\cdot\|_B$. Let $U(A)$ be the group of unitary elements in A and $\mathbb{Z}_+$ the set of nonnegative integers.

For a given mapping $f : A \to B$, we define

$$C_\lambda f(x,y) := 2f(\frac{\lambda x + \lambda y}{2}) - \lambda f(x) - \lambda f(y)$$

for all $\lambda \in \mathbb{T}^1 := \{\mu \in \mathbb{C} \mid |\mu| = 1\}$ and all $x, y \in A$.

THEOREM 2.1. ([18, 21, 25]) *Let p and θ be positive real numbers with $p < 1$, and let $f : A \to B$ be a multiplicative bijective mapping with $f(0) = 0$ such that*

$$\|C_\lambda f(x,y)\|_B \le \theta(\|x\|_A^p + \|y\|_A^p), \tag{2.1}$$

$$\|f(3^n u^*) - f(3^n u)^*\|_B \le 2 \cdot 3^{pn}\theta, \tag{2.2}$$

$$\lim_{n\to\infty} \frac{1}{3^n} f(3^n e) = e' \tag{2.3}$$

for all $\lambda \in \mathbb{T}^1$, all $x, y \in A$, all $u \in U(A)$ and all $n \in \mathbb{Z}_+$. Then the multiplicative bijective mapping $f : A \to B$ is a C^*-algebra isomorphism.

Proof. Let $\lambda = 1$ in (2.1). By Theorem 1 of [12], there is a unique additive mapping $L : A \to B$ such that

$$\|f(x) - L(x)\|_B \le \frac{3 + 3^p}{3 - 3^p}\theta\|x\|_A^p \tag{2.4}$$

for all $x \in A$. The additive mapping $L : A \to B$ is given by

$$L(x) := \lim_{n\to\infty} \frac{1}{3^n} f(3^n x) \tag{2.5}$$

for all $x \in A$.

By the same method as in the proof of Theorem 2.1 in [20], one can show that the mapping $L : A \to B$ is a C^*-algebra homomorphism and that $L = f$. Thus the multiplicative bijective mapping $f : A \to B$ is a C^*-algebra isomorphism. □

THEOREM 2.2. ([18, 21, 25]) *Let p and θ be positive real numbers with $p > 1$, and let $f : A \to B$ be a multiplicative bijective mapping satisfying $f(0) = 0$ and (2.1) such that*

$$\|f(\frac{u^*}{3^n}) - f(\frac{u}{3^n})^*\|_B \le \frac{2\theta}{3^{pn}}, \tag{2.6}$$

$$\lim_{n\to\infty} 3^n f(\frac{e}{3^n}) = e' \tag{2.7}$$

for all $u \in U(A)$ *and all* $n \in \mathbb{Z}_+$. *Then the multiplicative bijective mapping* $f : A \to B$ *is a* C^**-algebra isomorphism.*

Proof. Let $\lambda = 1$ in (2.1). By Theorem 6 of [12], there is a unique additive mapping $L : A \to B$ such that

$$\|f(x) - L(x)\|_B \le \frac{3^p + 3}{3^p - 3}\theta\|x\|_A^p \tag{2.8}$$

for all $x \in A$. The additive mapping $L : A \to B$ is given by

$$L(x) := \lim_{n\to\infty} 3^n f(\frac{x}{3^n}) \tag{2.9}$$

for all $x \in A$.

By the same method as in the proof of Theorem 2.1 in [20], one can show that the mapping $L : A \to B$ is a C^*-algebra homomorphism and that $L = f$. Thus the multiplicative bijective mapping $f : A \to B$ is a C^*-algebra isomorphism. □

Theorem 2.3. ([18, 21, 25]) *Let* p *and* θ *be positive real numbers with* $p < \frac{1}{2}$, *and let* $f : A \to B$ *be a multiplicative bijective mapping satisfying* $f(0) = 0$ *and* (2.3) *such that*

$$\|C_\lambda f(x, y)\|_B \le \theta \cdot \|x\|_A^p \cdot \|y\|_A^p, \tag{2.10}$$

$$\|f(3^n u^*) - f(3^n u)^*\|_B \le 3^{2pn}\theta \tag{2.11}$$

for all $\lambda \in \mathbb{T}^1$, *all* $x, y \in A$, *all* $u \in U(A)$ *and all* $n \in \mathbb{Z}_+$. *Then the multiplicative bijective mapping* $f : A \to B$ *is a* C^**-algebra isomorphism.*

Proof. Let $\lambda = 1$ in (2.10). By Theorem 1 of [12], there is a unique additive mapping $L : A \to B$ such that

$$\|f(x) - L(x)\|_B \le \frac{1 + 3^p}{3 - 3^{2p}}\theta\|x\|_A^{2p} \tag{2.12}$$

for all $x \in A$. The additive mapping $L : A \to B$ is given by

$$L(x) := \lim_{n\to\infty} \frac{1}{3^n} f(3^n x) \tag{2.13}$$

for all $x \in A$.

