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TORQUES AND RIEMANN'S MINIMAL SURFACES

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Abstract. In this article, we prove that a properly embedded minimal surface in $\mathbf{R}^3$ of genus zero must be one of Riemann's minimal examples if outside of a solid cylinder it is the union of planar ends with the same torques at all integer heights.

1. Introduction

An immersed surface in $\mathbf{R}^3$ is said to be *minimal* if its mean curvature vanishes identically. In 1867, Riemann [7] used elliptic functions to classify all minimal surfaces in $\mathbf{R}^3$ that are foliated by circles and straight lines in horizontal planes. He showed that these examples are the plane, the catenoid, the helicoid and one parameter family $\{R_t\}_{t>0}$ with infinity topology. The new surfaces R_t, called Riemann's minimal surfaces, intersect horizontal planes in lines at precisely integer heights. We have more characterizations of R_t by:

(a) Each R_t is invariant under the reflection of $\mathbf{R}^3$ in the (x_1, x_3)-plane and by the translation T by $(t, 0, 2)$.

(b) In the complement of a solid cylinder of $\mathbf{R}^3$, the surface R_t consists of planar ends at integer heights. *A planar end* means that a properly embedded finite total curvature minimal annulus with compact boundary, which is asymptotic to the end of a plane.

(c) Each R_t is conformally equivalent to $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$ punctured in a discrete set of points with 0 and ∞ being limit points of ends.

(d) The quotient surface R_t/T is a properly embedded finite total curvature minimal torus in $\mathbf{R}^3/T$ with two planar ends.

In 2001, the author studied the characterization problem of Riemann's minimal surfaces without assuming periodicity and showed that:

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Theorem 1.1. [2] *Let M be a properly embedded minimal surface in $\mathbf{R}^3$ of genus zero, which is the union of planar ends $\{E_n\}_{n\in\mathbf{N}}$ outside of a solid cylinder, at integer heights. Suppose that it is symmetric by the reflection in a plane, and there is a number $k \in \mathbf{Z}$ such that $Torque(E_{n+2k}) = Torque(E_n)$ for all $n \in \mathbf{Z}$. Then M must be one of Riemann's minimal examples.*

However, we realize that the assumption of the existence of symmetric plane is not necessarily, and can prove more elaborate result:

Theorem 1.2 (Main Theorem). *A properly embedded minimal surface in $\mathbf{R}^3$ of genus zero is a Riemann's minimal example if it is the union of planar ends with the same torques at all integer heights in the complement of a solid cylinder.*

Recall a torque vector of a closed curve points to the direction with the largest tendency of rotation of a surface in $\mathbf{R}^3$ around the curve. In particular, for a minimal surface of $\mathbf{R}^3$ the torque vector associated to a planar end whose Gauss map has ramification order 2, like as ends of R_t, describes the intersection of the surface with the limit affine tangent plane. Precisely, this intersection curve is asymptotic to a straight line in the direction of the torque. Recall in R_t, all of the torque vectors of the planar ends are the same. We use the maximum principle of a minimal surface and Liouville's theorem to prove that M is also periodic in such a setting. Then, by virtue of the previous result of Meeks, Pérez and Ros in [4], we can say that M is one of R_t.

2. Preliminaries

A minimal surface in the 3-dimensional Euclidean space is a conformal harmonic immersion $X : \tilde{S} \hookrightarrow \mathbf{R}^3$ where $\tilde{S}$ is a 2-dimensional smooth manifold, with or without boundary. The well known *Enneper-Weierstrass representation* of a minimal surface of $\mathbf{R}^3$ follows that

$$X(p) = \Re \int^p \left(\frac{1}{2} f\,(1-g^2),\, \frac{i}{2} f\,(1+g^2),\, fg \right) dz \tag{1}$$

where f is a holomorphic function and g is a meromorphic function on M, such that when a pole of order m of g occurs, f has a zero of order $2m$, and this is the only case where f can vanish. In fact, g is the stereographic projection of the Gauss map of X with respect to the north pole. Applying Stoke's theorem to the isometric (minimal) immersion X we have

$$\int_S \Delta_S X\, dA = \int_{\partial S} \nu ds = 0$$

where S is a compact domain of $\tilde{S}$, dA is the element of area on S, Δ_S is the Laplacian on S, ds is the line element on ∂S, and ν is the outward conormal. Define *the flux* of X along a closed curve $\gamma \subset S$ as

$$Flux(\gamma) := \int_\gamma \nu\, ds$$

which is well defined on the homology class of $[\gamma]$. Now let $R_{\vec{u}}$ be the Killing field associated with counter-clockwise rotation about the axis $\ell_{\vec{u}}$ in the $\vec{u}$ direction, then

$$\int_{\partial S} R_{\vec{u}} \cdot \nu\, ds = \int_{\partial S} (X \wedge \nu) \cdot \vec{u}\, ds = 0.$$

