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GEOMETRY OF $L^2(\Omega, g)$

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Abstract. Roh[1] derived 2D g-Navier-Stokes equations from 3D Navier-Stokes equations. In this paper, we will see the space $L^2(\Omega, g)$, which is the weighted space of $L^2(\Omega)$, as natural generalized space of $L^2(\Omega)$ which is mathematical setting for Navier-Stokes equations. Our future purpose is to use the space $L^2(\Omega, g)$ as mathematical setting for the g-Navier-Stokes equations.

In addition, we will see Helmoltz-Leray projection on $L^2_{per}(\Omega, g)$ and compare with the one on $L^2_{per}(\Omega)$.

1. Introduction

In this paper, we assume $g(x_1, x_2) \in C^\infty_{per}(\Omega)$ and $0 < m < g(x_1, x_2) < M$ for some constant m, M. Here, we consider only periodic boundary condition because one can get similar results for Dirichlet boundary condition. In the case of Navier-Stokes equations with periodic boundary condition , it is usual to use $L^2_{per}(\Omega)$ for mathematical setting. So for our problem, we study the geometry of $L^2_{per}(\Omega, g)$, which is generalized space of $L^2_{per}(\Omega)$. One note that $L^2_{per}(\Omega, g)$ is the space with the scalar product and the norm by, $\langle \mathbf{u}, \mathbf{v} \rangle_g = \int_\Omega (\mathbf{u} \cdot \mathbf{v})\ g\ d\mathbf{x}$, $\| \mathbf{u} \|_g^2 = \langle \mathbf{u}, \mathbf{u} \rangle_g$, where $\mathbf{x} = (x_1, x_2)$. One can see $L^2_{per}(\Omega, g) = L^2_{per}(\Omega)$ for $g = 1$. Also, two spaces have equivalent norms. Then, we define the functional space,

$$H^k(\Omega, g) = \{\mathbf{u} \in L^2_{per}(\Omega, g)\ :\ D^\alpha \mathbf{u} \in L^2(\Omega, g), \quad \text{for all} \quad |\alpha| \leq k\}$$

with the scalar product $\langle \mathbf{u}, \mathbf{v} \rangle_{H^k(\Omega, g)} = \sum_{|\alpha| \leq k} \langle D^\alpha \mathbf{u}, D^\alpha \mathbf{v} \rangle_g$.

In next chapter, we will see geometry of the space $L^2_{per}(\Omega)$ in several different view. We also define the g-Helmoltz-Leray orthogonal decomposition P_g on $L^2_{per}(\Omega, g)$ and prove $P_g \to P_1$, as $g \to 1$ in some sense, in

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the operator norm. For the references, one can refer Naylor and Sell[2], Sell and You[3] and Roh[1].

2. Main Theorems

Now we define spaces,

$$
\begin{aligned}
H_g &= CL_{L^2(\Omega,g)}\{\mathbf{u} \in C^\infty_{\text{per}}(\Omega) \ : \ \nabla \cdot g\mathbf{u} = 0\} \\
H_{g,\tilde g} &= CL_{L^2(\Omega,g)}\{\mathbf{u} \in C^\infty_{\text{per}}(\Omega) \ : \ \nabla \cdot g\mathbf{u} = 0, \ \int_\Omega \mathbf{u}\ \tilde g \ d\mathbf{x} = 0\} \\
V_g &= \{\mathbf{u} \in H^1_{\text{per}}(\Omega, g) \ : \ \nabla \cdot g\mathbf{u} = 0\} \\
V_{g,\tilde g} &= \{\mathbf{u} \in H^1_{\text{per}}(\Omega, g) \ : \ \nabla \cdot g\mathbf{u} = 0, \ \int_\Omega \mathbf{u}\ \tilde g \ d\mathbf{x} = 0\} \\
K_{g,\tilde g} &= \{\mathbf{u} \in L^2_{per}(\Omega, g) \ : \ \int_\Omega \mathbf{u}\ \tilde g \ d\mathbf{x} = 0\}
\end{aligned}
$$

and

$$
\begin{aligned}
Q &= CL_{L^2(\Omega)}\{\nabla p : p \in C^1_{per}(\Omega, R)\} \\
C_g &= \{\frac{\mathbf{m}}{g}, \quad \mathbf{m} \in R^2\}, \quad M_g\mathbf{u} = \frac{1}{g}\int_\Omega \mathbf{u}\ g\ d\mathbf{x}
\end{aligned}
$$

