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GENERIC SUBMANIFOLDS OF AN ALMOST CONTACT MANIFOLDS

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Abstract. In this paper, we are to study the generic submanifold M of a Kenmotsu manifold and consider the integrability condition of the almost complex structure induced on the even-dimensional product manifold $M \times R^p \times R^1$ where p is the codimension.

1. Introduction

S. Tanno [4] has classified connected almost contact Riemannian manifolds whose automorphism groups have the maximal dimension into three classes : the one is homogeneous normal contact Riemannian manifolds with $K(X, \xi) > 0$, a global Riemannian products of a line or a circle and a Kaehlerian manifold with constant holomorphic sectional curvature if $K(X, \xi) = 0$ and a warped product space $L \times_f CE^n$ if $K(X, \xi) < 0$, where $K(X, \xi)$ is the sectional curvature of plane sections containing ξ. It is well known that the first and second cases in the above statements are characterized by some tensor equations and they have Sasakian and cosympletic structures. For the thired case, K. Kenmotsu [2] characterized by tensor equations and studied their properties. Such a structure is normal, but not Sasakian.

On the other hand, many authors have studied the generic (or antiholomorphic) submanifolds of Kaehlerian or Sasakian manifolds.

In this paper, we are to study the generic submanifold M of a Kenmotsu manifold and consider the integrability condition of the almost complex structure induced on the even-dimensional product manifold

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$M \times R^p \times R^1$, where p is the codimension. Throughout this paper, the range of indices system are as follows:

$$A, B, C, D, \cdots : 1, 2, \cdots, 2m+2,$$
$$a, b, c, d, \cdots : 1, 2, \cdots, n,$$
$$x, y, z, w, \cdots : n+1, \cdots, n+p = 2m+1,$$
$$* \quad : 2m+2$$

Manifolds, submanifolds, all geometric objects and mappings we discuss in this paper are assumed to be differentiable and of class C^∞.

2. Kenmotsu manifold

Let N be an almost contact manifold with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is (1,1) tensor, ξ is a vector field, η is a 1-form and g is a Riemannian metric on X such that

$$\phi^2 X = -X + \eta(X) \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = 1, \tag{2.2}$$

$$\phi(\xi) = 0, \tag{2.3}$$

$$\xi(\phi X) = 0, \tag{2.4}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.5}$$

$$g(\phi X, Y) = -g(X, \phi Y), \tag{2.6}$$

$$g(X, \xi) = \eta(X) \tag{2.7}$$

for any vector fields X and Y on N An almost contact metric manifold

is called Kenmotsu manifold ([2]) if

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.8}$$

where ∇ is the covariant differentiation with respect to g. It is easily see that

$$\nabla_X \xi = -\phi^2 X = -X - \eta(X)\xi \tag{2.9}$$

from (2.3) and (2.8)

The fundamental 2-form Φ is defined by $\Phi(X, Y) = g(\Phi X, Y)$ is skew symmetric.

An almost contact structure (ϕ, ξ, η) on N is said to be normal if the almost complex structure J on $N \times R^1$ given by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}), \tag{2.10}$$

f being a c^∞-function to the condition

$$T(\phi, \phi) + 2d\eta \otimes \xi = 0,$$

where $T(\phi, \phi)$ denotes the Nijenhuis tensor

$$T(\phi, \phi)(X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] \tag{2.11}$$

of ϕ

An almost contact metric structure (ϕ, ξ, η, g) on N is said to be
(a) quasi Sasakian [1,3] if Φ is closed and (ϕ, ξ, η) is normal,
(b) cosymplectic [1,3] if Φ and η are closed and (ϕ, ξ, η) is normal,
(c) Sasakian [1,3] if $\Phi = d\eta$ and (ϕ, ξ, η) is normal.

It is well known that [2]

Proposition 2.1. The Kenmotsu manifold is normal but not quasi-Sasakian and hence not Sasakian.

