JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 19, No. 2, June 2006

MORITA EQUIVALENCE FOR HOMOGENEOUS C^*–ALGEBRAS OVER LOWER DIMENSIONAL SPHERES

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Abstract. All d-homogeneous C^*-algebras T^d over $\prod^{s_4} S^4 \times \prod^{s_2} S^2 \times \prod^{s_3} S^3 \times \prod^{s_1} S^1$ are constructed. It is shown that T^d are strongly Morita equivalent to $C(\prod^{s_4} S^4 \times \prod^{s_2} S^2 \times \prod^{s_3} S^3 \times \prod^{s_1} S^1)$.

1. Introduction

An important problem, in the bundle theory, is to compute the set $[M, BPU(d)]$ of homotopy classes of continuous maps of a compact CW-complex M into the classifying space $BPU(d)$ of the Lie group $PU(d)$. The set $[M, BPU(d)]$ is in bijective correspondence with the set of equivalence classes of principal $PU(d)$-bundles over M, which is in bijective correspondence with the set of d-homogeneous C^*-algebras over M (see [5, 7]).

In [4], the authors showed that two separable C^*-algebras A and B are stably isomorphic if and only if they are strongly Morita equivalent, i.e., there exists an A-B-equivalence bimodule defined in [8]. All d-homogeneous C^*-algebras T^d over $\prod^{s_2} S^2 \times \prod^{s_1} S^1$ were constructed in [1, Theorem 2.5], and it was shown in [1, Theorem 3.3] that T^d are strongly Morita equivalent to $C(\prod^{s_2} S^2 \times \prod^{s_1} S^1)$.

In this paper, all d-homogeneous C^*-algebras T^d over $\prod^{s_4} S^4 \times \prod^{s_2} S^2 \times \prod^{s_3} S^3 \times \prod^{s_1} S^1$ are constructed. It is shown that T^d are stably isomorphic to $C(\prod^{s_4} S^4 \times \prod^{s_2} S^2 \times \prod^{s_3} S^3 \times \prod^{s_1} S^1)$. Thus T^d are strongly Morita equivalent to $C(\prod^{s_4} S^4 \times \prod^{s_2} S^2 \times \prod^{s_3} S^3 \times \prod^{s_1} S^1)$. By modifying the construction given in [1, Theorem 3.3], we are going to construct a T^d-$C(\prod^{s_4} S^4 \times \prod^{s_2} S^2 \times \prod^{s_3} S^3 \times \prod^{s_1} S^1)$-equivalence bimodule.

Received by the editors on March 20, 2006.
2000 *Mathematics Subject Classifications*: Primary 46L05, 55R10.
Key words and phrases: equivalence bimodule.

2. Morita equivalence for homogeneous C^*-algebras over lower dimensional spheres

Krauss and Lawson [7, Proposition 2.10] proved that each d-homogeneous C^*-algebra over S^2 is isomorphic to one of the C^*-subalgebras $D_{d,k} = C_{g_k}(e_+^2 \amalg e_-^2, M_d(\mathbb{C}))$, $k \in \mathbb{Z}_d$, given as follows: if $f \in D_{d,k}$, then the following condition is satisfied

$$f_+(z) = U(z)^k f_-(z) U(z)^{-k}$$

for all $z \in S^1 \subset \mathbb{C}$, where $U(z) \in PU(d) = \mathrm{Inn}(M_d(\mathbb{C}))$ is defined as

$$U(z) = \begin{pmatrix} 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & z \end{pmatrix}$$

and e_+^n (resp. e_-^n) is the n-dimensional northern (resp. southern) hemisphere.

In [3, Theorem 3], an $A_{\frac{k}{d}}$-$C(\mathbb{T}^2)$-equivalence bimodule was constructed for $A_{\frac{k}{d}}$ a rational rotation algebra, and in [1, Theorem 3.3], by modifying the construction given in [3, Theorem 3], a $D_{d,k}$-$C(S^2)$-equivalence bimodule was constructed. But using a slightly different trick, we are going to construct a $D_{d,k}$-$C(S^2)$-equivalence bimodule.