We define $R_g : L^2_{per}(\Omega, g) \to K_{g,g}$ by $R_g\mathbf{u} = (I - M_g)\mathbf{u}$, for $\mathbf{u} \in L^2_{per}(\Omega)$, where I is the identity mapping on $L^2_{per}(\Omega, g)$. Then, we denote that $\mathcal{N}(M_g)$ is the null space of M_g and $\mathcal{R}(M_g)$ is the range of M_g. We note that M_g is a linear projection on $L^2_{per}(\Omega, g)$ and $M_g^2 = M_g$. In addition, the adjoint satisfies $M_g^* = M_1$ in $L^2_{per}(\Omega, g)$ and one has

$$\mathcal{N}(M_g) = K_{g,g} = \mathcal{R}(R_g), \qquad \mathcal{N}(M_1) = K_{g,1} = \mathcal{R}(R_1)$$

and

$$\mathcal{R}(M_g) = C_g = \mathcal{N}(R_g), \qquad \mathcal{R}(M_1) = R^2 = \mathcal{N}(R_1),$$

where R^2 is viewed as a subspace of $L^2_{per}(\Omega)$. Furthermore, we have

$$H_g \cap \mathcal{N}(M_g) = H_{g,g}, \qquad H_g \cap \mathcal{N}(M_1) = H_{g,1}, \qquad H_g \cap \mathcal{R}(M_g) = C_g.$$

Then we have the following theorem.

THEOREM 2.1. *M_1 and M_g are bounded projections on $L^2_{per}(\Omega, g)$. Furthermore the adjoint satisfy $M_g^* = M_1$ and $M_1^* = M_g$, which implies that*

$$\mathcal{N}(M_1) \perp_g \mathcal{R}(M_g) \qquad \textit{and} \qquad \mathcal{N}(M_g) \perp_g \mathcal{R}(M_1).$$

Note that M_g and R_g are orthogonal projection on $L^2_{per}(\Omega, g)$ if and only if $g = 1$. As a result we have

$$L^2(\Omega, g) = \mathcal{R}(M_g) \oplus \mathcal{N}(M_1) = \mathcal{R}(M_1) \oplus \mathcal{N}(M_g) \tag{1}$$

as well as

$$H_g = (H_g \cap \mathcal{R}(M_g)) \oplus (H_g \cap \mathcal{N}(M_1)) = C_g \oplus H_{g,1}.$$

Proof. Let $\mathbf{u}, \mathbf{v} \in L^2_{per}(\Omega, g)$. Then

$$\begin{aligned} \langle M_g \mathbf{u}, \mathbf{v} \rangle_g &= \int_\Omega M_g \mathbf{u} \cdot \mathbf{v} \; g \; d\mathbf{x} = \int_\Omega \frac{\int_\Omega \mathbf{u} \; g \; d\mathbf{x}}{g} \cdot \mathbf{v} \; g \; d\mathbf{x} \\ &= \int_\Omega \int_\Omega \mathbf{u} \; g d\mathbf{x} \cdot \mathbf{v} \; d\mathbf{x} = \int_\Omega \mathbf{u} g d\mathbf{x} \cdot \int_\Omega \mathbf{v} d\mathbf{x}. \end{aligned}$$

Also, we have

$$\langle \mathbf{u}, M_1 \mathbf{v} \rangle_g = \int_\Omega (\mathbf{u} \cdot M_1 \mathbf{v}) \; g \; d\mathbf{x} = \int_\Omega \mathbf{u} g \; d\mathbf{x} \cdot \int_\Omega \mathbf{v} \; d\mathbf{x}.$$

Thus one has $\langle M_g \mathbf{u}, \mathbf{v} \rangle_g = \langle \mathbf{u}, M_1 \mathbf{v} \rangle_g$, for $\mathbf{u}, \mathbf{v} \in L^2_{per}(\Omega, g)$.

