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THE HENSTOCK-PETTIS INTEGRAL OF BANACH SPACE-VALUED FUNCTIONS

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Abstract. In this paper, we study the Henstock-Pettis integral of Banach space-valued functions mapping an interval $[0,1]$ in $\mathbf{R}$ into a Banach space X. In particular, we show that a Henstock integrable function on $[0,1]$ is Henstock-Pettis integrable on $[0,1]$ and a Pettis integrable function is Henstock-Pettis integrable on $[0,1]$.

1. Introduction and Preliminaries

Let $[0,1]$ be the unit interval of the real line with the Lebesgue measure. If E is a measurable set, then |E| denotes the Lebesgue measure of E. A tagged partition of $[0,1]$ is a finite collection of pairs $\mathcal{P} = \{(I_1,t_1),(I_2,t_2),\cdots(I_n,t_n)\}$, where $I_1,\cdots,I_n$ are nonoverlapping subintervals in $[0,1]$ and t_i is a point of $I_i, i = 1,\cdots,p$, and $\cup_{i=1}^n I_i = [0,1]$. A gauge on E $\subset [0,1]$ is a positive function on E. For a given gauge δ, we say that a partition $\{(I_1,t_1),(I_2,t_2),\cdots(I_n,t_n)\}$ is δ-fine if $I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$, i=1,$\cdots$,n.

Throughout this paper, X is a Banach space with dual X^*.

A function $f : [0,1] \to X$ is said to be weakly measurable if for each $x^* \in X^*$ the real function x^*f is measurable; strongly measurable, or simply measurable if there is a sequence of simple functions f_n with $\lim_n \|f_n(t) - f(t)\| = 0$ for almost all $t \in [0,1]$.

Let $f : [0,1] \to X$ be function. We set

$$f(\mathcal{P}) = \sum_{i=1}^{n} f(t_i)|I_i|$$

for each partition $\mathcal{P} = \{(I_1,t_1),(I_2,t_2),\cdots(I_n,t_n)\}$ of $[0,1]$.

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2. The Henstock-Pettis Integral

Recall that a function $f : [0,1] \to X$ is said to be Henstock integrable on $[0,1]$ if there exists $w \in X$ with the following property: for every $\varepsilon > 0$ there exists a gauge δ on $[0,1]$ such that

$$\|f(\mathcal{P}) - w\| < \varepsilon$$

for each δ-fine partition $\mathcal{P}$ of $[0,1]$. We set $w = (H)\int_0^1 f$.

Definition 2.1. A function $f : [0,1] \to X$ is said to be Henstock-Pettis integrable on $[0,1]$ if for each $x^* \in X^*$ x^*f is Henstock integrable on $[0,1]$ and for each subinterval $[a,b]$ of $[0,1]$ there exists a vector $w_{[a,b]} \in X$ such that

$$x^*(w_{[a,b]}) = (H)\int_a^b x^*f.$$

We set $w_{[a,b]} = (HP)\int_a^b f$.

Theorem 2.1. *Let f and g be functions mapping $[0,1]$ into X.*

(a) If f is Henstock-Pettis integrable on $[0,1]$, then f is Henstock-Pettis integrable on every subinterval I of $[0,1]$.

(b) If f is Henstock-Pettis integrable on each of the intervals I_1 and I_2 in $[0,1]$, where I_1 and I_2 are nonoverlapping and $I_1 \cup I_2 = I$ is an interval, then f is Henstock-Pettis integrable on I and

$$(HP)\int_I f = (HP)\int_{I_1} f + (HP)\int_{I_2} f.$$

(c) If f and g are Henstock-Pettis integrable on $[0,1]$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is Henstock-Pettis integrable on $[0,1]$ and

$$(HP)\int_0^1 (\alpha f + \beta g) = \alpha(HP)\int_0^1 f + \beta(HP)\int_0^1 g.$$

Proof. (a) follows easily from the definition of the Henstock-Pettis integration.

(b) Since f is Henstock-Pettis integrable on each of the intervals I_1 and I_2 in $[0,1]$, for each $x^* \in X^*$ x^*f is Henstock integrable on each I_1 and I_2,

$$x^*(HP)\int_{I_1} f = (H)\int_{I_1} x^*f$$

and

$$x^*(HP)\int_{I_2} f = (H)\int_{I_2} x^* f.$$

Hence for each $x^* \in X^*$, $x^* f$ is Henstock integrable on I and

$$\begin{aligned} x^*\big((HP)\int_{I_1} f + (HP)\int_{I_1} f\big) &= (H)\int_{I_1} x^* f + (H)\int_{I_2} x^* f \\ &= (H)\int_{I} x^* f \end{aligned}$$

It follows that f is Henstock-Pettis integrable on I and

$$(HP)\int_{I} f = (HP)\int_{I_1} f + (HP)\int_{I_2} f,$$

(c) Since for each $x^* \in X^*$ $x^*\big(\alpha(HP)\int_0^1 f + \beta(HP)\int_0^1 g\big)$

$$\begin{aligned} &= \alpha x^*(HP)\int_0^1 f + \beta x^*(HP)\int_0^1 g \\ &= \alpha(H)\int_0^1 x^* f + \beta(H)\int_0^1 x^* g \\ &= (H)\int_0^1 x^*(\alpha f + \beta g), \end{aligned}$$

$\alpha f + \beta g$ is Henstock-Pettis integrable on $[0,1]$ and

$$(HP)\int_0^1 (\alpha f + \beta g) = \alpha(HP)\int_0^1 f + \beta(HP)\int_0^1 g.$$

□

THEOREM 2.2. *Let $f : [0,1] \to X$ be Henstock-Pettis integrable on $[0,1]$, and let $F(x) = (HP)\int_0^x f$ for each $x \in [0,1]$.*

(a) The function f is weakly measurable.