Lemma 1. *Each d-homogeneous C^*-algebra $D_{d,k}$ over S^2 is strongly Morita equivalent to $C(S^2)$.*

Proof. Let $V(z)^k = \begin{pmatrix} 1 & 0 \\ 0 & z^k \end{pmatrix}$. The d-homogeneous C^*-algebra $D_{d,k}$ over S^2 can be realized as the C^*-algebra of matrices $(f_{ij})_{i,j=1}^d$ of functions f_{ij} with

$$f_{ij} \in C(S^2) \text{ if } i,j \in \{1, \cdots, d-2\}$$
$$\begin{pmatrix} f_{(d-1)i} \\ f_{di} \end{pmatrix} \in \Omega \quad \& \quad \begin{pmatrix} f_{i(d-1)} & f_{id} \end{pmatrix} \in \Omega^* \text{ if } i \in \{1, \cdots, d-2\}$$
$$\begin{pmatrix} f_{(d-1)(d-1)} & f_{(d-1)d} \\ f_{d(d-1)} & f_{dd} \end{pmatrix} \in \Omega_0,$$

where Ω, Ω^* and Ω_0 are the $C(S^2)$-modules defined as

$$\begin{aligned}\Omega =&\{\begin{pmatrix} f \\ g \end{pmatrix} \mid f, g \in C((e^2_+ \amalg e^2_-)), (\begin{pmatrix} f \\ g \end{pmatrix} |_{e^2_+ \amalg e^2_-})_+(z) \\ &\qquad = V(z)^k (\begin{pmatrix} f \\ g \end{pmatrix} |_{e^2_+ \amalg e^2_-})_-(z) \quad \forall z \in S^1\} \\ \Omega^* =&\{(\, f \quad g\,) \mid f, g \in C((e^2_+ \amalg e^2_-)), \quad (\, f \quad g\,)^* \in \Omega\} \\ \Omega_0 =&\{\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \mid f_1, f_2, f_3, f_4 \in C((e^2_+ \amalg e^2_-)), (\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} |_{e^2_+ \amalg e^2_-})_+(z) \\ &\qquad = V(z)^k (\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} |_{e^2_+ \amalg e^2_-})_-(z) V(z)^{-k} \quad \forall z \in S^1\}.\end{aligned}$$

Let X be the complex vector space $\left(\oplus_1^{d-2} C(S^2)\right) \oplus \Omega$. We will consider the elements of X as $(d, 1)$ matrices where the first $(d-2)$ entries are in $C(S^2)$ and the last 2 entries are in Ω. If $x \in X$, denote by x^* the $(1, d)$ matrix resulting from x by transposition and involution so that $x^* \in \left(\oplus_1^{d-2} C(S^2)\right) \oplus \Omega^*$. The space X is a left $D_{d,k}$-module if module multiplication is defined by matrix multiplication $F \cdot x$, where $F = \left(g_{ij}\right)_{i,j=1}^d \in D_{d,k}$ and $x \in X$. If $g \in C(S^2)$ and $x \in X$, then $x \cdot g$ defines a right $C(S^2)$-module structure on X. Now we define a $D_{d,k}$-valued inner product $\langle \cdot, \cdot \rangle_{D_{d,k}}$ on X and a $C(S^2)$-valued inner product $\langle \cdot, \cdot \rangle_{C(S^2)}$ on X by

$$\langle x, y \rangle_{D_{d,k}} = x \cdot y^* \quad \text{and} \quad \langle x, y \rangle_{C(S^2)} = x^* \cdot y$$

for all $x, y \in X$, we have the matrix multiplication on the right. By the same reasoning as the proof given in [3, Theorem 3], equipped with this structure, X becomes a $D_{d,k}$-$C(S^2)$-equivalence bimodule, and hence $D_{d,k}$ is strongly Morita equivalent to $C(S^2)$, as desired. □

Krauss and Lawson [7, Proposition 2.13] proved that each d-homogeneous C^*-algebra over S^4 is isomorphic to one of the C^*-subalgebras $B_{d,k} = C_{g_k}(e^4_+ \amalg e^4_-, M_d(\mathbb{C}))$, $k \in \mathbb{Z}$, given as follows: if $f \in B_{d,k}$, then the following condition is satisfied

$$f_+(z, w) = U(z, w)^k f_-(z, w) U(z, w)^{-k}$$

for all $(z,w) \in S^3 \subset \mathbb{C}^2$, where $U(z,w) \in PU(d) = \mathrm{Inn}(M_d(\mathbb{C}))$ is defined as

$$U(z,w) = \begin{pmatrix} 1 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & 0 \\ 0 & \dots & 0 & \overline{z} & w \\ 0 & \dots & 0 & -\overline{w} & z \end{pmatrix}.$$

Each d-homogeneous C^*-algebra over $S^3 \times S^1$ corresponding to each element of $[S^3 \times S^1,\, BPU(d)]$ can be constructed.

