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UNIT KILLING VECTORS AND HOMOGENEOUS GEODESICS ON SOME LIE GROUPS

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Abstract. We find unit Killing vectors and homogeneous geodesics on the Lie group with Lie algebra $\mathfrak{a} \oplus_P \mathfrak{r}$, where $\mathfrak{a}$ and $\mathfrak{r}$ are abelian Lie algebra of dimension n and 1, respectively.

1. Introduction

Let $\mathfrak{a}$ and $\mathfrak{r}$ be abelian Lie algebras of dimension n and 1, respectively (so that $\mathfrak{a} = \mathbb{R}^n$ and $\mathfrak{r} = \mathbb{R}$). Let

$$P = (p_{ij}) \in \mathfrak{gl}(n, \mathbb{R})$$

be any real $(n \times n)$-matrix. A homomorphism $\varphi : \mathfrak{r} \to \mathrm{Endo}(\mathfrak{a})$ can be defined by

$$\varphi(\alpha)(x) = \alpha P x$$

for $\alpha \in \mathfrak{r}$ and $x \in \mathfrak{a}$.

One can form a semi-direct product of the Lie algebra $\mathfrak{a}$ by $\mathfrak{r}$ as follows: The underlying linear space is the direct sum $\mathfrak{a} \oplus \mathfrak{r}$, and the bracket operation is given by

$$[(a, \alpha), (b, \beta)] = (\varphi(\alpha)b - \varphi(\beta)a, [\alpha, \beta]) = (\varphi(\alpha)b - \varphi(\beta)a, 0).$$

It is trivial to see that this does satisfy the skew-symmetry and the Jacobi identity. We denote this new Lie algebra by $\mathfrak{a} \oplus_P \mathfrak{r}$.

Clearly, if the matrix P is nilpotent, then $\mathfrak{a} \oplus_P \mathfrak{r}$ is nilpotent. (If P is the zero matrix, then $\mathfrak{a} \oplus_P \mathfrak{r}$ is abelian). If P has trace 0, then $\mathfrak{a} \oplus_P \mathfrak{r}$ is unimodular. Otherwise, always $\mathfrak{a}$ will be the unimodular kernel. Moreover, if the matrix P is the identity matrix, then the associated simply

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connected Lie group is isometric to the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$([6]). This is not unimodular.

There are many geometrically interesting vector fields. In this paper we find all unit Killing vector fields and homogeneous geodesic vectors on the Lie group with Lie algebra $\mathfrak{a} \oplus_P \mathfrak{r}$.

Basic calculations are given in section 2. In section 3 we find the set of unit Killing vector fields and in the last section we give the set of homogeneous geodesic vectors.

2. Basic Calculations

Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{a} \oplus_P \mathfrak{r}$ defined in the introduction. Put $E_i = (0, \cdots, 1, \cdots, 0) \in \mathbb{R}^{n+1}$ and let $\{E_1, \cdots, E_{n+1}\}$ be orthonormal basis for $\mathfrak{g}$ and equip the left invariant metric on the associated Lie group with the Lie algebra $\mathfrak{g}$. Then we have the following.

PROPOSITION 2.1. *For $1 \leq i, j \leq n$, we have*

1. $[E_i, E_j] = 0$.
2. $[E_{n+1}, E_i] = \sum_{j=1}^{n} p_{ji} E_j$.
3. $[E_{n+1}, E_{n+1}] = 0$.

Let α_{ijk} be defined by

$$[E_i, E_j] = \sum_{k=1}^{n+1} \alpha_{ijk} E_k.$$

Then we have the following.

PROPOSITION 2.2. *For $1 \leq i, j, k \leq n$, we have*

1. $\alpha_{ijk} = 0$.
2. $\alpha_{(n+1)jk} = -\alpha_{j(n+1)k} = p_{kj}$.
3. $\alpha_{(n+1)j(n+1)} = \alpha_{(n+1)(n+1)k} = 0$.

By using the equation

$$\nabla_{E_i} E_j = \sum_{k=1}^{n+1} \frac{1}{2}(\alpha_{ijk} - \alpha_{jki} + \alpha_{kij}) E_k,$$

we have the following.

PROPOSITION 2.3. *For $1 \leq i, j \leq n$, we have*

1. $\nabla_{E_i} E_j = \frac{1}{2}(p_{ij} + p_{ji})E_{n+1}$.
2. $\nabla_{E_i} E_{n+1} = -\frac{1}{2}\sum_{k=1}^{n}(p_{ki} + p_{ik})E_k$.
3. $\nabla_{E_{n+1}} E_i = \frac{1}{2}\sum_{k=1}^{n}(p_{ki} - p_{ik})E_k$.
4. $\nabla_{E_{n+1}} E_{n+1} = 0$.