By the same method as in the proof of Theorem 2.1 in [20], one can show that the mapping $L : A \to B$ is a C^*-algebra homomorphism and that $L = f$. Thus the multiplicative bijective mapping $f : A \to B$ is a C^*-algebra isomorphism. □

THEOREM 2.4. ([18, 21, 25]) *Let p and θ be positive real numbers with $p > \frac{1}{2}$, and let $f : A \to B$ be a multiplicative bijective mapping satisfying $f(0) = 0$, (2.7) and (2.10) such that*

$$\|f(\frac{1}{3^n}u^*) - f(\frac{1}{3^n}u)^*\|_B \le \frac{\theta}{3^{2pn}} \tag{2.14}$$

for all $u \in U(A)$ and all $n \in \mathbb{Z}_+$. Then the bijective mapping $f : A \to B$ is a C^*-algebra isomorphism.

Proof. Let $\lambda = 1$ in (2.10). By Theorem 6 of [12], there is a unique additive mapping $L : A \to B$ such that

$$\|f(x) - L(x)\|_B \le \frac{3^p + 1}{3^{2p} - 3}\theta\|x\|_A^{2p} \tag{2.15}$$

for all $x \in A$. The additive mapping $L : A \to B$ is given by

$$L(x) := \lim_{n\to\infty} 3^n f(\frac{x}{3^n}) \tag{2.16}$$

for all $x \in A$.

By the same method as in the proof of Theorem 2.1 in [20], one can show that the mapping $L : A \to B$ is a C^*-algebra homomorphism and that $L = f$. Thus the multiplicative bijective mapping $f : A \to B$ is a C^*-algebra isomorphism. □

3. Approximate isomorphisms in Lie C^*-algebras

A unital C^*-algebra A, endowed with the Lie product $[x, y] = \frac{xy - yx}{2}$ on A, is called a *Lie C^*-algebra* (see [19, 20, 24, 26]).

Throughout this section, assume that A is a Lie C^*-algebra with unit e and norm $\|\cdot\|_A$ and that B is a Lie C^*-algebra with unit e' and norm $\|\cdot\|_B$.

DEFINITION 3.1. ([19, 20, 24]) A $\mathbb{C}$-linear bijective mapping $L : A \to B$ is called a *Lie C^*-algebra isomorphism* if $L : A \to B$ satisfies

$$\begin{aligned} L([x, y]) &= [L(x), L(y)], \\ L(x^*) &= L(x)^* \end{aligned}$$

for all $x, y \in A$.

THEOREM 3.1. ([20, 24]) *Let p and θ be positive real numbers with $p < 1$, and let $f : A \to B$ be a bijective mapping satisfying $f(0) = 0$,* *(2.1), (2.2), (2.3) and* $f(3^n ey) = f(3^n e)f(y)$ *for all* $y \in A$ *and all* $n \in \mathbb{Z}_+$, *such that*

$$\|f([x,y]) - [f(x), f(y)]\|_B \leq \theta(\|x\|_A^p + \|y\|_A^p) \tag{3.1}$$

for all $x, y \in A$. *Then the bijective mapping* $f : A \to B$ *is a Lie* C^*-*algebra isomorphism.*

Proof. Let $\lambda = 1$ in (2.1). By Theorem 1 of [12], there is a unique additive mapping $L : A \to B$ satisfying (2.4).