LEMMA 2. *Each d-homogeneous C^*-algebra over $S^3 \times S^1$ is isomorphic to one of the following C^*-subalgebras $A_{d,a}$, $k \in \mathbb{Z}$, of $C(S^3 \times [0,1],\, M_d(\mathbb{C}))$ given as follows: if $f \in A_{d,k}$, then the following condition is satisfied*

$$f((z,w),1) = U(z,w)^k f((z,w),0) U(z,w)^{-k}$$

for all $(z,w) \in S^3$, where $U(z,w) \in PU(d)$ is the unitary given above.

Proof. By the Woodward theorem [9], $[S^3 \times S^1, BPU(d)]$ is embedded into $H^2(S^3 \times S^1, \mathbb{Z}_d) \oplus H^4(S^3 \times S^1, \mathbb{Z}) \cong \mathbb{Z}$. Since there is a map of degree 1 from $S^3 \times S^1$ to S^4, the composite of the map of degree 1 and the map representing each element of $[S^4, BPU(d)]$ gives an element of $[S^3 \times S^1, BPU(d)]$. Hence each element of $[S^4, BPU(d)] \cong [S^3, PU(d)]$ representing a d-homogeneous C^*-algebra over S^4 induces an element of $[S^3, PU(d)] \subset [S^3 \times S^1, BPU(d)]$, and the d-homogeneous C^*-algebras $A_{d,k}$ over $S^3 \times S^1$ corresponding to the d-homogeneous C^*-algebras $B_{d,k}$ over S^4 are constructed in the statement, as desired. □

For a d-homogeneous C^*-algebra A over S^5 there is a matrix algebra $M_q(\mathbb{C})$ such that $A \otimes M_q(\mathbb{C})$ is isomorphic to $C(S^5) \otimes M_{dq}(\mathbb{C})$. Since there is a map of degree 1 from S^5 to $S^4 \times S^1$, there are d-homogeneous C^*-algebras over $S^4 \times S^1$ induced from d-homogeneous C^*-algebras over S^5. Also there are d-homogeneous C^*-algebras over $S^4 \times S^1$ induced from d-homogeneous C^*-algebras over S^4. But the tensor product of each d-homogeneous C^*-algebra over $S^4 \times S^1$ induced from a d-homogeneous C^*-algebra over S^5 with $M_q(\mathbb{C})$ has the trivial bundle structure for some integer q greater than 1 since $[S^5, BPU(dq)] \cong \{0\}$ (see [7]). And by the Woodward theorem [9] $[S^2 \times S^1, BPU(d)]$ is embedded into $H^2(S^2 \times S^1, \mathbb{Z}_d) \oplus H^4(S^2 \times S^1, \mathbb{Z}) \cong H^2(S^2 \times S^1, \mathbb{Z}_d) \cong H^2(S^2, \mathbb{Z}_d) \cong \mathbb{Z}_d$. This implies that each d-homogeneous C^*-algebra over $S^2 \times S^1$ is isomorphic to the tensor product of a d-homogeneous C^*-algebra over S^2 with $C(S^1)$. From now on, we assume that each d-homogeneous C^*-algebra over $S^4 \times S^1$ is isomorphic to the tensor product of a d-homogeneous C^*-algebra over S^4 with $C(S^1)$, and that each d-homogeneous C^*-algebra over $S^2 \times S^1$ is isomorphic to the tensor product of a d-homogeneous C^*-algebra over S^2 with $C(S^1)$.

