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ON THE ORDER AND RATE OF CONVERGENCE FOR PSEUDO-SECANT-NEWTON'S METHOD LOCATING A SIMPLE REAL ZERO

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Abstract. By combining the classical Newton's method with the pseudo-secant method, pseudo-secant-Newton's method is constructed and its order and rate of convergence are investigated. Given a function $f : \mathbb{R} \to \mathbb{R}$ that has a simple real zero α and is sufficiently smooth in a small neighborhood of α, the convergence behavior is analyzed near α for pseudo-secant-Newton's method. The order of convergence is shown to be cubic and the rate of convergence is proven to be $\left(\frac{f''(\alpha)}{2f'(\alpha)}\right)^2$. Numerical experiments show the validity of the theory presented here and are confirmed via high-precision programming in Mathematica.

1.. Introduction and Preliminaries

The aim of this paper is to establish the order and rate of convergence for a variant of Newton's method called pseudo-secant-Newton's method. Although an analysis for the convergence behavior of this method was done by Kasturiarachi [5], a slightly different approach will be presented here under for a sufficiently smooth function f whose typical zero is to be sought. Indeed, the rate of convergence for this method derived by Kasturiarachi has turned out to be incorrect. This incorrectedness motivates the current analysis which will provide us with a completely correct expression for the rate of convergence of the pseudo-secant-Newton's method. Furthermore, various numerical examples to be shown with the high-precision computability of Mathematica [12] will doubtlessly strengthen the validity of the current expression.

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Let $f : \mathbb{R} \to \mathbb{R}$ have a simple real zero α and be sufficiently smooth in a small neighborhood of α. Given $x_0 \in \mathbb{R}$, for $n \in \mathbb{N} \cup \{0\}$ Pseudo-secant-Newton's method is iteratively defined by

$$x_{n+1} = x_n - \frac{f(x_n)^2}{\{f(x_n) - f(\bar{x}_n)\}\, f'(x_n)}\ , \tag{1.1}$$

where

$$\bar{x}_n = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1.2}$$

Combining (1.1) with (1.2) immediately leads us to the iterative method below:

$$x_{n+1} = x_n - h \cdot \frac{f(x_n)}{f(x_n) - f(x_n - h)}\ , \tag{1.3}$$

where $f = f(x_n), f' = f'(x_n), h = f(x_n)/f'(x_n)$ are used for brevity and the symbol $'$ denotes the derivative with respect to x. Since $f(x_n)$ is approximately equal to 0 in a neighborhood of α, then the value of $\lim_{n\to\infty} h = 0$ is assumed.

We expand $f(x_n - h)$ by Talyor's series [1,9] about $x = x_n$, and take the first several terms up to the third-degree in h to obtain

$$\begin{aligned} f(x_n) - f(x_n - h) &= hf'(x_n) - \frac{h^2}{2} f''(x_n) + \frac{h^3}{6} f'''(x_n) + O(h^4) \\ &= hf'(x_n)(1-u)\ , \end{aligned} \tag{1.4}$$

where $O(h^4)$ is Landau notation for the bounded remainder term of the corresponding Taylor's series when divided by h^4, and

$$u = \frac{hf''(x_n)}{2f'(x_n)} - \frac{h^2 f'''(x_n)}{6f'(x_n)} + O(h^3). \tag{1.5}$$

Since $f(x_n) \approx 0$ near α, we find that h is small and $|u| < 1$. Therefore, the second term of the right side of (1.3) becomes, after expanding by Taylor series about $u = 0$,

$$\begin{aligned} h \cdot \frac{f(x_n)}{f(x_n) - f(x_n - h)} &= h \cdot (1 + u + u^2 + u^3 + \cdots) \\ &= h\left\{1 + \frac{h^2}{2} \cdot \frac{f''}{f'} - \frac{h^2}{6} \cdot \frac{f'''}{f'} + O(h^3)\right\} \\ &\quad + h^2 \left(\frac{h}{2} \cdot \frac{f''}{f'} - \frac{h^2}{6} \cdot \frac{f'''}{f'} + O(h^3)\right)^2 + hO(h^7) \\ &= h\left\{1 + \frac{h}{2} \cdot \frac{f''}{f'} - \frac{h^2}{6} \cdot \frac{f'''}{f'} + \frac{h^2}{4} \frac{f''^2}{f'^2}\right\} + O(h^4). \end{aligned} \tag{1.6}$$

Putting $\alpha = x_n + \alpha - x_n$ and expanding $f(\alpha)$ by Taylor series about x_n, we obtain

$$0 = f(\alpha) = f(\alpha - x_n + x_n) = f(x_n) + (\alpha - x_n)f'(x_n) + \frac{(\alpha - x_n)^2}{2!}f''(x_n) + \frac{(\alpha - x_n)^3}{3!}f'''(c),$$

where $c \in (a, b),\ a = min(\alpha, x_n),\ b = max(\alpha, x_n)$ and $\lim_{n\to\infty} c = \alpha$.

