JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **19**, No.2, June 2006

EXISTENCE OF PERIODIC SOLUTIONS TO LIAPUNOV CHARACTERISTIC INDEX EQUATIONS

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Abstract. In this paper, a numerical programming for Liapunov index of differential equations with periodic coefficients depending on parameters is given. The associated equations are given at first, then existence of periodic solutions to the associated equations is proved in this paper. And we compute periodic solution $u(t)$ of the associated equations. Finally, we give the error of approximate calculation.

1. Introduction

The number

$$\lambda_{x(t)} = \overline{\lim_{x\to\infty}} = \frac{1}{t}\ln x(t),$$

where $x(t) \neq 0$ is a solution of the equation $\dot{x}(t) = A(t)x(t)$ is called the Liapunov characteristic index (LCI) of $x(t)$. The complete resolution of LCI is reduced to the following questions:

1) The existence of LCI and how many are they?

2) Practicable computation of LCI.

The answer of the first question was given by Liapunov [4, 7]. His result generalized many authors about the conditions on the equation. While the second question far more from complete resolution even in the special case that $A(t) \equiv A(t+T)$, that is, the periodic case. But in the point of view of approximate computation methods, it is not so difficult to solve the second question. Indeed, we can calculate the fundamental solution matrix of the equation $\dot{x}(t) = A(t)x(t)$ with whatever small error we wish, and then approximately compute the LCI by its definition.

Received March 23, 2006.
2000 Mathematics Subject Classification: Primary 65H05, 65H99.
Key words and phrases: Liapunov, characteristic index.
Supported by the Natural Science Foundation of P.R. China (10171010) and Jilin Province Distinguish Youth Science Foundation (No.10201005), Key Project on Science and Technology of Education Ministry of People's Republic of China (No.01061).

However, the previous procedures are valid in practice only in the case that $A(t) \equiv A(0)$, i.e., the coefficient matrix $A(t)$ is a constant matrix. In this paper, we introduce a effective method to calculate the LCI for scalar equation

$$\varepsilon\ddot{x}(t) + [p_1 + c_1(t)]\dot{x}(t) = [p_2 + c_2(t)]x(t) \tag{1}$$

where $\varepsilon \neq 0, p_j = p_j(\varepsilon)$

$$c_j(t) = c_j(t,\varepsilon) = \sum_{|r|=1}^{N(\varepsilon)} c_{j,r}(\varepsilon)exp(irt), \tag{2}$$

$j = 1, 2$, and $p_j(\varepsilon), N_j(\varepsilon), c_{j,r}(\varepsilon)$ are all independent of t.

If we assume that

$$x = e^{-\int \frac{c_1(t)}{2\varepsilon}dt} \cdot W$$

then we get

$$\dot{x} = e^{-\int \frac{c_1(t)}{2\varepsilon}dt} \cdot \frac{-c_1(t)}{2\varepsilon} \cdot w + e^{-\int \frac{c_1(t)}{2\varepsilon}dt} \cdot \dot{w}$$

and

$$\ddot{x} = e^{-\int \frac{c_1(t)}{2\varepsilon}dt}[(\frac{-c_1(t)}{2\varepsilon})^2 w + (\frac{-c_1'(t)}{2\varepsilon})w - \frac{c_1(t)}{\varepsilon}\dot{w} + \ddot{w}].$$

And (1) becomes

$$\varepsilon\ddot{w} + p_1\dot{w} = [\frac{c_1'(t)}{2\varepsilon} - \frac{c_1^2(t)}{4\varepsilon^2} + c_1(t)(p_1 + c_1(t)) + p_2 + c_2(t)]w.$$

Thus, it is sufficient to study the case that $c_1(t) \equiv 0$ in the following.

