JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **19**, No.3, September 2006

ON THE QUADRATIC MAPPING IN GENERALIZED QUASI-BANACH SPACES

Choonkil Park*, Kil-Woung Jun**, and Gang Lu***

Abstract. In this paper, we prove the Hyers–Ulam–Rassias stability of the quadratic mapping in generalized quasi-Banach spaces, and of the quadratic mapping in generalized p-Banach spaces.

1. Introduction and preliminaries

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.1. ([1, 25]) Let X be a linear space. A *quasi-norm* is a real-valued function on X satisfying the following:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

(3) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on X.

A *quasi-Banach space* is a complete quasi-normed space.

A quasi-norm $\|\cdot\|$ is called a *p-norm* $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a *p-Banach space.*

Given a p-norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on X. By the Aoki–Rolewicz theorem [25] (see also [1]), each quasi-norm is equivalent to some p-norm. Since it is much

Received July 21, 2006.
2000 Mathematics Subject Classification: Primary 39B72, 47Jxx.
Key words and phrases: quadratic mapping, generalized quasi-Banach space, stability.
The third author was supported by the Brain Korea 21 Project in 2006.

easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms.

In [16], the author generalized the concept of quasi-normed spaces.

DEFINITION 1.2. Let X be a linear space. A *generalized quasi-norm* is a real-valued function on X satisfying the following:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

(3) There is a constant $K \geq 1$ such that $\|\sum_{j=1}^{\infty} x_j\| \leq \sum_{j=1}^{\infty} K\|x_j\|$ for all $x_1, x_2, \cdots \in X$ with $\sum_{j=1}^{\infty} x_j \in X$.

The pair $(X, \|\cdot\|)$ is called a *generalized quasi-normed space* if $\|\cdot\|$ is a generalized quasi-norm on X. The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$.

A *generalized quasi-Banach space* is a complete generalized quasi-normed space.

A generalized quasi-norm $\|\cdot\|$ is called a *p-norm* $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a generalized quasi-Banach space is called a *generalized p-Banach space*.

The stability problem of functional equations originated from a question of S.M. Ulam [29] concerning the stability of group homomorphisms: Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$? If the answer is affirmative, we would say that the equation of homomorphism $H(x * y) = H(x) \diamond H(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

D.H. Hyers [9] gave a first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f : X \to Y$ satisfies

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in X$ and some $\varepsilon \geq 0$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \varepsilon$$

for all $x \in X$.

Let X and Y be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Th.M. Rassias [17] introduced the following inequality, that we call *Cauchy–Rassias inequality* : Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \theta(||x||^p + ||y||^p)$$

for all $x, y \in X$. Th.M. Rassias [17] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}||x||^p$$

for all $x \in X$. The above inequality has provided a lot of influence in the development of what we now call *Hyers–Ulam–Rassias stability* of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians. Găvruta [8] generalized the Rassias' result.

A square norm on an inner product space satisfies the important parallelogram equality

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A Hyers–Ulam–Rassias stability problem for the quadratic functional equation was proved by Skof [27] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. In [4], Czerwik proved the Hyers–Ulam–Rassias stability of the quadratic functional equation. C. Borelli and G.L. Forti [2] generalized the stability result as follows: Let G be an abelian group, E a Banach space. Assume that a mapping $f : G \to E$ satisfies the functional inequality

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \varphi(x, y)$$

for all $x, y \in G$, and $\varphi : G \times G \to [0, \infty)$ is a function such that

$$\Phi(x, y) := \sum_{i=0}^{\infty} \frac{1}{4^{i+1}} \varphi(2^i x, 2^i y) < \infty$$

for all $x, y \in G$. Then there exists a unique quadratic mapping $Q : G \to E$ with the properties

$$\|f(x) - Q(x)\| \leq \Phi(x, x)$$

for all $x \in G$. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem. A large list of references can be found, for example, in the papers [5, 6, 7, 10, 11, 12, 13, 14, 15, 18, 19, 20, 21, 23, 24, 26, 28].

In this paper, we prove the Hyers–Ulam–Rassias stability of the quadratic mapping in generalized quasi-Banach spaces, and prove the Hyers–Ulam–Rassias stability of the quadratic mapping in generalized p-Banach spaces.

2. Stability of the quadratic mapping in generalized quasi-Banach spaces

Throughout this section, assume that X is a generalized quasi-normed vector space with generalized quasi-norm $||\cdot||$ and that Y is a generalized quasi-Banach space with generalized quasi-norm $\| \cdot \|$. Let K be the modulus of concavity of $\| \cdot \|$.