Lemma 3. *Let n and m be integers greater than 1. Each d-homogeneous C^*-algebra over $S^n \times S^m$ is isomorphic to a d-homogeneous C^*-algebra characterized by the unitary $U(Z)^a$ over S^{n-1} in a d-homogeneous C^*-algebra B_p over $e^n_+ \times S^m$ and $e^n_- \times S^m$, where $U(Z) \in PU(d)$ or $PU(p)$ if $M_p(\mathbb{C})$ is factored out of B_p.*

Proof. Since e^n_+ and e^n_- are contractible, each d-homogeneous C^*-algebra over $e^n_+ \times S^m$ and $e^n_- \times S^m$ is essentially induced by a d-homogeneous C^*-algebra over S^m. Each d-homogeneous C^*-algebra over $S^n \times S^m$ is characterized by a projective unitary over the boundaries $S^{n-1} \times S^m$ of $e^n_+ \times S^m$ and $e^n_- \times S^m$. But $\pi_1(S^n) = \{0\}$ and so the identification of the boundaries $S^m \hookrightarrow e^n_+ \times S^m$ and $S^m \hookrightarrow e^n_- \times S^m$ does give the trivial bundle structure. Hence the d-homogeneous C^*-algebra over $S^n \times S^m$ is characterized by the unitary $U(Z)^a$, $a \in \mathbb{Z}$ or $a \in \mathbb{Z}_d$, over S^{n-1} in the d-homogeneous C^*-algebra over $e^n_+ \times S^m$ and $e^n_- \times S^m$, where $U(Z) \in PU(d)$ or $PU(p)$. □

All d-homogeneous C^*-algebras T^d over $\prod^{s_4+s_2} S^2 \times \prod^{s_3+s_1} S^1$ were constructed in [1, Theorem 2.5]. Combining Lemma 2 and Lemma 3 yields that each d-homogeneous C^*-algebra over $\prod^{s_4} S^4 \times \prod^{s_2} S^2 \times \prod^{s_3} S^3 \times \prod^{s_1} S^1$ can be constructed by canonically replacing some d-homogeneous C^*-subalgebras over S^2 and some d-homogeneous C^*-subalgebras over $S^1 \times S^1$ in T^d with d-homogeneous C^*-algebras over S^4 and d-homogeneous C^*-algebras over $S^3 \times S^1$. If $s_3 + s_1$ is odd, then one can make the integer even by tensoring with $C(S^1)$. So one can assume that $s_3 + s_1$ is even, and that s_1 is greater than or equals to s_3 and big enough if needed.

THEOREM 1. *Each d-homogeneous C^*-algebra T^d over $\prod^{s_4} S^4 \times \prod^{s_2} S^2 \times \prod^{s_3} S^3 \times \prod^{s_1} S^1$ is isomorphic to one of the C^*-subalgebras $E^{(a)(b)}_{(c)(r)}$ of*

$$C(\prod^{s_4}(e^4_+ \amalg e^4_-) \times \prod^{s_2}(e^2_+ \amalg e^2_-) \times \prod^{s_3}(S^3 \times [0,1]) \times \prod^{\frac{s_1-s_3}{2}} (S^1 \times [0,1]), M_d(\mathbb{C}))$$

consisting of those functions f that satisfy

$$\begin{aligned}
(f|_{e^4_+ \amalg e^4_-})_+(Z_i) &= U(Z_i)^{a_i}(f|_{e^4_+ \amalg e^4_-})_-(Z_i)U(Z_i)^{-a_i}\\
(f|_{e^2_+ \amalg e^2_-})_+(w_i) &= U(w_i)^{b_i}(f|_{e^2_+ \amalg e^2_-})_-(w_i)U(w_i)^{-b_i}\\
(f|_{S^3\times[0,1]})(X_i,1) &= U(X_i)^{c_i}(f|_{S^3\times[0,1]})(X_i,0)U(X_i)^{-c_i}\\
(f|_{S^1\times[0,1]})(y_i,1) &= U(y_i)^{r_i}(f|_{S^1\times[0,1]})(y_i,0)U(y_i)^{-r_i}
\end{aligned}$$

