JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **19**, No.2, June 2006

ALMOST PERIODIC SOLUTIONS OF LINEAR DIFFERENCE SYSTEMS

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Abstract. In this paper, we present an elementary proof for the existence of almost periodic solutions of linear nonhomogeneous difference systems.

1. Introduction

The theory of almost periodic functions was created by H. Bohr [2] in 1923. Then in 1925, V. Stepanov generalized the class of almost periodic functions in the sense of Bohr without using the hypothesis of continuity. S. Bochner defined and studied the almost periodic functions with values in Banach spaces in 1933.

In recent years the theory of almost periodic functions has been developed in connection with problems of differential equations, stability theory, dynamical systems, and so on [4].

In this paper we prove the well-known Bochner's characterization of almost periodic sequences and the existence of almost periodic solutions of linear difference systems by the elementary methods.

2. Bochner's characterization

Definition 2.1. [5] A sequence $x = (x(n))_n,\ n \in \mathbb{Z}$, in $\mathbb{R}^k$ is called *almost periodic* if for every $\varepsilon > 0$ there exists $l = l(\varepsilon) > 0$ such that for any $m \in \mathbb{Z}$ there exists $\tau \in \{m, m+1, \cdots, m+l\}$ satisfying

$$|x(n+\tau) - x(n)| < \varepsilon,\ n \in \mathbb{Z}.$$

We call τ an *ε-translation number* of x.

Received March 27, 2006.
2000 Mathematics Subject Classification: Primary 39A10, 39A11.
Key words and phrases: almost periodic sequence, almost periodic solution.

The well-known Bochner's characterization of the almost periodic sequence is the following:

THEOREM 2.1. [6] *A sequence x in $\mathbb{R}^k$ is almost periodic if and only if for any sequence $(h_k^{'}) \subset \mathbb{Z}$ there exists a subsequence $(h_k) \subset (h_k^{'})$ such that $x(n+h_k)$ converges uniformly on $\mathbb{Z}$.*

The proof of Theorem 2.1 can be accomplished by means of the following lemmas:

LEMMA 2.2. [6] *A sequence x in $\mathbb{R}^k$ is almost periodic if and only if the function $f : \mathbb{R} \to \mathbb{R}^k$ defined by*

$$f(n+u) = x(n) + u[x(n+1) - x(n)],\ 0 \le u \le 1,\ n \in \mathbb{Z}$$

is almost periodic.

LEMMA 2.3. *Every almost periodic sequence is bounded.*

Proof. Let $x = (x(n))_n,\ n \in \mathbb{Z}$, be any almost periodic sequence. Then for every $\varepsilon > 0$, there exists an ε-translation number τ such that for every $n \in \mathbb{Z}$, $0 \le n + \tau \le l$ and

$$|x(n+\tau) - x(n)| < \varepsilon,\ n \in \mathbb{Z}.$$

Thus, when $\varepsilon = 1$, we have

$$\begin{aligned} |x(n)| &\le |x(n) - x(n+\tau)| + |x(n+\tau)| \\ &< 1 + \max_{0 \le n \le l} |x(n)| \\ &\equiv M. \end{aligned}$$

This completes the proof. □

Proof of Theorem 2.1

(⇒) Let $(h_k^{'})$ be any sequence in $\mathbb{Z}$. In view of Lemma 2.2, the function $f : \mathbb{R} \to \mathbb{R}^k$ defined by

$$f(n+u) = x(n) + u[x(n+1) - x(n)],\ 0 \le u \le 1,\ n \in \mathbb{Z}$$

is almost periodic. By the well-known Bochner's characterization [4] of the almost periodic function $f : \mathbb{R} \to \mathbb{R}^k$, there exists a sequence (h_k) such that $f(n+h_k)$ converges uniformly on $\mathbb{Z}$. Since $h_k \in \mathbb{Z}$, we have $f(n+h_k) = x(n+h_k)$.

(⇐) Note that x is bounded by Lemma 2.3. Let (h_k) be any sequence in $\mathbb{R}$. Then $h_k = m_k + p_k,\ m_k \in \mathbb{Z},\ 0 \le p_k < 1$. We show that $f : \mathbb{R} \to \mathbb{R}^k$ defined by

$$f(n+u) = x(n) + u[x(n+1) - x(n)],\ 0 \le u \le 1,\ n \in \mathbb{Z}$$

is almost periodic. Then $x(n) = f(n)$ is almost periodic by Lemma 2.2. To do this we use the Bochner's characterization of almost periodic functions.

