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EXISTENCE AND BOUNDEDNESS OF SOLUTIONS FOR VOLTERRA DISCRETE EQUATIONS

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Abstract. In this paper, we examine the existence and boundedness of the solutions of discrete Volterra equations

$$x(n) = f(n) + \sum_{j=0}^{n} g(n, j, x(j)),\ n \geq 0$$

and

$$x(n) = f(n) + \sum_{j=0}^{n} B(n, j)x(j),\ n \geq 0.$$

1. Introduction

Volterra difference equations arise in the mathematical modeling of some real phenomena, and also in numerical schemes for solving differential and integral equations.

Baker and Song [1] established the existence and uniqueness of the solutions of discrete Volterra equations via the discrete Volterra operators and the fixed point theorems. Moreover, they investigated the stability properties in the various sequence spaces in [5], using the representation of the solution by the resolvent for the kernel in Volterra equations.

In this paper, we examine the existence and boundedness of the solutions of discrete Volterra equations

$$x(n) = f(n) + \sum_{j=0}^{n} g(n, j, x(j)),\ n \geq 0$$

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and

$$x(n) = f(n) + \sum_{j=0}^{n} B(n,j)x(j),\ n \geq 0.$$

2. Some sequence spaces

Let $(X, |\cdot|)$ be a Banach space and $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$ be the set of all nonnegative integers. We denote by $S(X)$ the set of sequences in X:

$$S(X) = \{x : \mathbb{Z}_+ \to X :\ x(n) \in X, n \in \mathbb{Z}_+\}.$$

We employ the notation

$$\begin{aligned} x &= \{x(n)\}_{n=0}^{\infty}, \\ |x|_p &= (\sum_{n=0}^{\infty} |x(n)|^p)^{\frac{1}{p}},\ 1 \leq p < \infty, \\ |x|_\infty &= \sup_{n \geq 0} |x(n)|, \end{aligned}$$

and define the Banach space

$$S^p(X) = \{x \in S(X) :\ |x|_p < \infty\},\ 1 \leq p \leq \infty,$$

consisting of elements of $S(X)$ with the norm $|x|_p$. Also, we define the ball with radius r centered on the null sequence :

$$B^p(X, r) = \{x \in S^p(X) :\ |x|_p \leq r\}.$$

The corresponding ball with radius r but centered on y will be denoted $B^p(y; r) : z \in B^p(y; r)$ if and only if $z - y \in B^p(X, r)$. The case where $X = \mathbb{R}^d$, d-dimensional real Euclidean space,

$$l^p = l^p(\mathbb{R}^d) = S^p(\mathbb{R}^d).$$

Thus l^p denotes the Banach space comprising sequences of vectors with finite norm $|x|_p$ where

$$\begin{aligned} |x|_p &= \left(\sum_{n=0}^{\infty} |x(n)|^p\right)^{\frac{1}{p}},\ 1 \leq p < \infty, \\ |x|_\infty &= \sup_{n \geq 0} |x(n)|. \end{aligned}$$

We denote by $S_m(X), m \geq 0$, the linear space of terminating sequences

$$\{x(n)\}_{n=0}^{m} : x(0), x(1), \cdots, x(m)$$

and by $S^p_m(X)$ the corresponding Banach space with norm $|\cdot|^p_m$:

$$|x|^p_m = \left(\sum_{n=0}^{m}|x(n)|^p\right)^{\frac{1}{p}}, \ 1\le p<\infty,$$
$$|x|^\infty_m = \sup_{0\le n\le m}|x(n)|.$$

3. Existence

Definition 3.1. An operator $V : S(X)\to S(X)$ is called a (discrete) *Volterra operator* if there exists a family of mappings $v_i : S_i \to X, i\ge 0$, such that, for each $\phi=\{\phi(n)\}_{n=0}^\infty \in S(X)$

$$(V\phi)(n) = v_n(\phi(0),\phi(1),\cdots,\phi(n)), n=0,1,\cdots. \tag{3.1}$$

The mapping v_n is called the *n-th coordinate mapping* of V.

Example 3.2. We can consider the following Volterra operators on $S(\mathbb{R}^d)$:

$$(V\phi)(n) = f(n)+\sum_{j=0}^{n} g(n,j,\phi(j)),\ n\ge 0, \tag{3.2}$$

where $g:\mathbb{Z}_+\times\mathbb{Z}_+\times\mathbb{R}^d\to\mathbb{R}^d$ and $f=\{f(n)\}_{n=0}^\infty\in S(\mathbb{R}^d)$,

$$(V\phi)(n) = f(n)+\sum_{j=0}^{n} B(n,j)g(j,\phi(j)),\ n\ge 0, \tag{3.3}$$

where $\{B(n,m)\}_{0\le m\le n}, n\ge 0$, are $d\times d$ matrices and $g:\mathbb{Z}_+\times\mathbb{R}^d\to\mathbb{R}^d$, and

$$(V\phi)(n) = f(n)+\sum_{j=0}^{n} g(n-j,\phi(j)),\ n\ge 0. \tag{3.4}$$

The mapping $B:\mathbb{Z}_+\times\mathbb{Z}_+\to\mathbb{R}^d$ is called the *kernel* of V. The special form (3.4) is called a nonlinear discrete Volterra operator of *convolution type*.