This motivates defining the homologous invariant *torque* of a closed curve γ on S by

$$Torque_0(\gamma) := \int_\gamma X \wedge \nu.$$

In general, the torque is dependent on the base point of the position vector X. If we move the base point from 0 to $W \in R^3$, then the position vector based on W is $X - W$, and the torque is that

$$\begin{aligned} Torque_W(\gamma) &= Torque_0(\gamma) - W \wedge \int_\gamma \nu \\ &= Torque_0(\gamma) - W \wedge Flux(\gamma). \end{aligned} \tag{2}$$

Observe that we can define the flux and the torque associated to a planar end as that of one representative curve for the end.

3. Proof of main theorem

Since M is a properly embedded minimal surface of genus zero with two limit ends, by [6], it has the conformal type $\mathbf{C}^* \setminus K$ where K is the set of discrete punctures $p_n \in \mathbf{C}^*$, $n \in \mathbf{Z}$, corresponding to the planar ends E_n at integer height $x_3 = n$, respectively. Now we define a conformal harmonic embedding of M by

$$X : \mathbf{C}^* \setminus K \to \mathbf{R}^3.$$

Let $X = (X^1, X^2, X^3)$. Since M cuts transversally any horizontal plane, we can say that;

$$X^3(z) = \frac{1}{\log r} \log |z|$$

for some $r > 1$. Moreover, except at integer heights, M meets a horizontal plane in a compact Jordan curve, and hence X^3 can be continuously extended to the whole $\mathbf{C}^*$. In particular, since E_n is close to $\{x_3 = n\}$ at infinity, we have $X^3(p_n) = \frac{1}{\log r} \log |p_n| = n$ for all $p_n \in K$. It follows that $|p_n| = r^n$ for all $n \in \mathbf{Z}$. If $(g, f\,dz)$ is *the Weierstrass-data* of X, then from (1) we have;

$$f(z)\,g(z) := 2\frac{\partial X^3}{\partial z} = \frac{d}{dz}\left(\frac{1}{\log r}\log z\right) = \frac{1}{\log r}\,\frac{1}{z} \tag{3}$$

for all $z \in \mathbf{C}^*$. Since both z and dz have no zero or pole in $\mathbf{C}^*$, every planar end of M has the minimum branching order 2.

Let $C_R \subset \mathbf{R}^3$ be a solid cylinder with sufficiently large radius $R > 0$ such that M is the union of planar ends in complement of C_R, and let $(s, t, 1) \in \mathbf{R}^3$ be the direction of the axis line of the cylinder for some $s, t > 0$. Denote by $\tilde{M}$ the translation of M along the direction $2(s, t, 1)$. Since M and $\tilde{M}$ are conformally equivalent, with another suitable coordinate ζ, we have a minimal embedding

$$\tilde{X} : \mathbf{C}^* \setminus K \hookrightarrow \mathbf{R}^3$$

of $\tilde{M}$ such that $\tilde{X}^3(\zeta) = \frac{1}{\log r} \log |\zeta|$. Therefore, we can say that

$$X^3 \equiv \tilde{X}^3 \quad \text{on } \mathbf{C}^*. \tag{4}$$

Now let $(\tilde{g}, \tilde{f}dz)$ be the Weierstrass data of X and $\tilde{X}$, then similar to (3), we have $\tilde{f}(z)\,\tilde{g}(z) = \frac{1}{\log r}\,\frac{1}{z}$ for all $z \in \mathbf{C}^*$. Since all the planar ends have the minimum branching order 2, we have

$$g(z) = (z - p_n)^2\, h(z), \quad \tilde{g}(z) = (z - p_n)^2\, \tilde{h}(z)$$

on a sufficiently small neighborhood $D_n \subset \mathbf{C}^*$ of p_n, where $p_m \notin D_n$ if $m \neq n$, and h and $\tilde{h}$ are the holomorphic functions on D_n with $h(p_n) \neq 0$ and $\tilde{h}(p_n) \neq 0$. Therefore,