for all $(Z_1,\cdots,Z_{s_4},w_1,\cdots,w_{s_2},X_1,\cdots,X_{s_3},y_1,\cdots,y_{\frac{s_1-s_3}{2}}) \in \prod^{s_4} S^3 \times \prod^{s_2} S^1 \times \prod^{s_3} S^3 \times \prod^{\frac{s_1-s_3}{2}} S^1$, one of the tensor products of homogeneous C^*-algebras of the type above, or one of the C^*-algebras given by replacing $C(\prod^{s_4} S^4 \times \prod^{s_2} S^2 \times \prod^{s_3} S^3 \times \prod^{s_1} S^1)$ in $E^{(a)(b)}_{(c)(r)}$ or the tensor products with suitable d'-homogeneous C^*-algebras in the same sense as above, when $M_{d'}(\mathbb{C})$ are factored out of $E^{(a)(b)}_{(c)(r)}$ or the tensor products, where $(a) = (a_1,\cdots,a_{s_4}) \in \mathbb{Z}^{s_4}$, $(b) = (b_1,\cdots,b_{s_2}) \in \mathbb{Z}^{s_2}$, $(c) = (c_1,\cdots,c_{s_3}) \in \mathbb{Z}^{s_3}$, and $(r) = (r_1,\cdots,r_{\frac{s_1-s_3}{2}}) \in \mathbb{Z}^{\frac{s_1-s_3}{2}}$ and $U(Z_i), U(w_i), U(X_i)$ and $U(y_i) \in PU(d)$ are as defined before.

Proof. By Lemma 2, each d-homogeneous C^*-algebra over $S^3 \times S^1$ corresponds to a d-homogeneous C^*-algebra over S^4. By Lemma 3, each d-homogeneous C^*-algebra over the product space of two even dimensional spheres can be constructed. Combining Lemma 2 and Lemma 3 yields that replacing S^4 and S^3 with S^2 and S^1 does not give any change in the relation, associated with bundle structure, among the factors of $\prod^{s_4} S^4 \times \prod^{s_2} S^2 \times \prod^{s_3} S^3 \times \prod^{s_1} S^1$, and among the factors of $\prod^{s_4+s_2} S^2 \times \prod^{s_3+s_1} S^1$. Hence each d-homogeneous C^*-algebra over $\prod^{s_4} S^4 \times \prod^{s_2} S^2 \times \prod^{s_3} S^3 \times \prod^{s_1} S^1$

can be given by [1, Theorem 2.5], which is exactly the same as the statement for the case $s_4 = 0$ and $s_3 = 0$. □

For a compact CW-complex M, the Čech cohomology group $H^3(M,\mathbb{Z})$ classifies the tensor products of d-homogeneous C^*-algebras over M with the C^*-algebra $\mathcal{K}(\mathcal{H})$ of compact operators on a separable Hilbert space $\mathcal{H}$ (see [6]). The Čech cohomology group $H^3(M,\mathbb{Z})$ is isomorphic to the singular cohomology group $H^3(M,\mathbb{Z})$, when M is a triangularizable CW-complex (see [2, Theorem 15.8]).

THEOREM 2. *Each d-homogeneous C^*-algebra over $\prod^{s_4} S^4 \times \prod^{s_2} S^2 \times \prod^{s_3} S^3 \times \prod^{s_1} S^1$ is stably isomorphic to $C(\prod^{s_4} S^4 \times \prod^{s_2} S^2 \times \prod^{s_3} S^3 \times \prod^{s_1} S^1)$.*

Proof. Each non-trivial element in the cohomology group $H^3(\prod^{s_4} S^4 \times \prod^{s_2} S^2 \times \prod^{s_3} S^3 \times \prod^{s_1} S^1, \mathbb{Z})$ is given by a non-trivial element in $H^3((S^1)^3,\mathbb{Z})$, $H^3(S^2 \times S^1,\mathbb{Z})$, or $H^3(S^3,\mathbb{Z})$ if there exist such factors.