Let $L(X)$ be the Lie derivative along X , then we see that [2]

Proposition 2.2 On the Kenmotsu manifold,

$$(\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y)$$
$$L(\xi)g = 2(g - \eta \otimes \eta)$$
$$L(\xi)\phi = 0$$
$$L(\xi)\eta = 0$$

3. Generic submanifold of a Kenmotsu manifold

Let N be a $(2m+1)$-dimensional Kenmotsu manifold and M be an n-dimensional generic submanifold of N, that is the normal space of N is transformed into tangent space by ϕ

Let B_C and C_X be the local basis of the tangent space and normal space of N, then the induced metric on N and normal bundle are given by

$$g_{cb} = G(B_c, B_b),$$

$$\bar{g}_{xy} = G(C_x, C_y).$$

We can put

$$\begin{aligned} \phi_j{}^h B_c{}^j &= f_c{}^a B_a{}^h - f_c{}^x C_x{}^h \\ \phi_j{}^h C_X{}^j &= f_x{}^a B_a{}^h \\ \xi^h &= f^a B_a{}^h + f^x C_x{}^h \end{aligned} \tag{3.1}$$

where $f_c{}^a$ is (1,1)tensor field defined on N, $f_c{}^x$ is local 1-form for each fixed index X, f^a a vector fired, f^x a function for each fixed index X and

$$f_X{}^a = f_c{}^y g^{ac} \bar{g}_{yx}$$

By use of and (3.1)-(3.3), we have

$$\begin{aligned} f_c{}^a f_a{}^b &= -\delta_c{}^b + f_c{}^x f_x{}^b + \eta_c f^b \\ f_c{}^a f_a{}^x &= -\eta_c f^x \\ f_x{}^a f_a{}^b &= f_x f^b \\ f_x{}^a f_a{}^y &= \delta_x{}^y - f_x f^y \\ f^a f_a{}^b + f^x f_x{}^b &= 0 \\ f^a f_a{}^x &= 0 \\ f_a f^a + f_x f^x &= 1 \end{aligned} \tag{3.4}$$

We easily see that

$$f_{cb} = -f_{bc} \qquad \textit{and} \qquad f_{ax} = f_{xa}. \tag{3.5}$$

Let $\tilde{\nabla}$ be the operation of the covariant differentiation on M, then the induced connection ∇_c on N is given by

$$\nabla_c = B_c{}^j \tilde{\nabla}_j$$

and that we have the equations of Gauss and Weingarten for N

$$\nabla_c B_b{}^h = h_{cb}{}^x C_x{}^h, \tag{3.6}$$

$$\nabla_c C_x{}^h = -h_c{}^a{}_x B_a{}^h \tag{3.7}$$

respectively, where $h = (h_{cb}{}^x)$ is the second fundamental tensor with respect to C_x.
The structure equations of N are given by

$$R_{abcd} = K_{kjih} B_a{}^k B_b{}^j B_c{}^i B_d{}^h + h_{adx} h_{ac}{}^x, \tag{3.8}$$

$$K_{kji}{}^h B_b{}^k B_c{}^j B_a{}^i C_h{}^x = \nabla_b h_{ca}{}^x - \nabla_c h_{ba}{}^x, \tag{3.9}$$

$$K_{kji}{}^h B_b{}^k B_c{}^j C_x{}^i C_h{}^x = R_{bcx}{}^y - (h_{be}{}^y h_c{}^e{}_x - h_{ce}{}^y h_b{}^e{}_x). \tag{3.10}$$

where R_{abcd} is the curvature tensor of N and $R_{bcx}{}^y$ the curvature tensor of the connection induced in the normal bundle.

If we apply ∇_b to (3.1) and take account of (2.8),(2.9),(3.6) and (3.7), we have

$$\begin{aligned}
\nabla_d f_c{}^a &= -f_c f_d{}^a - f^a f_{dc} + h_{dc}{}^x f_x{}^a - h_d{}^a{}_x f_c{}^x \\
\nabla_d f_c{}^x &= -f_c f_d{}^x + f^x f_{dc} + h_{da}{}^x f_c{}^a \\
\nabla_d f_x{}^a &= -f_x f_d{}^a - f^a f_{xd} - h_d{}^b x f_b{}^a \\
f_x{}^a h_{da}{}^y &= f_x f_d{}^y - f^y f_{xd} + h_d{}^b x f_b{}^y \\
\nabla_d f^a &= \delta_d{}^a - f^a f_d + f^x h_d{}^a x \\
\nabla_d f^x &= -f^x f_d - f^a h_{da}{}^x
\end{aligned} \tag{3.11}$$

4. An almost complex structure

Let N be an n-dimensional generic submanifold of the Kenmotsu manifold. Denote the product manifold $N \times R^p \times R^1$ by M for the $p(= 2m+1-n)$-dimensional Euclidean space R^p and define on M a tensor field F of type (1,1) with local components $F_B{}^A$ given by

$$\textbf{(4.1)}\ (\mathbf{F_B{}^A}) = \begin{pmatrix} f_b{}^a & -f_b{}^x & -f_b \\ f_y{}^a & 0 & -f_y \\ f^a & f^x & 0 \end{pmatrix}$$

in (M, x^A) , (N, x^a) being a coordinate neighborhood of N and

$$(x^{n+1}, \cdots\cdots, x^{n+p} = x^{2m+1})$$

being a Cartesian coordinate in R^p and $x^* = x^{2m+2}$ a natural coordinate in R^1. Then taking account of (3.4), we see that $F^2 = -I$ holds on M. Thus we have

Proposition 4.1. Let N be a generic submanifold of the Kenmotsu manifolds. Then M is an almost complex manifold.