By the same method as in the proof of Theorem 5.1 in [24], one can show that the mapping $L : A \to B$ is a Lie C^*-algebra homomorphism and that $L = f$. Thus the bijective mapping $f : A \to B$ is a Lie C^*-algebra isomorphism. □

THEOREM 3.2. ([20, 24]) *Let* p *and* θ *be positive real numbers with* $p > 2$, *and let* $f : A \to B$ *be a bijective mapping satisfying* $f(0) = 0$, *(2.1), (2.6), (2.7), (3.1) and* $f(\frac{1}{3^n} ey) = f(\frac{1}{3^n} e)f(y)$ *for all* $y \in A$ *and all* $n \in \mathbb{Z}_+$. *Then the bijective mapping* $f : A \to B$ *is a Lie* C^*-*algebra isomorphism.*

Proof. Let $\lambda = 1$ in (2.1). By Theorem 6 of [12], there is a unique additive mapping $L : A \to B$ satisfying (2.8).

By the same method as in the proof of Theorem 5.1 in [24], one can show that the mapping $L : A \to B$ is a C^*-algebra homomorphism and that $L = f$. Thus the bijective mapping $f : A \to B$ is a Lie C^*-algebra isomorphism. □

THEOREM 3.3. ([20, 24]) *Let* p *and* θ *be positive real numbers with* $p < \frac{1}{2}$, *and let* $f : A \to B$ *be a bijective mapping satisfying* $f(0) = 0$, *(2.4), (2.11), (2.12) and* $f(3^n ey) = f(3^n e)f(y)$ *for all* $y \in A$ *and all* $n \in \mathbb{Z}_+$. *Then the bijective mapping* $f : A \to B$ *is a Lie* C^*-*algebra isomorphism.*

Proof. The proof is similar to the proofs of Theorems 2.4 and 3.2. □

THEOREM 3.4. ([20, 24]) *Let* p *and* θ *be positive real numbers with* $p > 1$, *and let* $f : A \to B$ *be a bijective mapping satisfying* $f(0) = 0$, *(2.7), (2.10), (2.15), (3.2) and* $f(\frac{1}{3^n} ey) = f(\frac{1}{3^n} e)f(y)$ *for all* $y \in A$ *and all* $n \in \mathbb{Z}_+$. *Then the bijective mapping* $f : A \to B$ *is a Lie* C^*-*algebra isomorphism.*

Proof. The proof is similar to the proofs of Theorems 2.5 and 3.3. □

4. Approximate isomorphisms in JC^*-algebras

A unital C^*-algebra A, endowed with the Jordan product $x \circ y = \frac{xy+yx}{2}$ on A, is called a *JC^*-algebra* (see [20, 24, 26]).

Throughout this section, assume that A is a JC^*-algebra with unit e and norm $\|\cdot\|_A$ and that B is a JC^*-algebra with unit e' and norm $\|\cdot\|_B$.

DEFINITION 4.1. ([20, 24]) A $\mathbb{C}$-linear bijective mapping $L : A \to B$ is called a *JC^*-algebra isomorphism* if $L : A \to B$ satisfies

$$\begin{aligned} L(x \circ y) &= L(x) \circ L(y), \\ L(x^*) &= L(x)^* \end{aligned}$$

for all $x, y \in A$.

THEOREM 4.1. ([20, 24, 26]) *Let p and θ be positive real numbers with $p < 1$, and let $f : A \to B$ be a bijective mapping satisfying $f(0) = 0$, (2.1), (2.2), (2.3) and $f(3^n e \circ y) = f(3^n e) \circ f(y)$ for all $y \in A$ and all $n \in \mathbb{Z}_+$, such that*

$$\|f(x \circ y) - f(x) \circ f(y)\|_B \leq \theta(\|x\|_A^p + \|y\|_A^p) \tag{4.1}$$

for all $x, y \in A$. Then the bijective mapping $f : A \to B$ is a JC^*-algebra isomorphism.

Proof. Let $\lambda = 1$ in (2.1). By Corollary 2.11 of [23], there is a unique additive mapping $L : A \to B$ satisfying (2.4).

By the same method as in the proof of Theorem 5.1 in [24], one can show that the mapping $L : A \to B$ is a Lie C^*-algebra homomorphism and that $L = f$. Thus the bijective mapping $f : A \to B$ is a JC^*-algebra isomorphism. $\square$

THEOREM 4.2. ([20, 24, 26]) *Let p and θ be positive real numbers with $p > 2$, and let $f : A \to B$ be a bijective mapping satisfying $f(0) = 0$, (2.1), (2.6), (2.7), (4.1) and $f(\frac{1}{3^n} e \circ y) = f(\frac{1}{3^n} e) \circ f(y)$ for all $y \in A$ and all $n \in \mathbb{Z}_+$. Then the bijective mapping $f : A \to B$ is a JC^*-algebra isomorphism.*

Proof. Let $\lambda = 1$ in (2.1). By Corollary 2.14 of [23], there is a unique additive mapping $L : A \to B$ satisfying (2.8).