Thus we have the followings.

$$\nabla E_i = \frac{1}{2}\sum_{j=1}^{n}(p_{ij} + p_{ji})E_{n+1} \otimes \theta_j + \frac{1}{2}\sum_{j=1}^{n}(p_{ji} - p_{ij})E_j \otimes \theta_{n+1}$$

$$\nabla E_{n+1} = -\frac{1}{2}\sum_{j=1}^{n}\left(\sum_{i=1}^{n}(p_{ij} + p_{ji})E_i\right) \otimes \theta_j$$

As for the sectional curvature κ, we have the following identity([5]).

$$\kappa(E_i, E_j) = \sum\{\frac{1}{2}\alpha_{ijk}(-\alpha_{ijk} + \alpha_{jki} + \alpha_{kij}) \\ -\frac{1}{4}(\alpha_{ijk} - \alpha_{jki} + \alpha_{kij})(\alpha_{ijk} + \alpha_{jki} - \alpha_{kij}) - \alpha_{kii} \cdot \alpha_{kjj}\}.$$

Thus we have the following.

PROPOSITION 2.4. *For $1 \leq i, j \leq n$, we have*

1. $\kappa(E_i, E_j) = \frac{1}{4}(p_{ij} + p_{ji})^2 - p_{ii}^2 \cdot p_{jj}^2$,
2. $\kappa(E_i, E_{n+1}) = -\frac{1}{4}\sum_{k=1}^{n}(p_{ki} + p_{ik}) \cdot (3p_{ki} - p_{ik})$.

3. Unit Killing vector fields

A unit vector field V is Killing if and only if $A_V = -\nabla V$ is skew-symmetric. In [1] it is shown that the vector E_{n+1} is Killing vector if and only if the matrix P is skew-symmetric. Now we want to find all unit Killing vectors.

For $V = \sum_{i=1}^{n+1} x_i E_i$, $\quad \sum_{i=1}^{n+1} x_i^2 = 1$, we have the followings.

$$\begin{aligned}
\nabla V &= \sum_{i=1}^{n+1} x_i \nabla E_i \\
&= \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} \frac{1}{2}(p_{ij}+p_{ji}) E_{n+1} \otimes \theta_j + \sum_{j=1}^{n} \frac{1}{2}(p_{ji}-p_{ij}) E_j \otimes \theta_{n+1} \right) \\
&- x_{n+1} \sum_{i=1}^{n} \frac{1}{2} \left(\sum_{j=1}^{n} (p_{ij}+p_j i) E_j \otimes \theta_i \right) \\
&= \frac{1}{2} \sum_{i,j=1}^{n} x_i (p_{ij}+p_{ji}) E_{n+1} \otimes \theta_j + \frac{1}{2} \sum_{i,j=1}^{n} x_i(-p_{ij}+p_{ji}) E_j \otimes \theta_{n+1} \\
&- \frac{1}{2} x_{n+1} \sum_{i,j=1}^{n} (p_{ij}+p_{ji}) E_j \otimes \theta_i .
\end{aligned}$$

So the necessary and sufficient condition for V to be unit Killing, i.e. $A_V = -\nabla V$ is skew-symmetric, is as follows.

$$\begin{aligned}
x_{n+1}(p_{ij}+p_{ji}) &= -x_{n+1}(p_{ij}+p_{ji}),\ 1 \le i,j \le n \\
\sum_{i=1}^{n} x_i (p_{ij}+p_{ji}) &= -\sum_{i=1}^{n} x_i(-p_{ij}+p_{ji}),\ 1 \le j \le n.
\end{aligned}$$

The above is equivalent to the following.

$$\begin{aligned}
x_{n+1}(p_{ij}+p_{ji}) &= 0,\ 1 \le i,j \le n \\
\sum_{i=1}^{n} x_i p_{ij} &= 0,\ 1 \le j \le n.
\end{aligned}$$

Thus we have the following.

THEOREM 3.1. 1. *Assume that the matrix (p_{ij}) is invertible.*
   (a) *If the matrix (p_{ij}) is skew-symmetric, then E_{n+1} is the unique unit Killing vector.*
   (b) *If the matrix (p_{ij}) is not skew-symmetric, then there is no unit Killing vector.*

2. *Assume that the matrix (p_{ij}) is not invertible.*
   (a) *If the matrix (p_{ij}) is skew-symmetric, then the set of unit Killing vectors is $\mathcal{S} \cap ker((p_{ij})^t)$*

(b) *If the matrix* (p_{ij}) *is not skew-symmetric, then the set of unique Killing vectors is* $\mathcal{S} \cap \ker((p_{ij})^t) \cap \{x_{n+1} = 0\}$.