The Nijenhuis tensor of the almost complex structure F has local components

$$[F,F]_{CB}{}^A = F_C{}^E \partial_E F_B{}^A - F_B{}^E \partial_E F_C{}^A - (\partial_C F_B{}^E - \partial_B F_C{}^E) F_E{}^A \tag{4.2}$$

We denote $[F,F]_{CB}{}^{A}$ by $N_{CB}{}^{A}$. Then, by use of (3.1), the non-vanishing components of $N_{CB}{}^{A}$ are given by

$$N_{cb}{}^{a} = f_c{}^e\partial_e f_b{}^a - f_b{}^e\partial_e f_c{}^a - (\partial_c f_b{}^e - \partial_b f_c{}^e)f_e{}^a + (\partial_c f_b{}^x - \partial_b f_c{}^x)f_x{}^a + (\partial_c f_b - \partial_b f_c)f^a \tag{4.3.1}$$

$$N_{cb}{}^{x} = -f_c{}^e\partial_e f_b{}^x + f_c{}^y\partial_y f_b{}^x + f_b{}^e\partial_e f_c{}^x - f_b{}^y\partial_y f_c{}^x + (\partial_c f_b{}^e - \partial_b f_c{}^e)f_e{}^x + (\partial_c f_b - \partial_b f_c)f^x \tag{4.3.2}$$

$$N_{cb}{}^{*} = -f_c{}^e\partial_e f_b + f_b{}^e\partial_e f_c + (\partial_c f_b{}^e - \partial_b f_c{}^e)f_e - (\partial_c f_b{}^y - \partial_b f_c{}^y)f_y \tag{4.3.3}$$

$$N_{cy}{}^{a} = f_c{}^e\partial_e f_y{}^a - f_b{}^x\partial_x f_y{}^a - f_y{}^e\partial_e f_c{}^a - (\partial_c f_y{}^e)f_e{}^a - (\partial_y f_c{}^x)f_x{}^a \tag{4.3.4}$$

$$N_{cy}{}^{x} = f_y{}^e\partial_e f_c + (\partial_c f_y{}^e)f_e{}^x \tag{4.3.5}$$

$$N_{cy}{}^{*} = f_c{}^z\partial_z f_y + f_y{}^e\partial_e f_c + (\partial_c f_y{}^e)f_e - (\partial_y f_c{}^z)f_z \tag{4.3.6}$$

$$N_{c*}{}^{a} = f_c{}^e\partial_e f^a - f^e\partial_e f_c{}^a - (\partial_c f^e)f_e{}^a - (\partial_c f^z)f_z{}^a \tag{4.3.7}$$

$$N_{c*}{}^{x} = -f_c{}^z\partial_z f^x + f^e f^z\partial_z f_c{}^x + (\partial_c f^e)f_e{}^x \tag{4.3.8}$$

$$N_{c*}{}^{*} = f^e\partial_e f_c + (\partial_c f^e)f_e \tag{4.3.9}$$

$$N_{zy}{}^{a} = f_z{}^e\partial_e f_y{}^a - f_y{}^e\partial_e f_z{}^a - (\partial_z f_y{}^e - \partial_y f_z{}^e)f_e{}^a + (\partial_z f_y - \partial_y f_z)f^a \tag{4.3.10}$$

$$N_{zy}{}^{x} = (\partial_z f_y{}^e - \partial_y f_z{}^e)f_e{}^x + (\partial_z f_y - \partial_y f_z)f^x \tag{4.3.11}$$

$$N_{zy}{}^{*} = (\partial_z f_y{}^e - \partial_y f_z{}^e)f_e \tag{4.3.12}$$

(4.3.13) $$N_{z*}{}^{a} = f_{z}{}^{e}\partial_{e}f^{a} - f^{e}\partial_{e}f_{z}{}^{a} - f^{x}\partial_{x}f_{z}{}^{a} - (\partial_{z}f^{x})f_{x}{}^{a}$$