By the same method as in the proof of Theorem 5.1 in [24], one can show that the mapping $L : A \to B$ is a C^*-algebra homomorphism and that $L = f$. Thus the bijective mapping $f : A \to B$ is a JC^*-algebra isomorphism. $\square$

THEOREM 4.3. ([20, 24, 26]) *Let p and θ be positive real numbers with $p < \frac{1}{2}$, and let $f : A \to B$ be a bijective mapping satisfying $f(0) = 0$, (2.3), (2.10), (2.11) and $f(3^n e \circ y) = f(3^n e) \circ f(y)$ for all $y \in A$ and all $n \in \mathbb{Z}_+$, such that*

$$\|f(x \circ y) - f(x) \circ f(y)\|_B \leq \theta \cdot \|x\|_A^p \cdot \|y\|_A^p \tag{4.2}$$

for all $x, y \in A$. Then the bijective mapping $f : A \to B$ is a JC^*-algebra isomorphism.

Proof. The proof is similar to the proofs of Theorems 2.4 and 4.2. □

THEOREM 4.4. ([20, 24, 26]) *Let p and θ be positive real numbers with $p > 1$, and let $f : A \to B$ be a bijective mapping satisfying $f(0) = 0$, (2.7), (2.10), (2.14, (4.2) and $f(\frac{1}{3^n} e \circ y) = f(\frac{1}{3^n} e) \circ f(y)$ for all $y \in A$ and all $n \in \mathbb{Z}_+$. Then the bijective mapping $f : A \to B$ is a JC^*-algebra isomorphism.*

Proof. The proof is similar to the proofs of Theorems 2.5 and 4.3. □

5. Stability of derivations on a C^*-algebra

Throughout this section, assume that A is a C^*-algebra with norm $\| \cdot \|_A$.

DEFINITION 5.1. A $\mathbb{C}$-linear involutive mapping $D : A \to A$ is called a *derivation* if $D : A \to A$ satisfies

$$\begin{aligned} D(xy) &= D(x)y + xD(y), \\ D(x^*) &= D(x)^* \end{aligned}$$

for all $x, y \in A$.

In [17, 25], the author proved the Hyers–Ulam–Rassias stability of derivations on a C^*-algebra.

THEOREM 5.1. ([17, 25]) *Let p and θ be positive real numbers with $p < 1$, and let $f : A \to A$ be a mapping with $f(0) = 0$ such that*

$$\|C_\lambda f(x, y)\|_A \leq \theta(\|x\|_A^p + \|y\|_A^p), \tag{5.1}$$

$$\|f(3^n u^*) - f(3^n u)^*\|_A \leq 2 \cdot 3^{pn}\theta, \tag{5.2}$$

$$\|f(xy) - f(x)y - xf(y)\|_A \leq \theta(\|x\|_A^p + \|y\|_A^p) \tag{5.3}$$

for all $\lambda \in \mathbb{T}^1$, all $x, y \in A$, all $u \in U(A)$ and all $n \in \mathbb{Z}_+$. Then there exists a unique derivation $D : A \to A$ such that

$$\|f(x) - D(x)\|_A \leq \frac{3 + 3^p}{3 - 3^p}\theta\|x\|_A^p \tag{5.4}$$

for all $x \in A$.

Proof. By the same reasoning as in the proof of Theorem 2.2, there is a unique $\mathbb{C}$-linear involutive mapping $D : A \to A$ satisfying (5.4). The mapping $D : A \to A$ is given by

$$D(x) := \lim_{n\to\infty} \frac{1}{3^n} f(3^n x) \tag{5.5}$$

for all $x \in A$.

By the same reasoning as in the proof of Theorem 11 in [25], the mapping $D : A \to A$ is a derivation satisfying (5.4). □

THEOREM 5.2. ([17, 25]) *Let p and θ be positive real numbers with $p > 2$, and let $f : A \to A$ be a mapping satisfying $f(0) = 0$, (5.1) and (5.3) such that*

$$\|f(\frac{1}{3^n}u^*) - f(\frac{1}{3^n}u)^*\|_A \le \frac{2\theta}{3^{pn}} \tag{5.6}$$

for all $u \in U(A)$ and all $n \in \mathbb{Z}_+$. Then there exists a unique derivation $D : A \to A$ such that

$$\|f(x) - D(x)\|_A \le \frac{3^p + 3}{3^p - 3}\theta\|x\|_A^p \tag{5.7}$$

for all $x \in A$.