Multiplying by $1/f'$ both sides of the above equation with the aid of (1.6) leads us to the following equations.

$$-h = (\alpha - x_n)[1 + \frac{(\alpha - x_n)}{2}\frac{f''}{f'} + \frac{(\alpha - x_n)^2}{6}\frac{f'''(c)}{f'}]. \tag{1.7}$$

$$h^2 = (\alpha - x_n)^2[1 + (\alpha - x_n)\frac{f''}{f} + \frac{(\alpha - x_n)^2}{4}\frac{f''^2}{f'^2} + O(a - x_n^3)]. \tag{1.8}$$

Subtracting α_n from both sides of (1.3) and using (1.6), we obtain the equation below:

$$x_{n+1} - \alpha = x_n - \alpha - h\left\{1 + \frac{h}{2}\cdot\frac{f''}{f'} - \frac{h^2}{6}\cdot\frac{f'''}{f'} + \frac{h^2}{4}\frac{f''^2}{f'^2}\right\} + O(h^4).$$

Substituting (1.7) into the first-order term in h on the the right side of the above equation immediately yields the equation below.

$$\begin{aligned} x_{n+1} - \alpha &= \frac{(\alpha - x_n)^2}{2}\frac{f''}{f'} + \frac{(\alpha - x_n)^3}{6}\frac{f'''(c)}{f'} \\ &\quad -h^2\left(\frac{f''}{2f'} - \frac{h}{6}\cdot\frac{f'''}{f'} + \frac{h}{4}\frac{f''^2}{f'^2}\right) + O(h^4) \\ &= \frac{(\alpha - x_n)^2}{2}\frac{f''}{f'} + \frac{(\alpha - x_n)^3}{6}\frac{f'''(c)}{f'} \\ &\quad -(\alpha - x_n)^2\left(1 + (\alpha - x_n)\frac{f''}{f'} + O(\alpha - x_n^2)\right)\left(\frac{f''}{2f'} + dh\right) \end{aligned} \tag{1.9}$$

where $d = \frac{f''^2}{4f'^2} - \frac{f'''}{6f'}$. Further calculation shows that

$$x_{n+1} - \alpha = (\alpha - x_n)^3\left\{\frac{f'''(c)}{6f'} + \frac{f''^2}{4f'^2} - \frac{f'''(x_n)}{6f'} - \frac{f''^2}{f'^2}\right\} + O(\alpha - x_n^4).$$

As a result, we obtain the following equation

$$\frac{(x_{n+1} - \alpha)}{(x_{n+1} - \alpha)^3} = \frac{f''^2}{4f'^2} + \frac{f'''(x_n)}{6f'} - \frac{f'''(c)}{6f'} + O(x_n - \alpha).$$

By taking absolute values of both sides of the above equation and passing to the limit as n approaches infinity, we find that $x_n \to \alpha$ and $c \to \alpha$. Hence the above equation now reduces to

$$\eta = \lim_{n\to\infty} \left| \frac{(x_{n+1} - \alpha)}{(x_{n+1} - \alpha)^3} \right| = \frac{f''^2(\alpha)}{4f'^2(\alpha),} \tag{1.10}$$

which is the rate of convergence (asymptotic error constant) [2,3,10] with third-order convergence. Other numerical methods of order three can be found in [4,7,8,11]. Kasturiarachi [5] showed a different rate of convergence

$$\eta = \frac{|3\ f''^2(\alpha) - 2f^{(3)}(\alpha)f'(\alpha)|}{6f'^2(\alpha)}$$

which seems incorrect due to the faulty manipulation of the Taylor series(1.7). The analysis discussed so far finally gives us the following theorem.

Theorem 1.1. Let $f : \mathbb{R} \to \mathbb{R}$ have a simple real zero α and be sufficiently smooth in a small neighborhood of α. Then pseudo-secant-Newton's method converges with order three and its rate of convergence is found to be $\eta = |c^2|/4$, where $c = f''(\alpha)/f'(\alpha)$.

2.. Algorithm, numerical results and discussions

In this section, we construct a zero-finding algorithm with the support of symbolic and computational ability of *Mathematica* on the basis of the analysis shown in Section 1.

Claim 2.2. Algorithm 2.1 (Zero-finding algorithm)
Step 1. For $k \in \mathbb{N} \cup \{0\}$, construct the iteration scheme (1.3) with the given function f having a simple zero α, according to the description in Section 1.