$H^3(S^2 \times S^1,\mathbb{Z}) \cong \mathbb{Z}$. By the Woodward theorem [9], $[S^2 \times S^1, BPU(d)]$ is embedded into $H^2(S^2 \times S^1,\mathbb{Z}_d) \oplus H^4(S^2 \times S^1,\mathbb{Z}) \cong H^2(S^2 \times S^1,\mathbb{Z}_d) \cong H^2(S^2,\mathbb{Z}_d) \cong \mathbb{Z}_d$. So each d-homogeneous C^*-algebra over $S^2 \times S^1$ is isomorphic to the tensor product of a d-homogeneous C^*-algebra over S^2 with $C(S^1)$, which is stably isomorphic to $C(S^2) \otimes C(S^1) \otimes M_d(\mathbb{C})$, since $H^3(S^2,\mathbb{Z}) = \{0\}$. Thus each d-homogeneous C^*-algebra over $S^2 \times S^1$ is stably isomorphic to $C(S^2 \times S^1) \otimes M_d(\mathbb{C})$.

Similarly, one can obtain the same result for the other cases.

Hence each d-homogeneous C^*-algebra over $\prod^{s_4} S^4 \times \prod^{s_2} S^2 \times \prod^{s_3} S^3 \times \prod^{s_1} S^1$ is stably isomorphic to $C(\prod^{s_4} S^4 \times \prod^{s_2} S^2 \times \prod^{s_3} S^3 \times \prod^{s_1} S^1)$. □

For each $k \in \mathbb{Z}$, let

$$U(z,w)^k = \begin{pmatrix} 1 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & 0 \\ 0 & \dots & 0 & \overline{z} & w \\ 0 & \dots & 0 & -\overline{w} & z \end{pmatrix}^k = \begin{pmatrix} 1 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & 0 \\ 0 & \dots & 0 & \overline{g_k(z,w)} & h_k(z,w) \\ 0 & \dots & 0 & -\overline{h_k(z,w)} & g_k(z,w) \end{pmatrix}$$

for all $(z,w) \in S^3 \subset \mathbb{C}^2$. Let $V(z,w)^k = \begin{pmatrix} \overline{g_k(z,w)} & h_k(z,w) \\ -\overline{h_k(z,w)} & g_k(z,w) \end{pmatrix}$.

LEMMA 4. *Each d-homogeneous C^*-algebra $B_{d,k}$ over S^4 is strongly Morita equivalent to $C(S^4)$.*

Proof. The d-homogeneous C^*-algebra $B_{d,k}$ over S^4 can be realized as the C^*-algebra of matrices $(f_{ij})_{i,j=1}^d$ of functions f_{ij} with

$$f_{ij} \in C(S^4) \text{ if } i,j \in \{1,\cdots,d-2\}$$

$$\begin{pmatrix} f_{(d-1)i} \\ f_{di} \end{pmatrix} \in \Omega \quad \& \quad \begin{pmatrix} f_{i(d-1)} & f_{id} \end{pmatrix} \in \Omega^* \text{ if } i \in \{1,\cdots,d-2\}$$

$$\begin{pmatrix} f_{(d-1)(d-1)} & f_{(d-1)d} \\ f_{d(d-1)} & f_{dd} \end{pmatrix} \in \Omega_0,$$

where Ω, Ω^* and Ω_0 are the $C(S^4)$-modules defined as

$$\begin{aligned}
\Omega =& \{\begin{pmatrix} f \\ g \end{pmatrix} \mid f,g \in C((e_+^4 \amalg e_-^4)), \\
& (\begin{pmatrix} f \\ g \end{pmatrix}|_{e_+^4 \amalg e_-^4})_+(z,w) = V(z,w)^k (\begin{pmatrix} f \\ g \end{pmatrix}|_{e_+^4 \amalg e_-^4})_-(z,w) \quad \forall (z,w) \in S^3\} \\
\Omega^* =& \{(f \quad g) \mid f,g \in C((e_+^4 \amalg e_-^4)), \quad (f \quad g)^* \in \Omega\} \\
\Omega_0 =& \{\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \mid f_1,f_2,f_3,f_4 \in C((e_+^4 \amalg e_-^4)), (\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}|_{e_+^4 \amalg e_-^4})_+(z,w) \\
& = V(z,w)^k (\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}|_{e_+^4 \amalg e_-^4})_-(z,w) V(z,w)^{-k} \quad \forall (z,w) \in S^3\}.
\end{aligned}$$