Proof. By the same reasoning as in the proof of Theorem 2.3, there is a unique $\mathbb{C}$-linear involutive mapping $D : A \to A$ satisfying (5.7). The mapping $D : A \to A$ is given by

$$D(x) := \lim_{n\to\infty} 3^n f(\frac{x}{3^n}) \tag{5.8}$$

for all $x \in A$.

By the same reasoning as in the proof of Theorem 11 in [25], the mapping $D : A \to A$ is a derivation satisfying (5.7). □

THEOREM 5.3. ([17, 25]) *Let p and θ be positive real numbers with $p < \frac{1}{2}$, and let $f : A \to A$ be a mapping with $f(0) = 0$ such that*

$$\|C_\lambda f(x, y)\|_A \le \theta \cdot \|x\|_A^p \cdot \|y\|_A^p, \tag{5.9}$$

$$\|f(3^n u^*) - f(3^n u)^*\|_A \le 3^{2pn}\theta, \tag{5.10}$$

$$\|f(xy) - f(x)y - xf(y)\|_A \le \theta \cdot \|x\|_A^p \cdot \|y\|_A^p \tag{5.11}$$

for all $\lambda \in \mathbb{T}^1$, all $x, y \in A$, all $u \in U(A)$ and all $n \in \mathbb{Z}_+$. Then there exists a unique derivation $D : A \to A$ such that

$$\|f(x) - D(x)\|_A \le \frac{1 + 3^p}{3 - 3^{2p}}\theta\|x\|_A^{2p} \tag{5.12}$$

for all $x \in A$.

Proof. By the same reasoning as in the proof of Theorem 2.4, there is a unique $\mathbb{C}$-linear involutive mapping $D : A \to A$ satisfying (5.12). The mapping $D : A \to A$ is given by

$$D(x) := \lim_{n\to\infty} \frac{1}{3^n} f(3^n x) \tag{5.13}$$

for all $x \in A$.

By the same reasoning as in the proof of Theorem 11 in [25], the mapping $D : A \to A$ is a derivation satisfying (5.12). □

THEOREM 5.4. ([17, 25]) *Let p and θ be positive real numbers with $p > 1$, and let $f : A \to A$ be a mapping satisfying $f(0) = 0$,* (5.9) *and* (5.11) *such that*

$$\|f(\frac{1}{3^n}u^*) - f(\frac{1}{3^n}u)^*\|_B \le \frac{\theta}{3^{2pn}} \tag{5.14}$$

for all $u \in U(A)$ and all $n \in \mathbb{Z}_+$. Then there exists a unique derivation $D : A \to A$ such that

$$\|f(x) - D(x)\|_A \le \frac{3^p + 1}{3^{2p} - 3}\theta\|x\|_A^{2p} \tag{5.15}$$

for all $x \in A$.

Proof. By the same reasoning as in the proof of Theorem 2.5, there is a unique $\mathbb{C}$-linear involutive mapping $D : A \to A$ satisfying (5.15). The mapping $D : A \to A$ is given by

$$D(x) := \lim_{n\to\infty} 3^n f(\frac{x}{3^n}) \tag{5.16}$$

for all $x \in A$.

By the same reasoning as in the proof of Theorem 11 in [25], the mapping $D : A \to A$ is a derivation satisfying (5.15). □

6. Stability of derivations on a Lie C^*-algebra

Throughout this section, assume that A is a Lie C^*-algebra with norm $\|\cdot\|_A$.

DEFINITION 6.1. ([19, 20]) A $\mathbb{C}$-linear mapping $D : A \to A$ is called a *Lie derivation* if $D : A \to A$ satisfies

$$D([x, y]) = [D(x), y] + [x, D(y)]$$

for all $x, y \in A$.

In [19, 20], the author proved the Hyers–Ulam–Rassias stability of derivations on a Lie C^*-algebra.