We investigate the existence of solutions of Volterra equation

$$x(n) = f(n)+\sum_{j=0}^{n} g(n,j,x(j)),\ n\ge 0, \tag{3.5}$$

where $f=\{f(n)\}_{n=0}^\infty\in S(\mathbb{R}^d), g:\mathbb{Z}_+\times\mathbb{Z}_+\times\mathbb{R}^d\to\mathbb{R}^d$ is a mapping.

Remark 3.3. Equation (3.5) is a discrete analogue of the Volterra integral equation

$$x(t) = f(f) + \int_0^t g(t, s, x(s))ds, \tag{3.6}$$

where f is continuous on $[0, a], a > 0$, and g is continuous on $[0, a] \times [0, a] \times B(f(t), b), b > 0$ and Lipschitzian in x, i.e.,

$$|g(t, s, x) - g(t, s, y)| \le L|x - y|.$$

Here $B(f(t), b)$ is an open ball with radius $b > 0$. Burton [2] use the operator $A : C([0, T], \mathbb{R}^n) \to C([0, T], \mathbb{R}^n)$ defined by

$$(A\phi)(t) = f(t) + \int_0^t g(t, s, \phi(s))ds,$$

where

$$T = \min\{a, \frac{b}{M}\}, \ M = \max |g(t, s, x)|,$$

to show that (3.6) has a unique solution on $[0, T]$.

Baker and Song [1] obtained one existence result for the Volterra operator V on $l^\infty(\mathbb{R}^d)$ in the following.

Theorem 3.4. [1] *Let $V : l^\infty(\mathbb{R}^d) \to l^\infty(\mathbb{R}^d)$ be a Volterra operator that is bounded (namely there is a number $M > 0$ such that $|V\phi|_\infty \le M$ for all $\phi \in l^\infty(\mathbb{R}^d)$). If all the coordinate mappings of V are continuous, then V has at least one fixed point in $l^\infty(\mathbb{R}^d)$.*

This theorem follows from the well-known fixed point theorem for finite dimensional spaces :

Theorem 3.5. [1] *(i) Any continuous mapping of a convex subset Ω of $\mathbb{R}^d$ into a bounded closed set inside Ω has one fixed point.*

(ii) Any continuous mapping of $\mathbb{R}^d$ into a bounded subset of $\mathbb{R}^d$ has a fixed point.

Now, we obtain the following result from Theorem 3.4 :

Theorem 3.6. *For equation (3.5), assume that*

(i) *$g(n, j, x)$ is continuous in x for every $(n, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ and $g(n, j) = 0$ when $j > n$,*
(ii) *$f \in l^\infty(\mathbb{R}^d)$,*
(iii) *$\sup_{n \ge 0} \sum_{j=0}^n |g(n, j, x(j))| \le M$ for some number $M > 0$.*

Then (3.5) has at least one bounded solution.

Proof. Define a Volterra operator $V : l^\infty(\mathbb{R}^d) \to l^\infty(\mathbb{R}^d)$ by

$$(Vx)(n) = f(n) + \sum_{j=0}^{n} g(n, j, x(j)),\ n \geq 0.$$

Then we have

$$\begin{aligned} |(Vx)(n)| &\leq |f(n)| + \sum_{j=0}^{n} |g(n, j, x(j))| \\ &\leq |f|_\infty + M. \end{aligned}$$

Thus

$$|Vx|_\infty \leq |f|_\infty + M.$$

This implies that V is a bounded Volterra operator on $l^\infty(\mathbb{R}^d)$. Clearly, each coordinate mapping $v_n : S_n^\infty(\mathbb{R}^d) \to \mathbb{R}^d$ is given by

$$v_n(x) = f(n) + \sum_{j=0}^{n} g(n, j, x(j)),\ x \in S_n^\infty(\mathbb{R}^d),$$

and it is continuous on $S_n^\infty(\mathbb{R}^d)$. Therefore, by Theorem 3.4, V has at least one fixed point in $l^\infty(\mathbb{R}^d)$ and the fixed point is a bounded solution of V. □

REMARK 3.7. To ensure the uniqueness of solution of (3.5), we use the Contraction Mapping Theorem. If each coordinate mapping $v_i : S_i^\infty(\mathbb{R}^d) \to \mathbb{R}^d$ is a contraction, then the Volterra operator V corresponding to equation (3.5) is also a contraction since the exists a number ρ such that $0 < \rho < 1$ and

$$|(V\phi) - (V\psi)|_\infty \leq \rho|\phi - \psi|_\infty$$

for any $\phi, \psi \in l^\infty(\mathbb{R}^d)$. Thus V has a unique fixed point in $l^\infty(\mathbb{R}^d)$ by the Contraction Mapping Theorem.