THEOREM 2.1. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y) := \sum_{j=1}^{\infty} 4^j \varphi(\frac{x}{2^j}, \frac{y}{2^j}) \quad < \quad \infty, \tag{2.1}$$

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \quad \leq \quad \varphi(x, y) \tag{2.2}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{K}{4} \widetilde{\varphi}(x, x) \tag{2.3}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.2), we get

$$\|f(2x) - 4f(x)\| \le \varphi(x, x) \tag{2.4}$$

for all $x \in X$. So

$$\|f(x) - 4f(\frac{x}{2})\| \le \varphi(\frac{x}{2}, \frac{x}{2})$$

for all $x \in X$. Hence

$$\|4^l f(\frac{x}{2^l}) - 4^m f(\frac{x}{2^m})\| \le K \sum_{j=l}^{m-1} 4^j \varphi(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}) \tag{2.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.1) and (2.5) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$.

By (2.2) and (2.1),

$$\begin{aligned}
&\|Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y)\| \\
&= \lim_{n\to\infty} 4^n \|f(\frac{x+y}{2^n}) + f(\frac{x-y}{2^n}) - 2f(\frac{x}{2^n}) - 2f(\frac{y}{2^n})\| \\
&\le \lim_{n\to\infty} 4^n \varphi(\frac{x}{2^n}, \frac{y}{2^n}) = 0
\end{aligned}$$

for all $x, y \in X$. So

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$$

for all $x, y \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.5), we get (2.3).

Now, let $Q' : X \to Y$ be another quadratic mapping satisfying (2.3). Then we have

$$\begin{aligned}
\|Q(x) - Q'(x)\| &= 4^n \|Q(\frac{x}{2^n}) - Q'(\frac{x}{2^n})\| \\
&\le 4^n K(\|Q(\frac{x}{2^n}) - f(\frac{x}{2^n})\| + \|Q'(\frac{x}{2^n}) - f(\frac{x}{2^n})\|) \\
&\le \frac{2K}{4} \cdot 4^n \widetilde{\varphi}(\frac{x}{2^n}, \frac{x}{2^n}),
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = Q'(x)$ for all $x \in X$. This proves the uniqueness of Q. □

COROLLARY 2.2. *Let $r > 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le \theta(||x||^r + ||y||^r)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{2K\theta}{2^r - 4}||x||^r$$

for all $x \in X$.

Proof. Define $\varphi(x, y) = \theta(||x||^r + ||y||^r)$, and apply Theorem 2.1. □

THEOREM 2.3. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y) := \sum_{j=0}^{\infty} \frac{1}{4^j}\varphi(2^j x, 2^j y) \quad < \quad \infty, \tag{2.6}$$

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \quad \le \quad \varphi(x, y) \tag{2.7}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{K}{4}\widetilde{\varphi}(x, x) \tag{2.8}$$

for all $x \in X$.

Proof. It follows from (2.4) that

$$\|f(x) - \frac{1}{4}f(2x)\| \le \frac{1}{4}\varphi(x, x)$$

for all $x \in X$. Hence

$$\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\| \le K\sum_{j=l}^{m-1} \frac{1}{4^{j+1}}\varphi(2^j x, 2^j x) \tag{2.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.6) and (2.9) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{1}{4^n}f(2^n x)$$

for all $x \in X$.

By (2.7) and (2.6),

$$\begin{aligned}&\|Q(x+y)+Q(x-y)-2Q(x)-2Q(y)\|\\&=\lim_{n\to\infty}\frac{1}{4^n}\|f(2^n(x+y))+f(2^n(x-y))-2f(2^nx)-2f(2^ny)\|\\&\le\lim_{n\to\infty}\frac{1}{4^n}\varphi(2^nx,2^ny)=0\end{aligned}$$

for all $x, y \in X$. So

$$Q(x+y)+Q(x-y)=2Q(x)+2Q(y)$$

for all $x, y \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.9), we get (2.8).

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let $r < 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\|\le\theta(||x||^r+||y||^r)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x)\|\le\frac{2K\theta}{4-2^r}||x||^r$$

for all $x \in X$.

Proof. Define $\varphi(x, y) = \theta(||x||^r + ||y||^r)$, and apply Theorem 2.3. □

3. Stability of the quadratic mapping in generalized p-Banach spaces

Throughout this section, assume that X is a generalized quasi-normed vector space with generalized quasi-norm $||\cdot||$ and that Y is a generalized p-Banach space with generalized quasi-norm $\|\cdot\|$.