So, $M_g^* \mathbf{v} = M_1 \mathbf{v}$, for all $\mathbf{v} \in L^2_{per}(\Omega, g)$. Similarly, $M_1^* \mathbf{u} = M_g \mathbf{u}$, for all $\mathbf{v} \in L^2_{per}(\Omega, g)$. The rest of proof comes by the definition of inner product in the space $L^2(\Omega, g)$. $\square$

We now will show that $L^2_{per}(\Omega, g) = H_g \oplus Q$ and $H_g = H_{g,1} \oplus C_g$. In fact, for every $\mathbf{v} \in L^2_{per}(\Omega, g)$, one has $\mathbf{v} = \mathbf{u} + \nabla p$ and $\mathbf{u} = \mathbf{w} + \frac{\mathbf{k}}{g}$, where $\mathbf{u} \in H_g$, $\mathbf{w} \in H_{g,1}$ and $\mathbf{k} = M_1(\frac{1}{g})^{-1} M_1(\mathbf{v})$.

Lemma 2.2. *For any $\mathbf{v} \in L^2_{per}(\Omega, g)$ there exist unique $\nabla p \in Q$ and $\mathbf{u} \in H_g$ such that $\mathbf{v} = \mathbf{u} \oplus \nabla p$.*

Proof. We define the Hilbert space $\mathcal{Z}$ as

$$\mathcal{Z} = \{p \in H^1_{per}(\Omega, g, R) : \int_\Omega p \; d\mathbf{x} = 0\}$$

with the norm $\| \, p \, \|^2_{\mathcal{Z}} = \| \, \nabla p \, \|^2_g$. Now, we define a bilinear form, $a(p, q) = \langle -\Delta_g p, q \rangle_g = \langle \nabla p, \nabla q \rangle_g$ for weak formulation. Then, one get

$$|a(p, q)| = |\langle \nabla p, \nabla q \rangle_g| \leq \| \, \nabla p \, \|_g \; \| \, \nabla q \, \|_g = \| \, p \, \|_{\mathcal{Z}} \; \| \, q \, \|_{\mathcal{Z}},$$

and $|a(p, p)| = |\langle \nabla p, \nabla p \rangle_g| = \| \, \nabla p \, \|^2_g = \| \, p \, \|^2_{\mathcal{Z}}$. Also, we have $b(q) = \langle \frac{1}{g} \nabla \cdot g \mathbf{v}, q \rangle_g = \langle \mathbf{v}, \nabla q \rangle_g$ which implies $|b(q)| \leq \| \, \mathbf{v} \, \|_g \; \| \, \nabla q \, \|_g$.

Therefore the bilinear form $a(p,q)$ satisfies the conditions for Lax-Milgram theorem. So for any $\mathbf{v} \in L^2_{per}(\Omega, g)$, there exist a unique solution $p \in \mathcal{Z}$ of the equation, $\frac{1}{g}(\nabla \cdot g\nabla p) = \frac{1}{g}(\nabla \cdot g\mathbf{v})$. Then, we set $\mathbf{u} = \mathbf{v} - \nabla p$ and note $\mathbf{u} \in H_g$. Also, integration by parts give

$$\langle \nabla p, \mathbf{u}\rangle_g = \int_\Omega \nabla p \cdot g\mathbf{u}\ d\mathbf{x} = \int_\Omega p\nabla \cdot (g\mathbf{u})\ d\mathbf{x} = 0.$$

$\square$

REMARK 2.1. We define $\mathbf{k} = [M_1(\frac{1}{g})]^{-1}\ M_1(\mathbf{v})$ and we set $\mathbf{w} = \mathbf{u} - \frac{\mathbf{k}}{g}$. Then $M_1(\mathbf{w}) = M_1(\mathbf{u}) - M_1(\frac{\mathbf{k}}{g}) = M_1(\mathbf{v}) - \mathbf{k}M_1(\frac{1}{g}) = 0$ by the definition of $\mathbf{k}$. Therefore, for $\mathbf{v} \in L^2_{per}(\Omega, g)$ there exists unique $\nabla p \in Q$, $\mathbf{w} \in H_{g,1}$ and $\mathbf{u} \in H_g$ such that $\mathbf{v} = \mathbf{u} + \nabla p = \mathbf{w} + \frac{\mathbf{k}}{g} + \nabla p$.

Now, the g-Helmoltz-Leray orthogonal projection $P_g : L^2_{per}(\Omega, g) \to H_g$ is defined by

$$P_g(\mathbf{v}) = \mathbf{u}, \qquad \text{for} \qquad \mathbf{v} \in L^2_{per}(\Omega)$$

where $\mathbf{v} = \mathbf{u} + \nabla p$ in lemma 2.2. We can also define another Helmoltz-Leray orthogonal projection $P_{g,1} : L^2_{per}(\Omega, g) \to H_{g,1}$ by

$$P_{g,1}(\mathbf{v}) = \mathbf{w}, \qquad \text{for} \qquad \mathbf{v} \in L^2_{per}(\Omega)$$

where $\mathbf{v} = \mathbf{w} + \frac{\mathbf{k}}{g} + \nabla p$ in remark 2.1.