If we assume that

$$\sup_{n\in\mathbb{Z}} |f(n+m_k) - f(n)| \leq 1/n$$

and

$$|p_k - p| \leq \frac{1}{2n|x|}, \quad 0 \leq p \leq 1,$$

then, for any $t \in \mathbb{R}$,

$$\sup_{t\in\mathbb{R}} |f(t+m_k) - f(t)| \leq 1/n$$

and

$$\begin{aligned}\sup_{t\in\mathbb{R}} |f(t+p_k) - f(t+p)| &\leq 2|x||p_k - p| \\ &\leq 1/n\end{aligned}$$

since $f(t)$ is Lipschitzian with Lipschitz constant $2|x|$. Therefore

$$\begin{aligned}&|f(t+h_k) - f(t+p)| \\ &\leq |f(t+h_k) - f(t+m_k+p)| + |f(t+m_k+p) - f(t+p)| \\ &\leq 2/n.\end{aligned}$$

This implies that $f(t+h_k)$ converges uniformly on $\mathbb{R}$ to $f(t+p)$. This completes the proof. □

3. Almost periodic solutions of difference systems

We consider the linear nonhomogeneous difference system

(1) $$x(n+1) = Ax(n) + g(n), \quad n \in \mathbb{Z},$$

where A is an $k \times k$ real matrix and $g : \mathbb{Z} \to \mathbb{R}$ is an almost periodic sequence. The general solution of (1) can be written as

$$\begin{aligned}x(n) &= A^n c + \sum_{l=1}^{k} A^{k-l} b(l-1), \quad k \in \mathbb{N} \\ &= A^k c - \sum_{l=k+1}^{0} A^{k-l} b(l-1), \quad k \leq 0,\end{aligned}$$

where $c \in \mathbb{R}^k$ [1].

The following existence theorem for almost periodic solutions of (1) is due to C. Corduneanu [3]. For this theorem we present the elementary proof.

THEOREM 3.1. *System (1) has an almost periodic solution $x(n)$ if and only if it is bounded.*

Proof. ($\Rightarrow$) It follows from Lemma 2.3.

($\Leftarrow$) From Linear Algebra, there exists a nonsingular matrix T such that

$$\begin{aligned} T^{-1}AT &= B, \quad \text{an upper triangular matrix} \\ &= \begin{pmatrix} \lambda_1 & b_{12} & \cdot & \cdot & \cdot & b_{1n} \\ 0 & \lambda_2 & & & & \cdot \\ 0 & & \cdot & & & \cdot \\ 0 & & & \cdot & & \cdot \\ 0 & & & & \cdot & \cdot \\ 0 & \cdot & & \cdot & 0 & \lambda_n \end{pmatrix}, \end{aligned}$$

where each $\lambda_i, i = 1, 2, \cdots, n$ is an eigenvalue of A.

Consider $y(n) = T^{-1}x(n)$. Then we have

$$y(n) \text{ is bounded} \Leftrightarrow x(n) \text{ is bounded},$$
$$y(n) \text{ is almost periodic} \Leftrightarrow x(n) \text{ is almost periodic}.$$

Furthermore, $y(n)$ is the solution of the triangular system

$$y(n+1) = By(n).$$

Note that if theorem holds when $n = 1$, then it always holds. Thus we consider the scalar equation

$$\begin{cases} x(n+1) &= ax(n) + g(n), \quad a \in \mathbb{R}, \\ x(n_0) &= x_0. \end{cases}$$

The solution $x(n)$ of the equation is given by

$$x(n) = a^n \left(x_0 + \sum_{j=0}^{n-1} \frac{g(j)}{a^{j+1}} \right). \tag{2}$$

We consider three cases:

(i) $|a| > 1$: Since $|a|^n \to \infty$ when $n \to \infty$, from (2), a bounded solution satisfies

$$x_0 = -\sum_{j=0}^{\infty} \frac{g(j)}{a^{j+1}}$$

and it is convergent since $(g(j))_j$ is bounded and $|a| > 1$. Thus the form of any bounded solution is

$$x(n) = a^n \sum_{j=n}^{\infty} \frac{g(j)}{a^{j+1}}.$$

From the inequality

$$|x(n+h) - x(n)| \leq \sup_{\tau} \frac{|g(\tau+h) - g(\tau)|}{|a|-1},$$

$(|a|-1)\varepsilon$-almost period of $g(n)$ equals ε-almost period of $x(n)$. This implies that $x(n)$ is almost periodic.

(ii) $|a| > 1$: By the same reasoning of (i), $x(n)$ is almost periodic.

(iii) $|a| = 1$: From (2),

$$x(n) \text{ is bounded} \Leftrightarrow \left(\sum_{j=0}^{n-1} \frac{g(j)}{a^{j+1}} \right) \text{ is bounded} \tag{3}$$

since $(a^n)_n$ is bounded. Also, $(a^n)_n$ is almost periodic. Thus (3) holds for almost periodic sequences. This completes the proof. □

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