2. The associated equations

Assuming that$x = exp(w), \dot{w} = u$, we have the equation

$$\varepsilon\dot{u} + \varepsilon u^2 + p_1 u = p_2 + c_2(t). \tag{3}$$

If $u(t)$ is a 2π-periodic solution of (3). then we can set

$$u(t) = \sum_{|m|=1}^{\infty} u_m exp(imt) + u_0 \tag{4}$$

and

$$w(t) = \sum_{|m|=1}^{\infty} \frac{u_m}{c_m} exp(imt) + u_0 t + w_0,$$

where the constants u_0, u_m and w_0 are independent of t.

Lemma 1. If $u(t)$ is a 2π-periodic solution of (3) and (4) holds, then Reu_0 is a **LCI** for equation (1).

Proof. Since $u(t)$ is a 2π-periodic solution of (3),

$$\begin{aligned} x(t) &= exp(w(t)) = e^{\int_0^t u(s)ds+w(0)} \\ &= e^{\sum\limits_{|m|=1}^{\infty} u_m \int_0^t exp(ims)ds+u_0t+w(0)} \end{aligned}$$

is a solution of (1). We have

$$\frac{1}{t}ln|x(t)| = \frac{1}{t}Re|\sum_{|m|=1}^{\infty} u_m \int_0^t exp(ims)ds + u_0t + w(0)|$$

Note that

$$\sum_{|m|=1}^{\infty} u_m \int_0^t exp(ims)ds + w(0)$$

is bounded, it follows that

$$\lambda_{x(t)} = \overline{\lim_{t\to\infty}}\frac{1}{t}\ln x(t) = Reu_0.$$

In order to find 2π-periodic solution to equation (3) and, then calculate the number u_0, we consider the associated equations

$$\varepsilon\dot{Z}_(k) + \varepsilon\sum_{p=0}^{k} Z_{(p)}Z_{(k-p)} + p_1Z_{(k)} = b_{(k)}, k = 0, 1, 2\cdots \tag{5}$$

where $b_0 = p_2, b_1 = c_1(t), b_{(k)} = 0, k \geq 2$. □

3. Existence of periodic solutions to the associated equations

If $k = 0$, then the associated equation (5) will take the form

$$\varepsilon\dot{Z}_(0) + \varepsilon(Z_{(0)})^2 + p_1Z_{(0)} = p_2 \tag{6}$$

Obviously, equation (6) has two 2π-periodic solutions: $\frac{-p_1+\alpha}{2\varepsilon}$ and $\frac{-p_1-\alpha}{2\varepsilon}$, where $\alpha = \sqrt{p_1^2 + 4p_2\varepsilon}$, and $\alpha \neq 0$.

Let $Z_{(0)}^{(j)}, j = 1, 2$, be one of the two 2π-periodic solutions of (6), respectively. Obviously, if$\alpha = 0$, then $Z_{(0)}^{(1)} \equiv Z_{(0)}^{(2)}$. If $k = 1$, then (5) will take the form

$$\varepsilon\dot{Z}_{(1)} + (-1)^j\alpha Z_{(1)} = c_2(t), j = 1, 2. \tag{7}$$

For fixed j, the equation (7) has one and only one 2π-periodic solution

$$Z_{(1)}^{(j)}(t) = \sum_{|m|=1}^{N(\varepsilon)} c_{2,m}(\varepsilon)(\varepsilon im + (-1)^j\alpha)^{-1}exp(imt) \tag{8}$$

if the condition

$$\varepsilon im \pm \alpha \neq 0, m = 0, \pm 1, \pm 2, \cdots \tag{9}$$

holds. If $k \geq 2$, then we have

$$\varepsilon \dot{z}_{(k)} + (-1)^j z_{(k)} = -\varepsilon \sum_{p=1}^{k-1} Z_{(j)}^{(p)} Z_{(j)}^{(k-p)} := f_{j,k} \tag{10}$$

The equation (10) has one and only one 2π-periodic solution for fixed j:

$$Z_{(k)}^{(j)} = \sum_{|m|=1}^{N(\varepsilon,k)} f_{k,m}(\varepsilon + \alpha)_{-1}exp(imt), j = 1, 2, \tag{11}$$

if (9) holds true.