REMARK 2.2. Since P_g is a orthogonal projection, by (1), one note that

$$H_g = P_g(\mathcal{R}(M_g)) \oplus P_g(\mathcal{N}(M_1)) = C_g \oplus H_{g,1},$$

as well as

$$H_g = P_g(\mathcal{R}(M_1)) \oplus P_g(\mathcal{N}(M_g)) = P_g(R^2) \oplus H_{g,g},$$

where R^2 is considered as a subset of $L^2_{per}(\Omega, g)$ and $\oplus$ is the standard orthogonal sum in the space $L^2_{per}(\Omega, g)$.

LEMMA 2.3. *We have*

$$P_1P_g\mathbf{u} = \mathbf{u}, \quad \textit{for} \quad \mathbf{u} \in H_1, \qquad \textit{and} \quad P_gP_1\mathbf{u} = \mathbf{u}, \quad \textit{for} \quad \mathbf{u} \in H_g.$$

Furthermore, one has

$$P_gP_1\mathbf{v} = P_g\mathbf{v}, \qquad P_1P_g\mathbf{v} = P_1\mathbf{v}, \qquad \textit{for all} \quad \mathbf{v} \in L^2_{per}(\Omega).$$

Proof. We skip the proof. $\square$

By lemma 2.2, for given $\mathbf{u} \in H_g$ there exist unique $\nabla p \in Q$ and $\mathbf{w} \in H_1$ such that

$$\mathbf{u} = \nabla p + \mathbf{w}, \quad \mathbf{w} = P_1 \mathbf{u}. \tag{2}$$

By using the fact that $\nabla \cdot \mathbf{w} = 0$, equation (2) implies that $\Delta p = \nabla \cdot \mathbf{u}$, where the derivatives are in the sense of distributions. Since $\mathbf{u} \in H_g$, one has $0 = \nabla \cdot g\mathbf{u} = \nabla g \cdot \mathbf{u} + g \nabla \cdot \mathbf{u}$ and we obtain

$$\Delta p = -\frac{1}{g} \nabla g \cdot \mathbf{u} \equiv f, \qquad \text{for} \quad \mathbf{u} \in H_{g,1}. \tag{3}$$

To find a strong solution for the Poisson equation $\Delta p = f$, we need the consistency property, $M_1 f = 0$. For our problem, where $f = -\frac{1}{g} \nabla g \cdot \mathbf{u}$, one needs

$$M_1\left(\frac{1}{g} \nabla g \cdot \mathbf{u}\right) = \int_\Omega \left(\frac{1}{g} \nabla g\right) \cdot \mathbf{u} = 0.$$

Since $\nabla \cdot (g\mathbf{u}) = 0$, we have $\mathbf{u} \in H_g \perp_g Q$. Hence

$$\int_\Omega \frac{1}{g} \nabla g \cdot \mathbf{u} \; d\mathbf{x} = \int_\Omega \frac{1}{g^2} \nabla g \cdot \mathbf{u} \; g \; d\mathbf{x} = -\int_\Omega \nabla\left(\frac{1}{g}\right) \cdot \mathbf{u} \; g \; d\mathbf{x} = 0.$$

Since $0 < m \le g \le M$ and $M_1 p = 0$, one has from Poincaré inequality that

$$\| \nabla p \| \le \| p \|_{H^1(\Omega)} \le \| p \|_{H^2(\Omega)} \le c \, \frac{1}{m} \| \nabla g \|_\infty \| \mathbf{u} \|, \tag{4}$$

for some positive constant c.