(b) If $f = 0$ almost everywhere on $[0,1]$, then f is Henstock-Pettis integrable on $[0,1]$ and $(HP)\int_0^1 f = 0$.

(c) If $f = g$ almost everywhere on $[0,1]$, then g is Henstock-Pettis integrable on $[0,1]$ and

$$(HP)\int_0^1 f = (HP)\int_0^1 g$$

Proof. (a) Since $x^* f$ is Henstock integrable for each $x^* \in X^*$, $x^* f$ is measurable. Hence f is weakly measurable.

(b) For each $x^* \in X^*$, $x^*f = 0$ almost everywhere on $[0,1]$. Hence, x^*f is Henstock integrable on $[0,1]$ and $(H)\int_0^1 x^*f = 0$. For each subinterval $[a,b]$ of $[0,1]$, $x^*(0) = 0 = (H)\int_a^b x^*f$. Hence, f is Henstock-Pettis integrable on $[a,b]$ and $(HP)\int_0^1 f = 0$.

(c) Since $f-g = 0$ almost everywhere on $[0,1]$, $f-g$ is Henstock-Pettis integrable on $[0,1]$ and $(HP)\int_0^1 (f-g) = 0$ by (b). Hence, $g = f-(f-g)$ is Henstock-Pettis integrable on $[0,1]$ and

$$(HP)\int_0^1 f - (HP)\int_0^1 g = (HP)\int_0^1 (f-g) = 0$$

□

LEMMA 2.3. *Let $f : [0,1] \to X$ be Henstock integrable on $[0,1]$. Then for each $x^* \in X^*$ the function x^*f is Henstock integrable on $[0,1]$ and*

$$x^*\big((H)\int_0^1 f\big) = (H)\int_0^1 x^*f$$

Proof. Since f is Henstock integrable on $[0,1]$, for every $\varepsilon > 0$ there exists a gauge δ on $[0,1]$ such that for any δ-fine partition $\mathcal{P} = \{(I,t)\}$ we have

$$\Big|\Big|f(\mathcal{P}) - (H)\int_0^1 f\Big|\Big| < \varepsilon.$$

For any $x^* \in X^*$, we have

$$\Big|x^*f(\mathcal{P}) - x^*\big((H)\int_0^1 f\big)\Big| \leq ||x^*||\Big|\Big|f(\mathcal{P}) - (H)\int_0^1 f\Big|\Big| < ||x^*||\varepsilon$$

for any δ-fine partition $\mathcal{P} = \{(I.t)\}$. Hence x^*f is Henstock integrable on $[0,1]$ and

$$(H)\int_0^1 x^*f = x^*\big((H)\int_0^1 f\big).$$

□

The following theorems show that the Henstock-Pettis integral is the generalization of the Henstock integral and the Pettis integral.

THEOREM 2.4. *If $f : [0,1] \to X$ is Henstock integrable on $[0,1]$, then f is Henstock-Pettis integrable on $[0,1]$ and*

$$(HP)\int_0^1 f = (H)\int_0^1 f.$$

Proof. For each $x^* \in X^*$, $x^* \in f$ is Henstock integrable on $[0,1]$ and $x^*\big((H)\int_0^1 f\big) = (H)\int_0^1 x^* f$ by lemma 2.3. Hence, f is Henstock-Pettis integrable on $[0,1]$ and

$$(HP)\int_0^1 f = (H)\int_0^1 f.$$

□

The following example shows that exists an Henstock-Pettis integrable function, but not Henstock integrable on $[0,1]$.

EXAMPLE 2.1. Let $A_n = [a_n, b_n] \subseteq [0,1]$ be a sequence of intervals such that $a_1 = 0$, $b_n < a_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_n = 1$ and define $f : [0,1] \to c_0$ by

$$f(t) = \Big(\frac{1}{2|A_{2n-1}|}\chi_{A_{2n-1}}(t) - \frac{1}{2|A_{2n}|}\chi_{A_{2n}}(t)\Big)_{n=1}^{\infty}$$

I_n [3]and [6], it is proved that f is a measurable Henstock-Pettis integrable function which is not Henstock integrable.

Recall that a function $f : [0,1] \to X$ is Pettis integrable on $[0,1]$ if f is weakly measurable such that $x^* f$ is Lebesgue integrable for all $x^* \in X^*$ and if for every measurable set $E \subset [0,1]$ there is an element $x_E \in X$ such that

$$x^*(x_E) = \int_E x^* f.$$

THEOREM 2.5. *If* $f : [0,1] \to X$*is Pettis integrable on* $[0,1]$, *then* f *is Henstock-Pettis integrable on* $[0,1]$ *and*

$$(HP)\int_0^1 f = (P)\int_0^1 f$$

Proof. Since a Lebesgue integrable function on $[0,1]$ is Henstock integrable on $[0,1]$, the result is obvious from the definition of Pettis integral and Henstock-Pettis integral. □

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