4. Periodic solution u(t) of (3)

Let

$$u^{(j)}(t) = \sum_{k=0}^{\infty} Z_{(k)}^{(j)}, j = 1, 2. \tag{12}$$

Let A be the Banach space of 2π-periodic functions with the norm

$$\|v\| := \sum_{|m|=0}^{\infty} |v_m| < \infty$$

$$v_m = \frac{1}{2\pi} \int_0^{2\pi} v(t)exp(-imt)dt.$$

It is clear that

$$Z_{(k)}^{(j)} \in A, j = 1, 2, k = 2, 3, \cdots .$$

Lemma 2. $l_k = \sum\limits_{p=1}^{k-1} \frac{1}{p^2} \frac{k^2}{(k-p)^2} < 6, k = 2, 3 \cdots .$

Proof. It is easy to calculate that $l_2 = 4, l_3 = 4.5, l_4 \approx 4.555, l_5 \approx 4.514, l_6 \approx 4.449, l_7 \approx 4.383, l_8 \approx 4.320, l_9 \approx 4.263$.

If $k \geq 10$, then we have

$$\frac{k^2}{(k-1)^2} \leq \frac{100}{81}, \frac{k^2}{(k-2)^2} \leq \frac{100}{(64}, \frac{k^2}{(k-3)}^2 \leq \frac{100}{49}.$$

It follows that

$$\begin{aligned} l_k &= \sum_{p=0}^{k-1} \tfrac{1}{p^2} \tfrac{k^2}{(k-p)^2} \leq \sum_{p=1}^{2\frac{k-1}{2}} \tfrac{1}{p^2} \tfrac{k^2}{(k-p)^2} \\ &\leq 2\{\tfrac{100}{81} + \tfrac{1}{4} \cdot \tfrac{100}{64} + \tfrac{1}{9} \cdot \tfrac{100}{49} + \sum_{p=3}^{2\frac{k-1}{2}} \tfrac{1}{p^2} \tfrac{k^2}{4}\} \\ &\leq 2\{\tfrac{100}{81} + \tfrac{1}{4} \cdot \tfrac{100}{64} + \tfrac{1}{9} \cdot \tfrac{100}{49} + 4[\sum_{p=1}^{\infty} \tfrac{1}{p^2} - (1 + \tfrac{1}{4} + \tfrac{1}{9})]\} \\ &\leq 2\{4 \cdot \tfrac{\pi}{6} - [5 + \tfrac{4}{9} - \tfrac{100}{81} - \tfrac{25}{64} - \tfrac{100}{441}]\} \approx 5.974 \leq 6, (k \geq 10) \end{aligned}$$

The proof of the lemma is complete. □

Let

$$\sum_{p=1}^{\infty} B_0^{(j)} = \min_{m \in Z} |\varepsilon i m + (-1)^j \alpha|, B_1^{(j)} = \|Z_1^{(j)}\|.$$

Lemma 3. If $6|\varepsilon|B_1^{(j)} \leq B_0^{(j)}$, then the series (12) is uniformly converges in $(-\infty, +\infty)$.

Proof. Assuming that for $p \in [1, k-1]$, the estimation

$$\|Z_p^{(j)}\| \leq B_1^{(j)}(d_j^{p-1})/p^2 \tag{13}$$

holds true, where $d_j = 6|\varepsilon|B_1^{(j)}/B_0^{(j)}$, we will prove that (13) is valid for each positive integer k.

By (11),(12)and Lemma 2, we have

$$\begin{aligned} \|Z_k^{(j)}\| &\leq \tfrac{1}{B_0^{(j)}} \|f_{(k)}\| \leq \tfrac{|\varepsilon|}{B_0^{(j)}} \sum_{p=1}^{k-1} \|Z_p^{(j)}\| \|Z_{k-p}^{(j)}\| \\ &\leq B_1^{(j)}(\tfrac{B_1^{(j)}}{B_0^{(j)}}) d_j^{k-2} (\sum_{p=1}^{k-1} (p^{-2})(k-p)^{-2}) \\ &\leq B_1^{(j)}(6\tfrac{B_1^{(j)}}{B_0^{(j)}}) d_j^{k-2}/k^2 = B_1^{(j)} d_j^{k-1}/k^2. \end{aligned}$$

Since (13) is obviously true for $k = 1$, it follows by induction that (13) is valid for every positive integer k.