Theorem 6.1. ([19, 20]) *Let p and θ be positive real numbers with $p < 1$, and let $f : A \to A$ be a mapping satisfying $f(0) = 0$ and* (5.1) *such that*

$$\|f([x,y]) - [f(x),y] - [x,f(y)]\|_A \le \theta(\|x\|_A^p + \|y\|_A^p) \tag{6.1}$$

for all $x, y \in A$. Then there exists a unique Lie derivation $D : A \to A$ such that

$$\|f(x) - D(x)\|_A \le \frac{3+3^p}{3-3^p}\theta\|x\|_A^p \tag{6.2}$$

for all $x \in A$.

Proof. By the same reasoning as in the proof of Theorem 3.2, there is a unique $\mathbb{C}$-linear mapping $D : A \to A$ satisfying (6.2). The mapping $D : A \to A$ is given by

$$D(x) := \lim_{n\to\infty} \frac{1}{3^n} f(3^n x) \tag{6.3}$$

for all $x \in A$.

By the same reasoning as in the proof of Theorem 4.1 in [20], the mapping $D : A \to A$ is a Lie derivation satisfying (6.2). □

Theorem 6.2. ([19, 20]) *Let p and θ be positive real numbers with $p > 2$, and let $f : A \to A$ be a mapping satisfying $f(0) = 0$,* (5.1) *and* (6.1). *Then there exists a unique Lie derivation $D : A \to A$ such that*

$$\|f(x) - D(x)\|_A \le \frac{3^p+3}{3^p-3}\theta\|x\|_A^p \tag{6.4}$$

for all $x \in A$.

Proof. By the same reasoning as in the proof of Theorem 3.3, there is a unique $\mathbb{C}$-linear mapping $D : A \to A$ satisfying (6.4). The mapping $D : A \to A$ is given by

$$D(x) := \lim_{n\to\infty} 3^n f(\frac{x}{3^n}) \tag{6.5}$$

for all $x \in A$.

By the same reasoning as in the proof of Theorem 4.1 in [20], the mapping $D : A \to A$ is a Lie derivation satisfying (6.4). □

Theorem 6.3. ([19, 20]) *Let p and θ be positive real numbers with $p < \frac{1}{2}$, and let $f : A \to A$ be a mapping satisfying $f(0) = 0$ and* (5.9) *such that*

$$\|f([x,y]) - [f(x),y] - [x,f(y)]\|_A \le \theta \cdot \|x\|_A^p \cdot \|y\|_A^p \tag{6.6}$$

for all $x, y \in A$. *Then there exists a unique Lie derivation* $D : A \to A$ *such that*

$$\|f(x) - D(x)\|_A \leq \frac{1+3^p}{3-3^{2p}}\theta\|x\|_A^{2p} \tag{6.7}$$

for all $x \in A$.

Proof. By the same reasoning as in the proof of Theorem 3.4, there is a unique $\mathbb{C}$-linear mapping $D : A \to A$ satisfying (6.7). The mapping $D : A \to A$ is given by

$$D(x) := \lim_{n\to\infty} \frac{1}{3^n} f(3^n x) \tag{6.8}$$

for all $x \in A$.

By the same reasoning as in the proof of Theorem 4.1 in [20], the mapping $D : A \to A$ is a Lie derivation satisfying (6.7). □

THEOREM 6.4. ([19, 20]) *Let* p *and* θ *be positive real numbers with* $p > 1$, *and let* $f : A \to A$ *be a mapping satisfying* $f(0) = 0$, (5.9) *and* (6.6). *Then there exists a unique Lie derivation* $D : A \to A$ *such that*

$$\|f(x) - D(x)\|_A \leq \frac{3^p+1}{3^{2p}-3}\theta\|x\|_A^{2p} \tag{6.9}$$

for all $x \in A$.

Proof. By the same reasoning as in the proof of Theorem 3.4, there is a unique $\mathbb{C}$-linear mapping $D : A \to A$ satisfying (6.9). The mapping $D : A \to A$ is given by

$$D(x) := \lim_{n\to\infty} 3^n f(\frac{x}{3^n}) \tag{6.10}$$

for all $x \in A$.

By the same reasoning as in the proof of Theorem 4.1 in [20], the mapping $D : A \to A$ is a Lie derivation satisfying (6.9). □

7. Stability of derivations on a JC^*-algebra

Throughout this section, assume that A is a JC^*-algebra with norm $\|\cdot\|_A$.

DEFINITION 7.1. ([20]) A $\mathbb{C}$-linear mapping $D : A \to A$ is called a *Jordan derivation* if $D : A \to A$ satisfies

$$D(x \circ y) = D(x) \circ y + x \circ D(y)$$

for all $x, y \in A$.