Step 2. Set the minimum number of precision digits. With exact zero α or most accurate zero, supply the theoretical asymptotic error constant η. Set the error range ϵ, the maximum iteration number n_{max} and the initial value x_0. Compute $f(x_0)$ and $|x_0 - \alpha\ |$.

Step 3. Compute x_{n+1} in (1.3) for $0 \le n \le n_{max}$ and display the computed values of $n,\ x_n, f(x_n), |x_n - \alpha|, |e_{n+1}/{e_n}^3|$ and η.

As a numerical example for the convergence of pseudo-secant-Newton's method, we first illustrate the order of convergence and asymptotic error constant with a function

$$f(x) = (x^2 + 1)\cos(\pi x/8)$$

having a simple real zero $\alpha = -4$. The symbolic computation of $f'(x)$ has been easily done with the aid of Mathematica. Table 1 lists the numerical results for approximated zeros of $f(x)$ computed with Mathematica programming. To obtain sufficient accuracy, the minimum number of precision digits was chosen as 250 by assigning *$MinPrecision=250* in Mathematica. The error bound ϵ for $\mid x_n - \alpha \mid < \epsilon$ was chosen as 0.5×10^{-235} for the current experiment. As can be seen in Table 1, the order of convergence has been confirmed to be cubic.

TABLE 1. Convergence of pseudo-secant-Newton method for $f(x) = (x^2 + 1)\cos(\pi x/8)$

n	x_n	$f(x_n)$	$\mid x_n - \alpha \mid$	$e_{n+1}/e_n{}^3$	η
0	-3.20000000000000	3.47335	0.800000		
1	-3.71842121657687	1.63613	0.281579	0.5499585614	
2	-3.99302114596102	0.0464371	0.00697885	0.3125966580	
3	-3.99999992409404	5.06739×10^{-7}	$7.59060 \times 1{-}^{-8}$	0.2233178953	0.2214532872
4	-4.00000000000009	6.46575×10^{-22}	9.68523×10^{-23}	0.2214533074	
5	-4.00000000000000	1.34314×10^{-66}	2.01192×10^{-67}	0.2214532872	
6	-4.00000000000000	$-1.20400 \times 10^{-200}$	1.80350×10^{-201}	0.2214532872	
7	-4.00000000000000	$-4.85718 \times 10^{-260}$	8.12987×10^{-261}		

As a second numerical example to confirm the convergence, we take

$$f(x) = z^{10} - 3\ z^3\ e^{\cos x} - 1$$

with a simple real zero

$$\begin{aligned}\alpha = 1.&2454283753596838267131847481004361733761068230618687830979702334524850709991\\&4183571669293744806752150642762504380678911657014836288641875371950827568473 10\\&4074191809464118054252715785040852090212428074580383195745177281569968061347 50\\&0405899693924955110430287653461606102499403101664390703677681259152509675476 43\\&3272702834572006826300587100943321989562086011440853109559440714016944353325 92\\&9617428575250112849921078690834404202626657119219842610270631238958875290767 64,\end{aligned}$$

which is accurate up to 250 significant decimal digits. Table 2 also shows a good accordance with the theoretical ideas presented in this paper. The computed asymptotic error constants were found in good agreement with the theoretical asymptotic error constants η up to 10 significant digits. The computed root was rounded to be accurate up to the 235 significant digits. The limited space allows us to list it only up

TABLE 2. Convergence of pseudo-secant-Newton method for $f(x) = z^{10} - 3\ z^3\ e^{\cos x} - 1$

n	x_n	$f(x_n)$	$\mid x_n - \alpha \mid$	$e_{n+1}/{e_n}^3$	η
0	1.10000000000000	-4.69109	0.145428		
1	1.13910928707585	-4.05959	0.106319	34.56716689	
2	1.00008863454853	-2.04922	0.0403693	33.59056353	
3	1.24383857765155	-0.0954153	0.00158980	24.16511149	
4	1.24542829965909	-4.57464×10^{-6}	7.57006×10^{-8}	18.83968769	18.64595504
5	1.24542837535968	-4.88810×10^{-19}	8.08877×10^{-21}	18.64596421	
6	1.24542837535968	-5.96335×10^{-58}	9.86808×10^{-60}	18.64595504	
7	1.24542837535968	$-1.08278 \times 10^{-174}$	1.79177×10^{-176}	18.64595504	
8	1.24542837535968	$-3.82104 \times 10^{-259}$	5.62073×10^{-261}		

to 15 significant digits. Additional numerical experiments clearly ensure cubic convergence of pseudo-secant-Newton's method.

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