(4.3.14) $$N_{z*}{}^{x} = 0$$

(4.3.15) $$N_{z*}{}^{*} = f^{y}(\partial_{y}f^{z}) - (\partial_{z}f^{y})f_{y}$$

The Nijenhuis tensor of F satisfies the condition [5]

(4.4) $$N_{CB}{}^{A}F_{B}{}^{E} + N_{CB}{}^{E}F_{E}{}^{A} = 0$$

Substituting (3.1) into (3.18), we obtain

(4.5.1) $$N_{ce}{}^{a}f_{b}{}^{e} - N_{cb}{}^{e}f_{e}^{a} - N_{cy}{}^{a}f_{b}^{x} + N_{cb}{}^{y}f_{y}{}^{a} - N_{c*}{}^{a}f_{b} + N_{cb}{}^{*}f^{a} = 0$$

(4.5.2) $$N_{ce}{}^{x}f_{b}{}^{e} - N_{cy}{}^{x}f_{b}^{y} - N_{c*}{}^{x}f_{b} - N_{cb}{}^{e}f_{e}{}^{x} + N_{cb}{}^{*}f^{x} = 0$$

(4.5.3) $$N_{ce}{}^{*}f_{b}{}^{e} - N_{cy}{}^{*}f_{b}^{y} - N_{c*}{}^{*}f_{b} - N_{cb}{}^{y}f_{y} = 0$$

(4.5.4) $$N_{ce}{}^{a}f_{y}{}^{e} - N_{c*}{}^{a}f_{y} + N_{cy}{}^{e}f_{e}^{a} + N_{cy}{}^{z}f_{z}{}^{a} + N_{cy}{}^{*}f^{a} = 0$$

(4.5.5) $$N_{ce}{}^{x}f_{y}{}^{e} - N_{c*}{}^{x}f_{y} - N_{cy}{}^{e}f_{e}{}^{x} + N_{cy}{}^{*}f^{x} = 0$$

(4.5.6) $$N_{ce}{}^{*}f_{y}{}^{e} - N_{c*}{}^{*}f_{y} - N_{cy}{}^{e}f_{e} - N_{cy}{}^{z}f_{z} = 0$$

(4.5.7) $$N_{ce}{}^{a}f^{e} + N_{cy}{}^{a}f^{y} + N_{c*}{}^{e}f_{e}{}^{a} + N_{c*}{}^{y}f_{y}{}^{a} + N_{c*}{}^{*}f^{a} = 0$$

(4.5.8) $$N_{ce}{}^{x}f^{e} + N_{cy}{}^{x}f^{y} - N_{c*}{}^{e}f_{e}{}^{x} + N_{c*}{}^{*}f^{x} = 0$$

(4.5.9) $$N_{ce}{}^{*}f^{e} + N_{cy}{}^{*}f^{y} - N_{c*}{}^{e}f_{e} - N_{c*}{}^{y}f_{y} = 0$$

(4.5.10) $$N_{ze}{}^{a}f_{b}{}^{e} - N_{zy}{}^{a}f_{b}^{y} - N_{z*}{}^{a}f_{b} + N_{zb}{}^{e}f_{e}{}^{a} + N_{zb}{}^{y}f_{y}{}^{a} + N_{zb}{}^{*}f^{a} = 0$$

(4.5.11) $$N_{ze}{}^{x}f_{b}{}^{e} - N_{zy}{}^{x}f_{b}{}^{y} - N_{z*}{}^{x}f_{b} - N_{zb}{}^{e}f_{e}{}^{x} + N_{zb}{}^{*}f^{x} = 0$$

(4.5.12) $$N_{ze}{}^{*}f_{b}{}^{e} - N_{zy}{}^{*}f_{b}{}^{y} - N_{z*}{}^{*}f_{b} - N_{zb}{}^{e}f_{e} - N_{zb}{}^{y}f_{y} = 0$$

(4.5.13) $$N_{ze}{}^{a}f_{y}{}^{e} - N_{z*}{}^{a}f_{y} + N_{zy}{}^{e}f_{w}{}^{a} + N_{zy}{}^{*}f^{a} = 0$$

(4.5.14) $$N_{ze}{}^{x}f_{y}{}^{e} - N_{z*}{}^{x}f_{y} - N_{zy}{}^{e}f_{e}^{x} + N_{zy}{}^{*}f^{x} = 0$$

(4.5.15) $$N_{ze}{}^{a}f^{e} + N_{zy}{}^{a}f^{y} + N_{z*}{}^{e}f_{y}{}^{a} + N_{z*}{}^{*}f^{a} = 0$$