In [20], the author proved the Hyers–Ulam–Rassias stability of derivations on a JC^*-algebra.

THEOREM 7.1. ([20]) *Let p and θ be positive real numbers with $p < 1$, and let $f : A \to A$ be a mapping satisfying $f(0) = 0$ and* (5.1) *such that*

$$\|f(x \circ y) - f(x) \circ y - x \circ f(y)\|_A \le \theta(\|x\|_A^p + \|y\|_A^p) \tag{7.1}$$

for all $x, y \in A$. Then there exists a unique Jordan derivation $D : A \to A$ such that

$$\|f(x) - D(x)\|_A \le \frac{3 + 3^p}{3 - 3^p}\theta\|x\|_A^p \tag{7.2}$$

for all $x \in A$.

Proof. By the same reasoning as in the proof of Theorem 3.2, there is a unique $\mathbb{C}$-linear mapping $D : A \to A$ satisfying (7.2). The mapping $D : A \to A$ is given by

$$D(x) := \lim_{n\to\infty} \frac{1}{3^n} f(3^n x) \tag{7.3}$$

for all $x \in A$.

By the same reasoning as in the proof of Theorem 4.1 in [20], the mapping $D : A \to A$ is a Jordan derivation satisfying (7.2). □

THEOREM 7.2. ([20]) *Let p and θ be positive real numbers with $p > 2$, and let $f : A \to A$ be a mapping satisfying $f(0) = 0$,* (5.1) *and* (7.1). *Then there exists a unique Jordan derivation $D : A \to A$ such that*

$$\|f(x) - D(x)\|_A \le \frac{3^p + 3}{3^p - 3}\theta\|x\|_A^p \tag{7.4}$$

for all $x \in A$.

Proof. By the same reasoning as in the proof of Theorem 3.3, there is a unique $\mathbb{C}$-linear mapping $D : A \to A$ satisfying (7.4). The mapping $D : A \to A$ is given by

$$D(x) := \lim_{n\to\infty} 3^n f(\frac{x}{3^n}) \tag{7.5}$$

for all $x \in A$.

By the same reasoning as in the proof of Theorem 4.1 in [20], the mapping $D : A \to A$ is a Jordan derivation satisfying (7.4). □

THEOREM 7.3. ([20]) *Let p and θ be positive real numbers with $p < \frac{1}{2}$, and let $f : A \to A$ be a mapping satisfying $f(0) = 0$ and* (5.9) *such that*

$$\|f(x \circ y) - f(x) \circ y - x \circ f(y)\|_A \le \theta \cdot \|x\|_A^p \cdot \|y\|_A^p \tag{7.6}$$

for all $x, y \in A$. Then there exists a unique Jordan derivation $D : A \to A$ such that

$$\|f(x) - D(x)\|_A \leq \frac{1 + 3^p}{3 - 3^{2p}} \theta \|x\|_A^{2p} \tag{7.7}$$

for all $x \in A$.

Proof. By the same reasoning as in the proof of Theorem 3.4, there is a unique $\mathbb{C}$-linear mapping $D : A \to A$ satisfying (7.7). The mapping $D : A \to A$ is given by

$$D(x) := \lim_{n\to\infty} \frac{1}{3^n} f(3^n x) \tag{7.8}$$

for all $x \in A$.

By the same reasoning as in the proof of Theorem 4.1 in [20], the mapping $D : A \to A$ is a Jordan derivation satisfying (7.7). □

THEOREM 7.4. ([20]) *Let p and θ be positive real numbers with $p > 1$, and let $f : A \to A$ be a mapping satisfying $f(0) = 0$, (5.9) and (7.6). Then there exists a unique Jordan derivation $D : A \to A$ such that*

$$\|f(x) - D(x)\|_A \leq \frac{3^p + 1}{3^{2p} - 3} \theta \|x\|_A^{2p} \tag{7.9}$$

for all $x \in A$.

Proof. By the same reasoning as in the proof of Theorem 3.5, there is a unique $\mathbb{C}$-linear mapping $D : A \to A$ satisfying (7.9). The mapping $D : A \to A$ is given by

$$D(x) := \lim_{n\to\infty} 3^n f(\frac{x}{3^n}) \tag{7.10}$$

for all $x \in A$.

By the same reasoning as in the proof of Theorem 4.1 in [20], the mapping $D : A \to A$ is a Jordan derivation satisfying (7.9). □

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