Let X be the complex vector space $\left(\oplus_1^{d-2} C(S^4)\right) \oplus \Omega$. We will consider the elements of X as $(d,1)$ matrices where the first $(d-2)$ entries are in $C(S^4)$ and the last 2 entries are in Ω. If $x \in X$, denote by x^* the $(1,d)$ matrix resulting from x by transposition and involution so that $x^* \in \left(\oplus_1^{d-2} C(S^4)\right) \oplus \Omega^*$. The space X is a left $B_{d,k}$-module if module multiplication is defined by matrix multiplication $F \cdot x$, where $F = \left(g_{ij}\right)_{i,j=1}^d \in B_{d,k}$ and $x \in X$. If $g \in C(S^4)$ and $x \in X$, then $x \cdot g$ defines a right $C(S^4)$-module structure on X. Now we define a $B_{d,k}$-valued inner product $\langle \cdot,\cdot \rangle_{B_{d,k}}$ on X and a $C(S^4)$-valued inner product $\langle \cdot,\cdot \rangle_{C(S^4)}$ on X by

$$\langle x,y \rangle_{B_{d,k}} = x \cdot y^* \quad \text{and} \quad \langle x,y \rangle_{C(S^4)} = x^* \cdot y$$

for all $x, y \in X$, we have the matrix multiplication on the right. By the same reasoning as the proof given in [3, Theorem 3], equipped with this structure, X becomes a $B_{d,k}$-$C(S^4)$-equivalence bimodule, and hence $B_{d,k}$ is strongly Morita equivalent to $C(S^4)$, as desired. $\square$

THEOREM 3. *Each d-homogeneous C^*-algebra T^d over $\prod^{s_4} S^4 \times \prod^{s_2} S^2 \times \prod^{s_3} S^3 \times \prod^{s_1} S^1$ is strongly Morita equivalent to $C(\prod^{s_4} S^4 \times \prod^{s_2} S^2 \times \prod^{s_3} S^3 \times \prod^{s_1} S^1)$.*

Proof. Let $S = \prod^{s_4} S^4 \times \prod^{s_2} S^2 \times \prod^{s_3} S^3 \times \prod^{s_1} S^1$, and $M = \prod^{s_4}(e^4_+ \amalg e^4_-) \times \prod^{s_2}(e^2_+ \amalg e^2_-) \times \prod^{s_3}(S^3 \times [0,1]) \times \prod^{\frac{s_1 - s_3}{2}}(S^1 \times [0,1])$. The d-homogeneous C^*-algebra $E^{(a)(b)}_{(c)(r)}$ over S can be realized as the C^*-algebra of matrices $(f_{ij})^d_{i,j=1}$ of functions f_{ij} with

$$f_{ij} \in C(S) \text{ if } i, j \in \{1, \cdots, d-2\}$$
$$\begin{pmatrix} f_{(d-1)i} \\ f_{di} \end{pmatrix} \in \Gamma \quad \& \quad \begin{pmatrix} f_{i(d-1)} & f_{id} \end{pmatrix} \in \Gamma^* \text{ if } i \in \{1, \cdots, d-2\}$$
$$\begin{pmatrix} f_{(d-1)(d-1)} & f_{(d-1)d} \\ f_{d(d-1)} & f_{dd} \end{pmatrix} \in \Gamma_0,$$

where Γ, Γ^* and Γ_0 are the $C(S)$-modules defined as

$$\begin{aligned}
\Gamma = \{ &\begin{pmatrix} f \\ g \end{pmatrix} \mid f, g \in C(M), \quad \forall i \\
&(\begin{pmatrix} f \\ g \end{pmatrix} |_{e^4_+ \amalg e^4_-})_+(Z_i) = V(Z_i)^{a_i}(\begin{pmatrix} f \\ g \end{pmatrix} |_{e^4_+ \amalg e^4_-})_-(Z_i), \quad \forall Z_i \in S^3 \\
&(\begin{pmatrix} f \\ g \end{pmatrix} |_{e^2_+ \amalg e^2_-})_+(w_i) = V(w_i)^{b_i}(\begin{pmatrix} f \\ g \end{pmatrix} |_{e^2_+ \amalg e^2_-})_-(w_i), \quad \forall w_i \in S^1 \\
&(\begin{pmatrix} f \\ g \end{pmatrix} |_{S^3 \times [0,1]})(X_i, 1) = V(X_i)^{c_i}(\begin{pmatrix} f \\ g \end{pmatrix} |_{S^3 \times [0,1]})(X_i, 0), \quad \forall X_i \in S^3 \\
&(\begin{pmatrix} f \\ g \end{pmatrix} |_{S^1 \times [0,1]})(y_i, 1) = V(y_i)^{r_i}(\begin{pmatrix} f \\ g \end{pmatrix} |_{S^1 \times [0,1]})(y_i, 0), \quad \forall y_i \in S^1 \} \\
\Gamma^* = \{ &\begin{pmatrix} f & g \end{pmatrix} \mid f, g \in C(M), \quad \begin{pmatrix} f & g \end{pmatrix}^* \in \Gamma \}
\end{aligned}$$