(4.5.16) $$N_{ze}{}^{x}f^{e} + N_{zy}{}^{x}f^{y} - N_{z*}{}^{e}f_{e}{}^{x} + N_{z*}{}^{*}f^{x} = 0$$

(4.5.17) $$N_{ze}{}^{*}f^{e} + N_{zy}{}^{*}f^{y} - N_{z*}{}^{e}f_{e} - N_{z*}{}^{y}f_{y} = 0$$

(4.5.18) $$N_{ze}{}^{*}f^{e} + N_{zy}{}^{*}f^{y} - N_{z*}{}^{e}f_{e} - N_{z*}{}^{y}f_{y} = 0$$

(4.5.19) $$N_{*e}{}^{a}f_{b}{}^{e} - N_{*y}{}^{a}f_{b}{}^{y} + N_{*b}{}^{e}f_{e}^{a} + N_{*b}{}^{y}f_{y}{}^{a} + N_{*b}{}^{*}f^{a} = 0$$

(4.5.20) $$N_{*e}{}^{x}f_{b}{}^{e} - N_{*y}{}^{x}f_{b}^{y} - N_{*b}{}^{e}f_{e}{}^{x} = 0$$

(4.5.21) $$N_{*e}{}^{*}f_{b}{}^{e} - N_{*y}{}^{*}f_{b}^{y} - N_{*b}{}^{e}f_{e} - N_{*b}{}^{y}f_{y} = 0$$

(4.5.22) $$N_{*e}{}^{a}f_{y}{}^{e} + N_{*x}{}^{e}f_{e}{}^{a} + N_{*y}{}^{z}f_{z}{}^{a} + N_{*y}{}^{*}f^{a} = 0$$

(4.5.23) $$N_{*e}{}^{x}f_{y}{}^{e} - N_{*y}{}^{e}f_{e}{}^{x} + N_{*y}{}^{*}f^{x} = 0$$

(4.5.24) $$N_{*e}{}^{*}f_{y}{}^{e} - N_{*y}{}^{e}f_{e} - N_{*y}{}^{z}f_{z} = 0$$

(4.5.25) $$N_{*e}{}^{a}f^{e} + N_{*y}{}^{a}f^{y} = 0$$

(4.5.26) $$N_{*e}{}^{x}f^{e} + N_{*y}{}^{x}f^{y} = 0$$

(4.5.27) $$N_{*e}{}^{*}f^{e} + N_{*y}{}^{*}f^{y} = 0$$

Assume that $N_{cb}{}^{a} = 0$, $N_{cb}{}^{x} = 0$, $N_{cb}{}^{*} = 0$. Then the equation (4.5.1) is reduced to

(4.6) $$N_{cy}{}^{a}f_{b}{}^{y} + N_{c*}{}^{a}f_{b} = 0$$

Transvecting f^{b}, we obtain $N_{c*}{}^{a} = 0$ if $\lambda(1-\lambda^{2}) \neq 0$ almost every where for $\lambda^{2} = f_{x}f^{x}$. This fact (3.1) and (4.6), we easily see that $N_{cz}{}^{a} = 0$.

By the same methord, we get $N_{c*}{}^{x} = 0$ and $N_{cz}{}^{x} = 0$ from (4.5.2). Moreover we see that $N_{c*}{}^{*} = 0$, $N_{cz}{}^{*} = 0$ from (4.5.3) and $N_{z*}{}^{a} = 0$ and $N_{zy}{}^{a} = 0$ from (4.5.10) and $N_{z*}{}^{x} = 0$ and $N_{zy}{}^{x} = 0$ from(4.5.11) and $N_{zy}{}^{*} = 0$ and $N_{z*}{}^{*} = 0$ from (4.5.12) and finally we have $N_{zy}{}^{x} = 0$ from (4.5.13).

Hence we can state THEOREM 4.1 Let N be a generic submanifold of Kenmotsu manifolds. If the components $N_{cb}{}^{a} = 0$, $N_{cb}{}^{x} = 0$ and $N_{cb}{}^{*} = 0$ and $\lambda(1-\lambda^{2}) \neq 0$ a.e, then all components of the Nijenhuis tensor all vanish

It is well Known that [5] the necessary and sufficient condition an almost complex structure F to be integrable is the components of N formed by F are all vanish. Thus we have

THEOREM 4.2. Let N be a generic submanifold of Kenmotsu manifolds. Then the almoat complex structure F is integrable if and only if the components ${N_{cb}}^a = 0$, ${N_{cb}}^x = 0$ and ${N_{cb}}^* = 0$ and $\lambda(1-\lambda^2) \neq 0$ a.e.

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