We now define the operator $L : H_{g,1} \to Q$ by $L\mathbf{u} = \nabla p$, where ∇p is given as a strong solution of (3). We note from equation (2) that $(I - L)\mathbf{u} = \mathbf{w}$, that is, $I - L = P_{1|_{H_{g,1}}}$, where I is considered as the identity mapping on the space $H_{g,1}$. From inequality (4), the operator norm of L satisfies

$$\| (I - P_1)_{|_{H_{g,1}}} \|_{op} = \| L \|_{op} \le \frac{c}{m} \| \nabla g \|_\infty.$$

Similarly, for given $\mathbf{w} \in H_1$, there exist unique $\nabla q \in Q$ and $\mathbf{u} \in H_g$ such that

$$\mathbf{w} = \nabla q + \mathbf{u}, \quad \mathbf{u} = P_g \mathbf{w}. \tag{5}$$

Now, we can define the operator $K : H_{1,1} \to Q$ by $K\mathbf{w} = \nabla q$, where ∇q is given by (5). By using the fact $\frac{1}{g} \nabla \cdot g\mathbf{u} = 0$, equation (5) implies that

$$\frac{1}{g}(\nabla \cdot g \nabla q) = \frac{1}{g} \nabla \cdot g\mathbf{w}.$$

Since $\mathbf{w} \in H_{1,1}$, we obtains

$$\frac{1}{g}(\nabla \cdot g\nabla q) = \frac{1}{g}\nabla g \cdot \mathbf{w}, \qquad \text{for} \quad \mathbf{w} \in H_{1,1}. \tag{6}$$

Then as we did in the proof of lemma 2.2, by using Lax-Milgram theorem, for given $\mathbf{w} \in H_{1,1}$, one can find a strong solution $\nabla q \in \mathcal{Z}$ of (6) with the following estimate,

$$\| \nabla q \| \leq \frac{\| \nabla g \|_\infty}{m} \| \mathbf{w} \|. \tag{7}$$

Also we note that $I - K = P_{g|_{H_{1,1}}}$, where I is considered as the identity mapping on the space $H_{1,1}$. Therefore, from inequality (7), one has the operator norm of K by

$$\| (I - P_g)_{|_{H_{1,1}}} \|_{op} = \| K \|_{op} \leq \frac{\| \nabla g \|_\infty}{m}.$$

For the operator $P_g - P_1 : L^2_{per}(\Omega) \to L^2_{per}(\Omega)$, we have the following result concerning the operator norm $\| P_g - P_1 \|_{op}$.

Theorem 2.4. *As $\| \nabla g \|_\infty \to 0$, we have $\| (P_g - P_1)_{|_{L^2}} \|_{op} \to 0$.*

Proof. For $\mathbf{u} \in L^2_{per}(\Omega)$ we obtain

$$\mathbf{u} = \mathbf{v}_g + \nabla p_g, \quad \mathbf{u} = \mathbf{v}_1 + \nabla p_1, \quad \mathbf{v}_g = P_g\mathbf{u}, \text{ and } \mathbf{v}_1 = P_1\mathbf{u},$$

where $\mathbf{v}_g \in H_g$, $\mathbf{v}_1 \in H_1$, and $\nabla p_g, \nabla p_1 \in Q$. Since $\nabla \cdot \mathbf{v}_1 = 0$ and $\nabla \cdot g\mathbf{v}_g = 0$, we have

$$\Delta p_1 = \nabla \cdot \mathbf{u}, \quad \Delta_g p_g = \frac{1}{g}(\nabla \cdot g\mathbf{u}) = \nabla \cdot \mathbf{u} + (\frac{\nabla g}{g} \cdot \mathbf{u}),$$

where $\Delta_g p_g = \Delta p_g + (\frac{\nabla g}{g} \cdot \nabla)p_g$. So, one obtains

$$\Delta(p_g - p_1) = \frac{\nabla g}{g} \cdot \mathbf{u} - (\frac{\nabla g}{g} \cdot \nabla)p_g.$$

Let us denote $p = p_g - p_1$. Since we have $M_1(p) = 0$, and $\| \nabla p_g \|_g^2 \leq \| \mathbf{u} \|_g^2$ which implies $m\| \nabla p_g \|^2 \leq M\| \mathbf{u} \|^2$, due to Poincaré inequality we have some constant c such that

$$\begin{aligned}
\| P_g\mathbf{u} - P_1\mathbf{u} \| = \| \nabla p \| &\leq \| p \|_{H^2} \\
&\leq c \left(\| \frac{\nabla g}{g} \|_\infty \| \mathbf{u} \| + \| \frac{\nabla g}{g} \|_\infty \| \nabla p_g \| \right) \\
&\leq c\,(1 + \sqrt{\frac{M}{m}}) \| \frac{\nabla g}{g} \|_\infty \| \mathbf{u} \|.
\end{aligned}$$

□

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