Therefore, the series (12) converges in norm of A, that is, it converges absolutely and uniformly. □

Let $u^{(j)}(t)j = 1, 2$, be the limit function of the series (12).

Lemma 4. If $6|\varepsilon|B_1^{(j)} \leq B_0^{(j)}$, then $u^{(j)}(t), j = 1, 2$ is a 2π-periodic solution of equation (3).

Proof. Let

$$U_n^{(j)} = \sum_{k=0}^{n} Z_{(k)}^{(j)}, j = 1, 2.$$

We have

$$\varepsilon \dot{U}_n^{(j)} + \varepsilon \sum_{k=0}^{n} \sum_{p=0}^{k} Z_p^{(j)} Z_{k-p}^{j} + p_1 U_n^{(j)} = p_2 + c_2(t), n \geq 2.$$

Then

(14)

$$\begin{aligned} \varepsilon U_n^{(j)}(t) &= \varepsilon U_n^{j}(0) + \textstyle\int_0^t \{[p_2 + c_2(s)] - p_1 U_n^{(j)}(s) - \varepsilon \sum\limits_{k=0}^{n} \sum\limits_{p=0}^{k} Z_p^{(j)}(s) Z_{k-p}^{(j)}(s)\} ds \\ &= \varepsilon U_n^{(j)}(0) + \textstyle\int_0^t \{[p_2 + c_2(s)] - p_1 U_n^{(j)}(s) - \varepsilon[(U_n^{(j)}(s))^2 \\ &+ \sum\limits_{k=0}^{n} \sum\limits_{p=0}^{k} Z_p^{(j)}(s) Z_{k-p}^{j}(s) - (U_n^{(j)}(s))^2]\} ds \end{aligned}$$

One can check that

$$\begin{aligned} &|(\sum_{k=0}^{n} Z_k^(j))^2 - \sum_{k=0}^{n} \sum_{p=0}^{k} Z_p^{(j)} Z_{k-p}^{j}| \leq |\sum_{p=1}^{n} [Z_p^{(j)} (\sum_{k=0}^{p-1} Z_{n-k}^{(j)})]| \\ &\leq \sum_{p=1}^{n} [|Z_p^{j}| (\sum_{k=0}^{p-1} |Z_{n-k}^{(j)}|)] \\ &\leq \sum_{p=1}^{[\frac{n}{2}+1]} [|Z_p^{(j)}| (\sum_{k=0}^{p-1} |Z_{n-k}^{(j)}|)] + \sum_{p=[\frac{n}{2}+1]}^{n} [|Z_p^{(j)}| (\sum_{k=0}^{p-1} |Z_{n-k}^{(j)}|)] \\ &= (\sum_{p=1}^{n} |Z_p^{(j)}|)^2 - (\sum_{p=1}^{[\frac{n}{2}]} |Z_p^{(j)}|)^2 \\ &= (\sum_{p=1}^{\infty} |Z_p^{(j)}|)^2 - (\sum_{p=1}^{[\frac{n}{2}]} |Z_p^{(j)}|)^2 \end{aligned}$$

By Lemma 3 we have

$$(\sum_{p=1}^{\infty} |Z_p^{(j)}|)^2 - (\sum_{p=1}^{[\frac{n}{2}]} |Z_p^{(j)}|)^2 \to 0, \quad n \to \infty$$

It follows that

$$|(\sum_{p=1}^{\infty} |Z_p^{(j)}|)^2 - \sum_{k=0}^{n} \sum_{p=0}^{k} Z_p^{(j)} Z_{k-p}^{j}| \to 0, \quad n \to \infty$$