$$f(z) = \frac{1}{\log r}\,\frac{1}{z\,g(z)} = \frac{1}{\log r}\,\frac{1}{p_n\,h(p_n)}\,\frac{1}{(z - p_n)^2} + F(z)$$
$$\tilde{f}(z) = \frac{1}{\log r}\,\frac{1}{z\,\tilde{g}(z)} = \frac{1}{\log r}\,\frac{1}{p_n\,\tilde{h}(p_n)}\,\frac{1}{(z - p_n)^2} + \tilde{F}(z)$$

where F and $\tilde{F}$ are holomorphic functions on D_n, and hence from (1);

$$(X^1 - i\,X^2\,)\,(z) = \frac{-1}{2\log r}\,\frac{1}{p_n\,h(p_n)}\,\frac{1}{z-p_n} + O(|z-p_n|)$$

$$(\tilde{X}^1 - i\tilde{X}^2)(z) = \frac{-1}{2\log r}\,\frac{1}{p_n\,\tilde{h}(p_n)}\,\frac{1}{z-p_n} + O(|z-p_n|).$$

Take a representative curve γ_n for the planar end E_n of M by

$$\gamma_n(\theta) = \left(R\cos\theta,\ R\sin\theta,\ \frac{1}{R}(\gamma_n^1\cos\theta + \gamma_n^2\sin\theta) + O(R^{-2})\right)$$

where $0 \le \theta \le 2\pi$ and

$$\gamma_n^1 X^1 + \gamma_n^2 X^2 = \Re\left(\frac{1}{-2(\log r)^2\,{p_n}^2\,h(p_n)}(X^1 + iX^2)\right).$$

Then the conormal is $\nu(\theta) = (\cos\theta, \sin\theta, 0) + O(R^{-2})$, and so we can compute the flux $Flux(E_n) = (0,0,0)$ and the torque by

$$Torque(E_n) = \pi(-\gamma_n^2, \gamma_n^1, 0) = -\frac{i\,\pi}{2\,(\log r)^2}\,\left(\overline{({p_n}^2\,h(p_n))^{-1}},\ 0\right)$$

which is independent of the base point, see (2). Similarly, the torque of the end $\tilde{E}_n$ of $\tilde{M}$ at the height $x_3 = n$ is

$$Torque\,(\tilde{E}_n) = -\frac{i\,\pi}{2\,(\log r)^2}\,\left(\overline{({p_n}^2\,\tilde{h}(p_n))^{-1}},\ 0\right).$$

Since both E_n and $\tilde{E}_n$ have the same torques, $\tilde{h}(p_n) = h(p_n)$ and

$$(X^1 - i\,X^2\,)\,(z) - (\tilde{X}^1 - i\,\tilde{X}^2\,)\,(z) = O(|z-p_n|)$$

on the small neighborhood of p_n. Together with (4), it shows that each $p_n \in K$ is the removable singularity of $X - \tilde{X}$. Hence, we can obtain the extended harmonic map

$$Y : \mathbf{C}^* \hookrightarrow \mathbf{R}^3$$

of $X - \tilde{X}$ such that $Y(p_n) = 0$ for all $n \in \mathbf{Z}$. Take a pairwise disjoint connected neighborhood $U_n \subset \mathbf{C}^*$ of p_n, $n \in \mathbf{Z}$, respectively, such that both $M \setminus \bigcup_{n\in\mathbf{Z}} X(U_n)$ and $\tilde{M} \setminus \bigcup_{n\in\mathbf{Z}} \tilde{X}(U_n)$ are contained in C_R, then;

$$\|Y(z)\| = \|(X^1 - i\,X^2)(z) - (\tilde{X}^1 - i\,\tilde{X}^2)(z)\| \le 2R\sqrt{1+s^2+t^2}$$

on ∂U_n. By the maximum principle of the harmonic map, we have;

$$\|Y\| \le 2R\sqrt{1+s^2+t^2} \quad \text{on } U_n$$

for all $n \in \mathbf{Z}$. By the definition of U_n, the above inequality also holds in the complement of $\bigcup_{n\in\mathbf{Z}} U_n$, so Y is a bounded harmonic map on the punctured plane $\mathbf{C}^*$. By virtue of the Liouville's theorem, Y is then a constant map $Y \equiv 0$ by $Y(p_n) = 0$. Hence $M = \tilde{M}$ and M is periodic under a translation. However, we know the result of [4]: A properly periodic embedded minimal annulus in $\mathbf{R}^3$ must be one of Riemann's minimal examples. Thus M is also one of Riemann's minimal examples, and we have proved the theorem.

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