4. Boundednesss

In this section, we examine the boundedness of solutions of discrete Volterra equation

$$x(n) = f(n) + \sum_{j=0}^{n} B(n, j)x(j),\ n \geq 0. \tag{4.1}$$

DEFINITION 4.1. We define the *fundamental matrices* $\{\Phi(n,m)\}$ for the kernel B in (4.1) as the unique solution of the equations

$$\begin{aligned} \Phi(n,m) &= I + \sum_{j=m}^{n} B(n,j)\Phi(j,m),\ 0 \le m \le n, \\ \Phi(n,m) &= I \text{ (the } d \times d \text{ identity matrix)},\ 0 \le n \le m. \end{aligned} \tag{4.2}$$

LEMMA 4.2. [5, Lemma 2.12] *The solution of (4.1) has the representation*

$$x(n) = \Phi(n,0)f(0) + \sum_{j=0}^{n} \Phi(n,j)\nabla f(j), \tag{4.3}$$

where $\nabla f(j) = f(j) - f(j-1)$ *for* $n \ge 1$ *and* $\nabla f(0) = 0$.

DEFINITION 4.3. For $\phi = \{\phi(n)\}_{n=0}^{\infty} \in S(X)$, we define

$$\nabla\phi(n) = \phi(n) - \phi(n-1),\ n \ge 0$$

and

$$|\phi|_{\nabla} = |\phi(0)| + |\nabla\phi|_{\infty}.$$

THEOREM 4.4. *For (4.1), assume that*

(i) $|\Phi(n,0)| \le M$ *for some* $M > 0$,
(ii) $\sup_{n\ge 0} \sum_{j=0}^{n} |\Phi(n,j)| \le M$,
(iii) $|f|_{\nabla} \le C$ *for some* $C > 0$.

Then the solution x *of (4.1) belongs to* $l^{\infty}(\mathbb{R}^d)$.

Proof. From (4.3), we have

$$\begin{aligned} |x(n)| &\le |\Phi(n,0)||f(0)| + \sum_{j=0}^{n} |\Phi(n,j)|\nabla f(j)| \\ &\le M|f|_{\nabla} + M|f|_{\nabla} = 2M|f|_{\nabla} \\ &\le 2MC. \end{aligned}$$

Hence

$$|x|_{\infty} = \sup_{n\ge 0} |x(n)| \le 2MC \equiv M'.$$

□

THEOREM 4.5. *For (4.1), we assume that*

(i) $|\nabla f|_{\nabla}^{p} \le C$ *for some* $C > 0, 1 \le p < \infty$,
(ii) $(\sum_{n=0}^{\infty}(\sum_{j=0}^{n} |\Phi(n,j)|^{q})^{\frac{p}{q}})^{\frac{1}{p}} = M < \infty$ *for some* $M > 0$,

where $1 \leq p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then for the solution* x *of (4.1),* $x \in l^p(\mathbb{R}^d)$.

Proof. In view of (4.3), we have

$$|x(n)| \leq |\Phi(n,0)||f(0)| + \sum_{j=0}^{n} |\Phi(n,j)||\nabla f(j)|, \ n \geq 1.$$

By the well-known Minkowski inequality and Hölder inequality, we obtain

$$\begin{aligned}
\left(\sum_{n=0}^{N} |x(n)|^p\right)^{\frac{1}{p}} &\leq \left(\sum_{n=0}^{N} |\Phi(n,0)||f(0)|^p\right)^{\frac{1}{p}} + \left(\sum_{n=0}^{N}\sum_{j=0}^{n} |\Phi(n,j)||\nabla f(j)|^p\right)^{\frac{1}{p}} \\
&\leq |f(0)| \left(\sum_{n=0}^{N} \left(\sum_{j=0}^{n} |\Phi(n,0)|^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} \\
&+ \left(\sum_{n=0}^{N} \left(\sum_{j=0}^{n} |\Phi(n,j)|^q\right)\right)^{\frac{p}{q}} \left(\sum_{j=0}^{n} |\nabla f(j)|^p\right)^{\frac{1}{p}} \\
&\leq |f(0)|M + M|\nabla f|_p \\
&= M|\nabla f|_{\nabla}^{p} \\
&\leq CM.
\end{aligned}$$

It follows that

$$|x|_p = \left(\sum_{n=0}^{\infty} |x(n)|^p\right)^{\frac{1}{p}} \leq CM.$$

This completes the proof. □

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