From (14), letting $n \to \infty$, we get

$$\varepsilon U_n^{(j)}(t) = \varepsilon U_n^{(j)}(0) + \int_0^t \{[p_2 + c_2(s)] - p_1 U_n^{(j)}(s) - \varepsilon (U^{(j)}(s))^2\} ds.$$

Hence, $U^{(j)}$ is a solution of equation (3), and it is obviously 2π-periodic. The proof is complete. $\square$

5. Approximate calculation of LCI

By Lemma 1 we assert that

$$\sum_{k=0}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} Re Z_k^{(j)}(t) dt, j = 1, 2,$$

is a LCI for equation (1), if $6|\varepsilon| B_1^{(j)} \leq B_0^{(j)}, j = 1, 2$. We can take

$$\sum_{k=0}^{M} \frac{1}{2\pi} \int_0^{2\pi} Re Z_k^{(j)}(t) dt, j = 1, 2, \tag{15}$$

as an approximate value of LCI.

Assume that

$$Z_k^j(t) = \sum_{|m|=0}^{\infty} Z_{k,m}^{(j)} \exp imt, k = 0, 1, 2, \cdots,$$

where

$$Z_{k,m}^{(j)} = \frac{1}{2\pi} \int_0^{2\pi} Z_k^{(j)} \exp imt(t) dt, j = 1, 2, k, m = 0, 1, 2, \cdots.$$

By (15) the approximate value of LCI is

$$Re(\sum_{k=0}^{M} Z_{k,0}^{(j)}). \tag{16}$$

From (8) we have $Z_{1,m}^{(j)} = 0$ for $|m| \in [1, N(\varepsilon)]$. Therefore, the sequence $\{Z_{1,m}^{(j)} : m \in \mathbf{Z}\}$ is known. From (11) we get

$$(\varepsilon i m + (-1)^j \alpha) Z_{k,m}^{(j)} = -\varepsilon \sum_{p=1}^{k-1} (\sum_{|l|=0}^{N(\varepsilon,p)} Z_{p,l}^{(j)} Z_{k-p,m-l}^{(j)}), k = 2, 3, \cdots. \tag{17}$$

Since there are only finite members deferent from zero in the sequence $\{Z_{1,m}^{(j)} : m \in \mathbf{Z}\}$, all sums in the formula (17) are finite. Thus, from the point of view of approximate calculation, it is not difficult to find all sequences $\{Z_{k,m}^{(j)} : m \in \mathbf{Z}\}, k = 2, 3, \cdots, M$, and then calculate the value of $Re(\sum_{k=0}^{M})$, the approximate value of LCI for equation (1).

Since equation (1) has two and only two LCIs, by our method we can find the approximate values of the LCIs of equation (1).

6. The error of approximate calculation

We assume that $d_j = 6|\varepsilon|B_1^{(j)} \leq B_0^{(j)} \leq 1, j = 1, 2$.

Lemma 5. $\sum_{p=M+1}^{\infty} p^{-2} \leq \frac{1}{M}$

Proof.

$$\sum_{p=M+1}^{\infty} p^{-2} = \sum_{p=M+1}^{\infty} \frac{1}{p^2} \leq \sum_{p=M+1}^{\infty} \frac{1}{(p-1)p} = \sum_{p=M+1}^{\infty} [\frac{1}{p-1} - \frac{1}{p}] = \frac{1}{M}$$

Since $\|Z_k^{(j)}\| \leq B_1^{(j)} d_j^{k-1}$, we have

$$|\sum_{k=0}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} ReZ_K^{(j)}(t)dt - \sum_{k=0}^{M} \frac{1}{2\pi} \int_0^{2\pi} ReZ_k^{(j)}(t)dt| \leq$$

$$\leq \sum_{k=M+1}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} |Z_k^{(j)}(t)|dt \leq \sum_{k=M+1}^{\infty} \|Z_K^{(j)}\|$$

$$\leq B_1^{(j)} d_j^M \sum_{p=M+1}^{\infty} p^{-2} \leq \frac{B_1^{(j)} d_j^M}{M}, j = 1, 2.$$

The error of approximate calculation is estimated. □

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