$$\Gamma_0 = \{ \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \mid f_1, f_2, f_3, f_4 \in C(M), \quad \forall i$$

$$(\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} |_{e_+^4 \amalg e_-^4})_+(Z_i)$$

$$= V(Z_i)^{a_i}(\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} |_{e_+^4 \amalg e_-^4})_-(Z_i)V(Z_i)^{-a_i}, \forall Z_i \in S^3$$

$$(\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} |_{e_+^2 \amalg e_-^2})_+(w_i)$$

$$= V(w_i)^{b_i}(\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} |_{e_+^2 \amalg e_-^2})_-(w_i)V(w_i)^{-b_i}, \forall w_i \in S^1$$

$$(\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} |_{S^3 \times [0,1]})(X_i, 1)$$

$$= V(X_i)^{c_i}(\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} |_{S^3 \times [0,1]})(X_i, 0)V(X_i)^{-c_i}, \forall X_i \in S^3$$

$$(\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} |_{S^1 \times [0,1]})(y_i, 1)$$

$$= V(y_i)^{r_i}(\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} |_{S^1 \times [0,1]})(y_i, 0)V(y_i)^{-r_i}, \forall y_i \in S^1\}.$$

Let Y be the complex vector space $\left(\oplus_1^{d-2} C(S)\right) \oplus \Gamma$. We will consider the elements of Y as $(d, 1)$ matrices where the first $(d-2)$ entries are in $C(S)$ and the last 2 entries are in Γ. If $y \in Y$, denote by y^* the $(1, d)$ matrix resulting from y by transposition and involution so that $y^* \in \left(\oplus_1^{d-2} C(S)\right) \oplus \Gamma^*$. The space Y is a left $E_{(c)(r)}^{(a)(b)}$-module if module multiplication is defined by matrix multiplication $F \cdot y$, where $F = \left(\, g_{ij} \,\right)_{i,j=1}^{d} \in E_{(c)(r)}^{(a)(b)}$ and $y \in Y$. If $g \in C(S)$ and $y \in Y$, then $y \cdot g$ defines a right $C(S)$-module structure on Y. Now we define an $E_{(c)(r)}^{(a)(b)}$-valued inner product $\langle \cdot, \cdot \rangle_{E_{(c)(r)}^{(a)(b)}}$ on Y and a $C(S)$-valued inner product $\langle \cdot, \cdot \rangle_{C(S)}$ on Y by

$$\langle x, y \rangle_{E_{(c)(r)}^{(a)(b)}} = x \cdot y^* \quad \text{and} \quad \langle x, y \rangle_{C(S)} = x^* \cdot y$$

for all $x, y \in Y$, we have the matrix multiplication on the right. By the same reasoning as the proof given in [1, Theorem 3.3], equipped with this structure, Y becomes an $E_{(c)(r)}^{(a)(b)}$-$C(S)$-equivalence bimodule, and hence $E_{(c)(r)}^{(a)(b)}$ is strongly Morita equivalent to $C(S)$. By a finite step of the above process, one can obtain the result.

Therefore, each C^*-algebra T^d over $\prod^{s_4} S^4 \times \prod^{s_2} S^2 \times \prod^{s_3} S^3 \times \prod^{s_1} S^1$ is strongly Morita equivalent to $C(\prod^{s_4} S^4 \times \prod^{s_2} S^2 \times \prod^{s_3} S^3 \times \prod^{s_1} S^1)$. □

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