Theorem 3.1. *Let $r > 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\|\le\theta(||x||^r+||y||^r) \tag{3.1}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x)\|\le\frac{2\theta}{(2^{pr}-4^p)^{\frac{1}{p}}}||x||^r \tag{3.2}$$

for all $x \in X$.

Proof. Letting $y = x$ in (3.1), we get

$$\|f(2x) - 4f(x)\| \le 2\theta||x||^r \tag{3.3}$$

for all $x \in X$. So

$$\|f(x) - 4f(\frac{x}{2})\| \le \frac{2}{2^r}\theta||x||^r$$

for all $x \in X$. Since Y is a generalized p-Banach space,

$$\begin{aligned} \|4^l f(\frac{x}{2^l}) - 4^m f(\frac{x}{2^m})\|^p &\le \sum_{j=l}^{m-1} \|4^j f(\frac{x}{2^j}) - 4^{j+1} f(\frac{x}{2^{j+1}})\|^p \\ &\le \sum_{j=l}^{m-1} \frac{4^{pj}}{2^{prj}} \cdot \frac{2^p}{2^{pr}}\theta^p||x||^{pr} \end{aligned} \tag{3.4}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. So the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$.

By (3.1),

$$\begin{aligned} &\|Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y)\| \\ &= \lim_{n\to\infty} 4^n \|f(\frac{x+y}{2^n}) + f(\frac{x-y}{2^n}) - 2f(\frac{x}{2^n}) - 2f(\frac{y}{2^n})\| \\ &\le \lim_{n\to\infty} \frac{4^n}{2^{rn}}\theta(||x||^r + ||y||^r) = 0 \end{aligned}$$

for all $x, y \in X$. So

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$$

for all $x, y \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.4), we get (3.2).

Now, let $Q' : X \to Y$ be another quadratic mapping satisfying (3.2). Then we have

$$\begin{aligned} \|Q(x) - Q'(x)\|^p &= 4^{pn}\|Q(\frac{x}{2^n}) - Q'(\frac{x}{2^n})\|^p \\ &\le 4^{pn}(\|Q(\frac{x}{2^n}) - f(\frac{x}{2^n})\|^p + \|Q'(\frac{x}{2^n}) - f(\frac{x}{2^n})\|^p) \\ &\le 2 \cdot \frac{4^{pn}}{2^{prn}} \cdot \frac{2^p\theta^p}{2^{pr} - 4^p}||x||^{pr}, \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = Q'(x)$ for all $x \in X$. This proves the uniqueness of Q. □

REMARK 3.1. The result for the case $K = 1$ in Corollary 2.2 is the same as the result for the case $p = 1$ in Theorem 3.1.

THEOREM 3.2. *Let $r < 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping such that*

$$(3.5)\ \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \theta(||x||^r + ||y||^r)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{(4^p - 2^{pr})^{\frac{1}{p}}} ||x||^r \tag{3.6}$$

for all $x \in X$.

Proof. Letting $y = x$ in (3.5), we get

$$\|f(2x) - 4f(x)\| \leq 2\theta||x||^r \tag{3.7}$$

for all $x \in X$. So

$$\|f(x) - \frac{1}{4}f(2x)\| \leq \frac{1}{2}\theta||x||^r$$

for all $x \in X$. Since Y is a generalized p-Banach space,

$$\begin{aligned} \|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\|^p &\leq \sum_{j=l}^{m-1} \|\frac{1}{4^j}f(2^j x) - \frac{1}{4^{j+1}}f(2^{j+1}x)\|^p \\ &\leq \sum_{j=l}^{m-1} \frac{2^{prj}}{4^{pj}} \cdot \frac{\theta^p}{2^p}||x||^{pr} \end{aligned} \tag{3.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. So the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$.

By (3.5),

$$
\begin{aligned}
&\|Q(x+y)+Q(x-y)-2Q(x)-2Q(y)\|\\
&=\lim_{n\to\infty}\frac{1}{4^n}\|f(2^n(x+y))+f(2^n(x-y))-2f(2^nx)-2f(2^ny)\|\\
&\le \lim_{n\to\infty}\frac{2^{rn}}{4^n}\theta(||x||^r+||y||^y)=0
\end{aligned}
$$

for all $x, y \in X$. So

$$
Q(x+y)+Q(x-y)=2Q(x)+2Q(y)
$$

for all $x, y \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get (3.6).