4. Homogeneous geodesic

Let $(M, g) = G/H$ be a homogeneous Riemannian manifold and $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ an ad(H)-invariant decomposition([2]). A geodesic $\gamma(t)$ through the origin $o \in M$ is homogeneous if and only if

$$\gamma(t) = \exp(tX)(o), \qquad t \in \mathbb{R},$$

where $X \in \mathfrak{g} - (0)$ is a nonzero vector. A nonzero vector $X \in \mathfrak{g} - (0)$ for which $\gamma(t) = \exp(tX)(o), t \in \mathbb{R}$, is a geodesic is called a *geodesic vector*. Thus geodesic vectors are in one-to-one correspondence with homogeneous geodesic through the origin o. There are many interesting results about homogeneous geodesics([2], [3], [4], [7]).

The following is a simple criterion for a vector to be a geodesic vector([3]).

LEMMA 4.1. *A vector* $X \in \mathfrak{g} - (0)$ *is a geodesic vector if and only if*

$$< [X, Y]_m, X_m >= 0, \quad \textit{for all } Y \in \mathfrak{m},$$

where $<,>$ *is the ad*(H)*-invariant scalar product on* $\mathfrak{m}$ *induced by the Riemannian scalar product on* T_0M *and the subscripts indicate the corresponding projection* $\mathfrak{g} \to \mathfrak{m}$.

We are going to calculate the above criterion for the semi-direct product $\mathfrak{a} \oplus_P \mathfrak{r}$.

Let $X = \sum_{i=1}^{n+1} x_i E_i$ be a vector in $\mathfrak{a} \oplus_P \mathfrak{r}$. Then we have

$$(1) \qquad \left\langle \left[\sum_{i=1}^{n+1} x_i E_i, E_j \right], \sum_{k=1}^{n+1} x_k E_k \right\rangle = 0, \quad \text{for } j = 1, \dots, n+1.$$

For $1 \le j \le n$, we have

$$\begin{aligned} \text{left hand side of (1)} &= \left\langle x_{n+1} \left[E_{n+1}, E_j \right], \sum_{k=1}^{n+1} x_k E_k \right\rangle \\ &= x_{n+1} \sum_{i=1}^{n} p_{ij} x_i = 0 \end{aligned}$$

And for $j = n+1$, we have

$$\begin{aligned} \text{left hand side of (1)} &= \left\langle \sum_{i=1}^{n} x_i \left(- \sum_{l=1}^{n} p_{li} E_l \right), \sum_{k=1}^{n+1} x_k E_k \right\rangle \\ &= - \sum_{k=1}^{n} x_k \cdot \left(\sum_{i=1}^{n} x_i p_{ki} \right) \\ &= - \sum_{i,k=1}^{n} p_{ki} x_i x_k = 0 \end{aligned}$$

Thus we have

$$x_{n+1} \cdot \sum_{i=1}^{n} p_{ij} x_i = 0, \text{ for } j = 1, \ldots, n \tag{2}$$

$$\sum_{i,k=1}^{n} p_{ki} x_i x_k = 0 \tag{3}$$

Assume that $x_{n+1} \neq 0$. Then we have $\sum_{i=1}^{n} p_{ij} x_i = 0$, for $j = 1, \ldots, n$.

In other words, the vector $\{x_1, x_2, \cdots, x_n\}$ lies in the null-space of P^t.

Thus if $\det(p_{ij}) \neq 0$, we have $x_1 = x_2 = \cdots = x_n = 0$. Thus x_{n+1} is the only geodesic vector.

Next assume that $x_{n+1} = 0$. Then we have $\sum_{i,k=1}^{n} p_{ki} x_i x_k = 0$. Thus if (p_{ij}) is skew-symmetric then all $x_1, \ldots, x_n$ satisfy equations (2) and (3). And if (p_{ij}) is skew-symmetric matrix plus a positive diagonal matrix, i.e., $p_{ij} = -p_{ji}, i \neq j$ and $p_{ii} > 0, i = 1, \ldots, n$, then we have x_i's are all zero. Thus we have the following.

THEOREM 4.2. *If the matrix (p_{ij}) is skew-symmetric matrix plus a positive diagonal matrix and non-singular, then the Lie group with Lie algebra $\mathfrak{a} \oplus_P \mathfrak{r}$ has only one geodesic vector.*

COROLLARY 4.3. *For the Lie group isometric to the hyperbolic space H^n, there is only one homogeneous geodesic.*

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