Now, let $Q' : X \to Y$ be another quadratic mapping satisfying (3.6). Then we have

$$
\begin{aligned}
\|Q(x)-Q'(x)\|^p &= \frac{1}{4^{pn}}\|Q(2^nx)-Q'(2^nx)\|^p\\
&\le \frac{1}{4^{pn}}(\|Q(2^nx)-f(2^nx)\|^p+\|Q'(2^nx)-f(2^nx)\|^p)\\
&\le 2\cdot\frac{2^{prn}}{4^{pn}}\cdot\frac{2^p\theta^p}{4^p-2^{pr}}||x||^{pr},
\end{aligned}
$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = Q'(x)$ for all $x \in X$. This proves the uniqueness of Q. □

Remark 3.2. The result for the case $K = 1$ in Corollary 2.4 is the same as the result for the case $p = 1$ in Theorem 3.3.

References

[1] Y. Benyamini and J. Lindenstrauss *Geometric Nonlinear Functional Analysis, Vol. 1*, Colloq. Publ. **48**, Amer. Math. Soc., Providence, 2000.

[2] C. Borelli and G.L. Forti, *On a general Hyers–Ulam stability result*, Internat. J. Math. Math. Sci. **18** (1995), 229–236.

[3] P.W. Cholewa, *Remarks on the stability of functional equations*, Aeq. Math. **27** (1984), 76–86.

[4] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg **62** (1992), 59–64.

[5] S. Czerwik, *The stability of the quadratic functional equation*, in 'Stability of Mappings of Hyers–Ulam Type' (edited by Th.M. Rassias and J. Tabor), Hadronic Press, Palm Harbor, Florida, 1994, pp. 81–91.

[6] S. Czerwik, *Stability of Functional Equations of Ulam–Hyers–Rassias Type*, Hadronic Press, Palm Harbor, Florida, 2003.

[7] G.L. Forti, *Comments on the core of the direct method for proving Hyers–Ulam stability of functional equations*, J. Math. Anal. Appl. **295** (2004), 127–133.

[8] P. Găvruta, *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
[9] D.H. Hyers, *On the stability of the linear functional equation*, Pro. Nat'l. Acad. Sci. U.S.A. **27** (1941), 222–224.
[10] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.
[11] D.H. Hyers and Th.M. Rassias, *Approximate homomorphisms*, Aeq. Math. **44** (1992), 125–153.
[12] K. Jun and H. Kim, *Ulam stability problem for quadratic mappings of Euler–Lagrange*, Nonlinear Anal.–TMA **61** (2005), 1093–1104.
[13] K. Jun and H. Kim, *On the generalized A-quadratic mappings associated with the variance of a discrete-type distribution*, Nonlinear Anal.–TMA **62** (2005), 975–987.
[14] C. Park, *Generalized quadratic mappings in several variables*, Nonlinear Anal.–TMA **57** (2004), 713–722.
[15] C. Park, *Cauchy–Rassias stability of a generalized Trif's mapping in Banach modules and its applications*, Nonlinear Anal.–TMA **62** (2005), 595–613.
[16] C. Park, *Generalized quasi-Banach spaces and quasi-*$(2;p)$*-normed spaces*, (preprint).
[17] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[18] Th.M. Rassias, *On the stability of the quadratic functional equation and its applications*, Studia Univ. Babes-Bolyai **XLIII** (1998), 89–124.
[19] Th.M. Rassias, *The problem of S.M. Ulam for approximately multiplicative mappings*, J. Math. Anal. Appl. **246** (2000), 352–378.
[20] Th.M. Rassias, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. **251** (2000), 264–284.
[21] Th.M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. **62** (2000), 23–130.
[22] Th.M. Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers, Dordrechtm, Boston and London, 2003.
[23] Th.M. Rassias and P. Šemrl, *On the Hyers–Ulam stability of linear mappings*, J. Math. Anal. Appl. **173** (1993), 325–338.
[24] Th.M. Rassias and K. Shibata, *Variational problem of some quadratic functionals in complex analysis*, J. Math. Anal. Appl. **228** (1998), 234–253.
[25] S. Rolewicz, *Metric Linear Spaces*, PWN-Polish Sci. Publ., Reidel and Dordrecht, 1984.
[26] P. Šemrl, *On quadratic functionals*, Bull. Austral. Math. Soc. **37** (1987), 27–28.
[27] F. Skof, *Proprietà locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.
[28] J. Tabor, *Stability of the Cauchy functional equation in quasi-Banach spaces*, Ann. Polon. Math. **83** (2004), 243–255.
[29] S.M. Ulam, *Problems in Modern Mathematics*, Wiley, New York, 1960.

*

Department of Mathematics
Hanyang University
Seoul 133-791, Republic of Korea
E-mail: baak@hanyang.ac.kr

**

Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: kwjun@cnu.ac.kr

***

Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea