JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **19**, No. 4, December 2006

ON FIBRED KAEHLERIAN SPACES

Jin Hyuk Choi *

Abstract. In this paper, we are to construct a new fibred Riemannian space with almost complex structure from the lift of an almost contact structures of the base space and that of each fibre. Moreover, we deal with the fibred Riemannian space with various Kaehlerian structure.

1. Introduction

Fibred Riemannian space was first considered by Y.Muto [5] and treated by B.L. Reinhart [8] in the name of foliated Riemannian manifolds. B.O'Neill[7] called such a foliation a Riemannian submersion and gave its structure equations and in the almost same time K.Yano and S. Ishihara[11] developed an extensive theory of fibred Riemannian space.

M.Ako[1] and T.Okubo[6] studied fibred space with almost complex or almost Hermitian structure.These work were synthetically reported in S.Ishihara and M.Konishi's monograph[3].

In connection with almost contact structures, S.Tanno[9] investigated principal bundles over almost complex spaces having a 1-dimensional structure group. Generalizing Calabi-Eikmann's example, A.Morimoto[4] defined an almost complex structure in the product of two almost contact spaces and obtained a condition on the normality, and S. Ishihara and K.Yano[11] researched similar properties for the product of two framed manifolds. Y.Tashiro and B.H.Kim[10]have studied fibred Riemannian spaces with almost Hermitian or almost contact metric structure. They applied these results to the study of tangent bundle of Riemannian spaces.

Received December 5, 2006.
2000 Mathematics Subject Classification: Primary 53B20, 53B35.
Key words and phrases: conformal, conharmonic transformation, almost complex structure, kaehlerian space, fibred Riemannian space.
* This paper was partially supported by Korea Science and Engineering Foundation Grant(R05-2004-000-11588).

In this point of view, we constructed a new almost complex structure on the fibred Riemannian space from the lift of an almost complex structures of the base space and that each fibre. Moreover, we shall deal with the fibred Riemannian space with almost Kaehlerian or nearly Kaehlerian structure.

2. Fibred Riemannian space

Let $\{M,\ B,\ \tilde{g},\ \pi\}$ be a fibred Riemannian space,that is, M an m-dimensional total space with projectable Riemannian metric $\tilde{g}, B$ an n-dimensional base space, and $\pi : M \to B$ a projection with maximal rank n. The fibre passing through a point $\tilde{P} \in M$ is denoted by $F(\tilde{P})$ or generally F, which is a p-dimensional submanifold of M, where $p = m - n$. Throughout this paper, manifolds, geometric objects and mappings are supposed to be of C^∞ class and manifolds are assumed to be connected. Also, unless stated otherwise, the ranges of indices are as follows ;

$$\begin{aligned}&A, B, C, D, E : 1, 2, \cdots, m,\\ &h, i, j, k, l : 1, 2, \cdots, m,\\ &a, b, c, d, e : 1, 2, \cdots, n,\\ &x, y, z, w, u : n+1, \cdots, n+p = m.\end{aligned}$$

If we take coordinate neighborhoods $(\tilde{U},\ z^h)$ in M and $(U,\ x^a)$ in B such that $\pi(\tilde{U})$=U, then the projection π is expressed by equations

$$x^a = x^a(z^h) \tag{2.1}$$

with Jacobian $(\frac{\partial x^a}{\partial z^h})$ of maximum rank n. There is a local coordinate system y^x in $F \cap \tilde{U} \neq \emptyset$, (x^a, y^x) form a coordinate system in $\tilde{U}$ and each fibre $F(\tilde{P})$ at $\tilde{P}$ in $F \cap \tilde{U}$ is parametrized as $z^h = z^h(x^a, y^x)$. Then we can choose a local frame (E_a, C_x) and its dual frame (E^a, C^x) in $\tilde{U}$, where the components of E^a and C^x are given by

$$E_i{}^a = \frac{\partial x^a}{\partial z^i} \qquad and \qquad C^x{}_h = \frac{\partial y^x}{\partial z^h} \tag{2.2}$$

The vector fields E_a span the horizontal distribution and C_x the tangent space of each fibre. The metric tensor g in the base space B is given by

(2.3) $$g_{cb} = \tilde{g}(E_c, E_b)$$

and the induced metric tensor $\bar{g}$ in each fibre F by

(2.4) $$\bar{g}_{xy} = \tilde{g}(C_x, C_y).$$

We write (E_B) for the frame (E_b, C_x) in all, if necessary. Let $h = (h_{xy}{}^a)$ be components of the second fundamental tensor with respect to the normal vector E_a and $L = (L_{cb}{}^x)$ the nomal connection of each fibre F. Then we have

(2.5) $$h_{xy}{}^a = h_{yx}{}^a \qquad \textit{and} \qquad L_{cb}{}^x + L_{bc}{}^x = 0.$$

Denoting by $\tilde{\nabla}$ the Riemannian connection of the total space M, we have the following equations[3].

(2.6.1) $$\tilde{\nabla}_j E^h{}_b = \Gamma_{cb}{}^a E_j{}^c E^h{}_a - L_{cb}{}^x E_j{}^c C^h{}_x + L_b{}^a{}_y C_j{}^y E^h{}_a - h_y{}^x{}_b C_j{}^y C^h{}_x,$$

(2.6.2) $$\tilde{\nabla}_j C^h{}_x = L_c{}^a{}_x E_j{}^c E^h{}_a - (h_x{}^y{}_c - P_{cx}{}^y) E_j{}^c C^h{}_y + h_{zx}{}^a C_j{}^z E^h{}_a + \bar{\Gamma}_{zx}{}^y C_j{}^z C^h{}_y,$$

(2.6.3) $$\tilde{\nabla}_j E_i{}^a = -\Gamma_{cb}{}^a E_j{}^c E_i{}^b - L_c{}^a{}_x (E_j{}^c C_i{}^x + C_j{}^x E_i{}^c) - h_{yx}{}^a C_j{}^y C_i{}^x,$$

(2.6.4) $$\tilde{\nabla}_j C_i{}^x = L_{cb}{}^x E_j{}^c E_i{}^b + (h_y{}^x{}_c - P_{cy}{}^x) E_j{}^c C_i{}^y + h_z{}^x{}_b C_j{}^z E_i{}^b \bar{\Gamma}_{zy}{}^x C_j{}^z C_i{}^y,$$

where $\Gamma_{cb}{}^a$ are connection coefficients of the projection $\nabla = p\tilde{\nabla}$ in B, $\bar{\Gamma}_{zy}{}^x$ those of the induced connection $\bar{\nabla}$ in F, $L_c{}^a{}_y = L_{cb}{}^x g^{ba} \bar{g}_{xy}, h_y{}^x{}_b = h_{yz}{}^a \bar{g}^{zx} g_{ba}$ and $P_{cy}{}^x$ are local functions in $\tilde{U}$ defined by $[E_b, C_y] = P_{by}{}^x C_x$.

From(2.6.1), we see that $[E_c, E_b] = -2L_{cb}{}^x C_x$, and so the horizontal distribution is integrable if and only if the structure tensor L vanishes identically.

Let γ be a curve through a point P in the base space B and X be the tangent vector field of γ. There is a unique curve $\tilde{\gamma}$ through a point $\tilde{P} \in \pi^{-1}(P)$ such that the tangent vector field is the lift X^L. The curve $\tilde{\gamma}$ is called the *horizontal lift* of γ passing through $\tilde{P}$. If a curve γ joins points P and Q in B, then the horizontal lifts of γ through all points of the fibre $F(P)$ define a fibre mapping $\Phi_\gamma : F(P) \to F(Q)$, called the *horizontal mapping covering* γ.

If the horizontal mapping covering any curve in B is an isometry of fibres, then $\{M, B, \tilde{g}, \pi\}$ is called a fibred Riemannian space with *isometric fibres.* A necessary and sufficient condition for M to have isometric fibres is $(\pounds_{X^L}\tilde{g}^V)^V = 0$ for any vector field X in B, or equivalently $h_{xy}{}^a = 0$. Here and hereafter A^H and A^V indicate the horizontal and vertical parts of A respectively. The model space of the fibred Riemannian space with isometric fibre can be seen in [3]. If the horizontal mapping covering any curve in B is conformal mapping of fibres, then $\{M, B, \tilde{g}, \pi\}$ is called a fibred Riemannian space with *conformal fibres.* A condition for M to have conformal fibres is $h_{xy}{}^a = \bar{g}_{xy}A^a$, where $A = A^a E_a$ is the mean curvature vector along each fibre in M. The following theorem is well known[3].

THEOREM 2.1. *If the normal connection $L = (L_{cb}{}^x)$ and second fundamental form $h = (h_{xy}{}^a)$ vanish identically in a fibred Riemannian space, then the fibred space is locally the Riemannian product of the base and a fibre.*

The curvature tensor of a fibred Riemannian space M is defined by

$$\tilde{K}(\tilde{X}, \tilde{Y})\tilde{Z} = \tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z} - \tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z} - \tilde{\nabla}_{[\tilde{X},\tilde{Y}]}\tilde{Z} \tag{2.7}$$

for any vector fields $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ in M. We put

$$\tilde{K}(E_D, E_C)E_B = \tilde{K}_{DCB}{}^A E_A = \tilde{K}_{DCB}{}^a E_a + \tilde{K}_{DCB}{}^x C_x, \tag{2.8}$$

then $\tilde{K}_{DCB}{}^A$ are components of the curvature tensor with respect to the basis (E_B). Denoting by $\tilde{K}_{kji}{}^h$ components of the curvature tensor in $(\tilde{U}, z^h)$, we have the relations

$$\tilde{K}_{DCB}{}^A = \tilde{K}_{kji}{}^h E^k{}_D E^j{}_C E^i{}_B E_h{}^A \tag{2.9}$$

Substituting (2.6.2) into the definition (2.7) of the curvature tensor, we have the structure equations of a fibred Riemannian space as follows [2,3,7,11]:

$$\tilde{K}_{dcb}{}^a = K_{dcb}{}^a - L_d{}^a{}_x L_{cb}{}^x + L_c{}^a{}_x L_{db}{}^x + 2L_{dc}{}^x L_b{}^a{}_x, \tag{2.10}$$

$$\tilde{K}_{dcb}{}^x = -{}^*\nabla_d L_{cb}{}^x + {}^*\nabla_c L_{db}{}^x - 2L_{dc}{}^y h_y{}^x{}_b, \tag{2.11}$$

$$\begin{aligned}\tilde{K}_{dcy}{}^x = {}&{}^*\nabla_c h_y{}^x{}_d - {}^*\nabla_d h_y{}^x{}_c + 2{}^{**}\nabla_y L_{dc}{}^x + L_{de}{}^x \\ &L_c{}^e{}_y - L_{ce}{}^x L_d{}^e{}_y - h_z{}^x{}_d h_y{}^z{}_c + h_z{}^x{}_c h_y{}^z{}_d,\end{aligned} \tag{2.12}$$

$$\tilde{K}_{dzb}{}^a = {}^*\nabla_d L_b{}^a{}_z - L_d{}^a{}_x h_z{}^x{}_b + L_{db}{}^x h_{zx}{}^a - L_d{}^a{}_x h_z{}^x{}_d, \tag{2.13}$$

$$(2.14)\qquad \tilde{K}_{dzb}{}^{x} = -{}^{*}\nabla_d h_z{}^x{}_b + {}^{**}\nabla_z L_{db}{}^x + L_d{}^e{}_z L_{eb}{}^x + h_z{}^y{}_d h_y{}^x{}_b,$$

$$(2.15)\qquad \tilde{K}_{zyb}{}^{a} = L_{zyb}{}^a + h_z{}^x{}_b h_{yx}{}^a - h_y{}^x{}_b h_{zx}{}^a,$$

$$(2.16)\qquad \tilde{K}_{zyx}{}^{a} = {}^{**}\nabla_z h_{yx}{}^a - {}^{**}\nabla_y h_{zx}{}^a,$$

$$(2.17)\qquad \tilde{K}_{zyx}{}^{w} = \bar{K}_{zyx}{}^w + h_{zx}{}^e h_y{}^w{}_e - h_{yx}{}^e h_z{}^w{}_e,$$

where we have put

$$(2.18)\qquad K_{dcb}{}^a = \partial_d \Gamma_{cb}{}^a - \partial_c \Gamma_{db}{}^a + \Gamma_{de}{}^a \Gamma_{cb}{}^e - \Gamma_{ce}{}^a \Gamma_{db}{}^e,$$

$$(2.19)\qquad {}^{*}\nabla_d L_{cb}{}^x = \partial_d L_{cb}{}^x - \Gamma_{dc}{}^e L_{eb}{}^x - \Gamma_{db}{}^e L_{ce}{}^x + Q_{dy}{}^x L_{cb}{}^y,$$

$$(2.20)\qquad {}^{*}\nabla_d L_c{}^a{}_y = \partial_d L_c{}^a{}_y + \Gamma_{de}{}^a L_c{}^e{}_y - \Gamma_{dc}{}^e L_e{}^a{}_y - Q_{dy}{}^z L_c{}^a{}_z,$$

$$(2.21)\qquad {}^{*}\nabla_d h_{zy}{}^a = \partial_d h_{zy}{}^a + \Gamma_{de}{}^a h_{zy}{}^e - Q_{dz}{}^x h_{xy}{}^a - Q_{dy}{}^x h_{zx}{}^a,$$

$$(2.22)\qquad {}^{*}\nabla_d h_y{}^x{}_b = \partial_d h_y{}^x{}_b - \Gamma_{db}{}^e h_y{}^x{}_e + Q_{dz}{}^x h_y{}^z{}_b - Q_{dy}{}^z h_z{}^x{}_b,$$

$Q_{cy}{}^x$ being defined by $\qquad Q_{cy}{}^x = P_{cy}{}^x - h_y{}^x{}_c$

and

$$(2.23)\qquad {}^{**}\nabla_y L_{cb}{}^x = \partial_y L_{cb}{}^x + \bar{\Gamma}_{yz}{}^x L_{cb}{}^z - L_c{}^e{}_y L_{eb}{}^x - L_b{}^e{}_y L_{ce}{}^x,$$

$$(2.24)\qquad {}^{**}\nabla_y L_b{}^a{}_x = \partial_y L_b{}^a{}_x - \bar{\Gamma}_{yx}{}^z L_b{}^a{}_z + L_e{}^a{}_y L_b{}^e{}_x - L_b{}^e{}_y L_e{}^{ax},$$

$$(2.25)\qquad {}^{**}\nabla_z h_{yx}{}^a = \partial_z h_{yx}{}^a - \bar{\Gamma}_{zy}{}^w h_{wx}{}^a - \bar{\Gamma}_{zx}{}^w h_{yw}{}^a + L_e{}^a{}_z h_{yx}{}^e,$$

$$(2.26)\qquad {}^{**}\nabla_z h_y{}^x{}_b = \partial_z h_y{}^x{}_b + \bar{\Gamma}_{zw}{}^x h_y{}^w{}_b - \bar{\Gamma}_{zw}{}^y h_w{}^x{}_b - L_b{}^e{}_z h_y{}^x{}_e,$$

$$(2.27)\qquad L_{yxb}{}^a = \partial_y L_b{}^a{}_x - \partial_x L_b{}^a{}_y + L_e{}^a{}_w L_b{}^e{}_x - L_e{}^a{}_x L_b{}^e{}_y,$$

$$(2.28)\qquad \bar{K}_{zyx}{}^w = \partial_z \bar{\Gamma}_{yx}{}^w - \partial_y \bar{\Gamma}_{zx}{}^w + \bar{\Gamma}_{zu}{}^w \bar{\Gamma}_{yx}{}^u - \bar{\Gamma}_{yu}{}^w \bar{\Gamma}_{zx}{}^u.$$

Among these, the functions $K_{dcb}{}^a$ are projectable in $\tilde{U}$ and its projections, denoted by $K_{dcb}{}^a$ too, are components of the curvature tensor of the base space $\{B, g\}$. On each fibre F, the functions $\bar{K}_{zyx}{}^w$ are components of the curvature tensor of the induced Riemannian metric $\bar{g}$ and $L_{yxb}{}^a$ those of the curvature tensor of the normal connection of F in M. The components $\tilde{K}_{DCB}{}^A$ satisfy the same algebraic equations as those $\tilde{K}_{kji}{}^h$ satisfy. Denote by $\tilde{K}_{CB}$ components of the Ricci tensor of $\{M, \tilde{g}\}$ with respect to the basis (E_B) in $\tilde{U}$, and by K_{cb} and $\bar{K}_{yx}$ componets of the Ricci tensors of the base space $\{B, g\}$ in (U, x^a) and each fibre $\{F, \bar{g}\}$ in $(\bar{U}, y^x)$ respectively. Then we have

$$(2.29)\qquad \tilde{K}_{cb} = K_{cb} - 2L_{ce}{}^x L_b{}^e{}_x - h_y{}^x{}_c h_x{}^y{}_b + \tfrac{1}{2}({}^{*}\nabla_c h_x{}^x{}_b + {}^{*}\nabla_b h_x{}^x{}_c),$$

$$(2.30)\qquad \tilde{K}_{xb} = {}^{**}\nabla_x h_y{}^y{}_b - {}^{**}\nabla_y h_x{}^y{}_b + {}^{*}\nabla_e L_b{}^e{}_x - 2h_x{}^y{}_e L_b{}^e{}_y,$$

(2.31) $$\tilde{K}_{yx} = \bar{K}_{yx} - h_{yx}{}^e h_z{}^z{}_e +^* \nabla_e h_{yx}{}^e - L_a{}^e{}_y L_e{}^a{}_x,$$

Denoting by $\tilde{K}, K$ and $\bar{K}$ the scalar curvatures of M, B and each fibre F respectively, we have the relation

(2.32) $$\tilde{K} = K^L + \bar{K} - L_{cbz}L^{cbz} - h_{yxe}h^{yxe} - h_y{}^y{}_e h_u{}^{ue} + 2^*\nabla_e h_z{}^{ze}.$$

3. Fibred Riemannian space with various Kaehlerian structures

In this chapter, we consider a fibred Riemannian space M such that the base space B and each fibre F are almost contact spaces with almost contact structures $(\phi_b{}^a, \eta_b, \xi^a)$ and $(\bar{\phi}_x{}^y, \bar{\eta}_x, \bar{\xi}^y)$ respectively. The structure (ϕ, η, ξ) satisfies

$$\phi^2 = -I + \eta \otimes \xi,\ \phi(\xi) = 0,\ \eta \otimes \phi = 0,\ \eta(\xi) = 1,$$

where I is the identity map. An almost contact metric structure (ϕ, η, ξ, g) is said to be

(a) nearly cosymplectic if ϕ is killing

(b) almost cosymplectic if Φ and η are closed where $\Phi(X,Y) = g(\phi X, Y)$. If we define

(3.1) $$J_j{}^i = \phi_b{}^a E_j{}^b E^i{}_a - \eta_b \bar{\xi}^y E_j{}^b C^i{}_y + \bar{\eta}_x \xi^a C_j{}^x E^i{}_a + \bar{\phi}_x{}^y C_j{}^x C^i{}_y,$$

then we can easily see that $J^2 = -I$ and that we can construct almost Hermitian structure with almost complex structure J on the total space M, which will be called fibred almost Hermitian space. Thus we can state

THEOREM 3.1. *Let B and F be an almost contact metric spaces. Then the fibred Riemannian space M admits an almost Hermitian structure.*

The equation (3.1) is rewritten as

(3.2) $$J_j{}^i E^j{}_d = \phi_d{}^a E^i{}_a - \eta_d \bar{\xi}^y C^i{}_y,$$

(3.3) $$J_j{}^i C^j{}_z = \bar{\eta}_z \xi^a E^i{}_a + \bar{\phi}_z{}^y C^i{}_y.$$

Components of the covariant derivative $\tilde{\nabla} J$ with respect to the frame (E_B)=(E_b, C_y) are given by $(\tilde{\nabla}_j J_{ih}) E^j{}_C E^i{}_B E^h{}_A$ and we can obtain the following expressions by means of (2.6), (3.2) and (3.3).

(3.4.1) $$(\tilde{\nabla}_k J_{ji}) E^k{}_c E^j{}_d E^i{}_e = \nabla_c \phi_{de} + L_{cd}{}^x \bar{\eta}_x \eta_e - L_{cey} \eta_d \bar{\xi}^y$$

$$(3.4.2)\qquad (\tilde{\nabla}_k J_{ji})E^k{}_c E^j{}_d C^i{}_z = -(\nabla_c \eta_d)\bar{\eta}_z - \eta_d({}^*\nabla_c \bar{\eta}_z) + L_{cd}{}^x \bar{\phi}_{xz} - L_{caz}\phi_d{}^a$$

$$(3.4.3)\qquad (\tilde{\nabla}_k J_{ji})E^k{}_c C^j{}_z E^i{}_d = ({}^*\nabla_c \bar{\eta}_z)\eta_d + (\nabla_c \eta_d)\bar{\eta}_z + L_{cdy}\bar{\phi}_z{}^y - L_c{}^a{}_z \phi_{ad}$$

$$(3.4.4)\qquad (\tilde{\nabla}_k J_{ji})E^k{}_c C^j{}_z C^i{}_w =^* \nabla_c \bar{\phi}_{zw} - L_{caw}\bar{\eta}_z \xi^a + L_c{}^a{}_z \eta_a \bar{\eta}_w + (\nabla_z \bar{\eta}_w)\eta_c$$

$$(3.4.5)\qquad (\tilde{\nabla}_k J_{ji})C^k{}_z E_j{}^d E^i{}_e =^{**} \nabla_z \phi_{de} - \eta_d \bar{\xi}^y h_{zye} + h_z{}^y{}_d \bar{\eta}_y \eta_e$$

$$(3.4.6)\qquad (\tilde{\nabla}_k J_{ji})C^k{}_z E^j{}_d C^i{}_w = -({}^{**}\nabla_z \eta_d)\bar{\eta}_w - \eta_d \bar{\nabla}_z \bar{\eta}_w + h_z{}^y{}_d \bar{\phi}_{yw} - h_{zwa}\phi_d{}^a$$

$$(3.4.7)\qquad (\tilde{\nabla}_k J_{ji})C^k{}_z C^j{}_x E^i{}_c = (\bar{\nabla}_z \bar{\eta}_x)\eta_c + \bar{\eta}_x({}^{**}\nabla_z \eta_c) + h_{zyc}\bar{\phi}_x{}^y - h_{zx}{}^b \phi_{bc}$$

$$(3.4.8)\qquad (\tilde{\nabla}_k J_{ji})C^k{}_z C^j{}_x C^i{}_y = \bar{\nabla}_z \bar{\phi}_{xy} + h_{zx}{}^b \eta_b \bar{\eta}_y - h_{zyb}\xi^b \bar{\eta}_x$$

From the equations (3.4.1)-(3.4.8), we get

$$(3.5.1)\qquad (\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ik} + \tilde{\nabla}_i J_{kj})E^k{}_c E^j{}_d E^i{}_e = \nabla_c \phi_{de} + \nabla_d \phi_{ec} + \nabla_e \phi_{cd} + 2(L_{cd}{}^x \bar{\eta}_x \eta_e + L_{ec}{}^x \bar{\eta}_x \eta_d + L_{de}{}^x \bar{\eta}_x \eta_c),$$

$$(3.5.2)\qquad (\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ik} + \tilde{\nabla}_i J_{kj})E^k{}_c E^j{}_d C^i{}_z = -(\nabla_c \eta_d)\bar{\eta}_z - \eta_d{}^*\nabla_c \bar{\eta}_z + 2L_{cd}{}^x \bar{\phi}_{xz} - L_{caz}\phi_d{}^a + ({}^*\nabla_d \bar{\eta}_z)\eta_c + (\nabla_d \eta_c)\bar{\eta}_z - L_d{}^a{}_z \phi_{ac} +^{**} \nabla_z \phi_{cd} - \eta_c \bar{\xi}^y h_{zyd} + h_z{}^y{}_c \bar{\eta}_y \eta_d,$$

$$(3.5.3)\qquad (\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ik} + \tilde{\nabla}_i J_{kj})E^k{}_c C^j{}_z E^i{}_d = ({}^*\nabla_c \bar{\eta}_z)\eta_d + (\nabla_c \eta_d)\bar{\eta}_z + 2L_{cdy}\bar{\phi}_z{}^y - L_c{}^a{}_z \phi_{ad} +^{**} \bar{\nabla}_z \phi_{dc} - \eta_d \bar{\xi}^y h_{zyc} + h_z{}^y{}_d \bar{\eta}_y \eta_c - (\nabla_d \eta_c)\bar{\eta}_z - \eta_c^* \nabla_d \bar{\eta}_z - L_{daz}\phi_c{}^a,$$

$$(3.5.4)\qquad (\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ik} + \tilde{\nabla}_i J_{kj})E^k{}_c C^j{}_z C^i{}_w =^* \nabla_c \bar{\phi}_{zw} - L_{caw}\bar{\eta}_z \eta^a + L_{cz}^{\ a} \eta_a \bar{\eta}_w + (\nabla_z \bar{\eta}_w)\eta + \bar{\eta}_w({}^{**}\nabla_z \eta_c) + h_{zyc}\bar{\phi}_w{}^y - ({}^{**}\nabla_w \eta_c)\bar{\eta}_z - \eta_c r \nabla_w \bar{\eta}_z + h_w{}^y{}_c \bar{\phi}_{yz},$$

$$(3.5.5)\qquad (\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ik} + \tilde{\nabla}_i J_{kj})C^k{}_z E^j{}_d E^i{}_e$$

$$
\begin{aligned}
&={}^{**}\nabla_z\phi_{de}-\eta_d\bar{\xi}^y h_{zye}+h_z{}^y{}_d\bar{\eta}_y\eta_e\\
&\quad-(\nabla_d\eta_e)\bar{\eta}_z-\eta^*_e\nabla_d\bar{\eta}_z+2L_{de}{}^x\bar{\phi}_{xz}\\
&\quad-L_{daz}\phi_e{}^a+({}^*\nabla_e\bar{\eta}_z)\eta_d+(\nabla_e\eta_d)\bar{\eta}_z-L_e{}^a{}_z\phi_{ad},
\end{aligned}
$$

$$
\begin{aligned}
(3.5.6)\qquad &(\tilde{\nabla}_k J_{ji}+\tilde{\nabla}_j J_{ik}+\tilde{\nabla}_i J_{kj})C^k{}_z E^j{}_d C^i{}_w\\
&=-({}^{**}\nabla_z\eta_d)\bar{\eta}_w-\eta_d\nabla_z\bar{\eta}_w+h_z{}^y{}_d\bar{\phi}_{yw}\\
&\quad-h_{zwa}\phi_d{}^a+{}^*\nabla_d\bar{\phi}_{wz}-L_{daz}\bar{\eta}_w\xi^a\\
&\quad+L_d{}^a{}_w\eta_a\bar{\eta}_z+(\nabla_w\bar{\eta}_z)\eta_d+\bar{\eta}^{**}_z\nabla_w\eta_d+h_{wyd}\bar{\phi}_z{}^y,
\end{aligned}
$$

$$
\begin{aligned}
(3.5.7)\qquad &(\tilde{\nabla}_k J_{ji}+\tilde{\nabla}_j J_{ik}+\tilde{\nabla}_i J_{kj})C^k{}_z C^j{}_x E^i{}_c\\
&=(\nabla_z\bar{\eta}_x)\eta_c+\bar{\eta}^{**}_x\nabla_z\eta_c+h_{zyc}\bar{\phi}_x{}^y\\
&\quad-({}^{**}\nabla_x\eta_c)\bar{\eta}_z-\eta_c(\nabla_x\bar{\eta}_z)+h_x{}^y{}_c\bar{\phi}_{yz}\\
&\quad+{}^*\nabla_c\bar{\phi}_{zx}-L_{cax}\bar{\eta}_z\xi^a+L_c{}^a{}_z\eta_a\bar{\eta}_x,
\end{aligned}
$$

$$
\begin{aligned}
(3.5.8)\qquad &(\tilde{\nabla}_k J_{ji}+\tilde{\nabla}_j J_{ik}+\tilde{\nabla}_i J_{kj})C^k{}_z C^j{}_x C^i{}_y=\nabla_z\bar{\phi}_{xy}\\
&\quad+\nabla_x\bar{\phi}_{yz}+\nabla_y\bar{\phi}_{zx}.
\end{aligned}
$$

We suppose that the induced almost complex structure J on M is almost Kaehlerian, that is $dJ=0$, equivalently in component, $\tilde{\nabla}_k J_{ji}+\tilde{\nabla}_j J_{ik}+\tilde{\nabla}_i J_{kj}=0$. Then we have

$$(3.6.1)\quad \nabla_c\phi_{de}+\nabla_d\phi_{ec}+\nabla_e\phi_{cd}+2(L_{cd}{}^x\bar{\eta}_x\eta_e+L_{ec}{}^x\bar{\eta}_x\eta_d+L_{de}{}^x\bar{\eta}_x\eta_c)=0,$$

$$(3.6.2)\qquad \nabla_x\bar{\phi}_{yz}+\nabla_y\bar{\phi}_{zx}+\nabla_z\bar{\phi}_{xy}=0,$$

$$
\begin{aligned}
(3.6.3)\qquad &{}^{**}\nabla_z\phi_{dc}+({}^*\nabla_c\bar{\eta}_z)\eta_d-({}^*\nabla_d\bar{\eta}_z)\eta_c+(\nabla_c\eta_d-\nabla_d\eta_c)\bar{\eta}_z\\
&+(L_d{}^a{}_z\phi_{ac}-L_c{}^a{}_z\phi_{ad})+(h_z{}^y{}_d\eta_c-h_z{}^y{}_c\eta_d)\bar{\eta}_y=0
\end{aligned}
$$

$$
\begin{aligned}
(3.6.4)\qquad &{}^*\nabla_d\bar{\phi}_{wz}+({}^{**}\nabla_w\eta_d)\bar{\eta}_z-({}^{**}\nabla_z\eta_d)\bar{\eta}_w+(\nabla_w\bar{\eta}_z-\nabla_z\bar{\eta}_w)\eta_d\\
&+(L_d{}^a{}_w\bar{\eta}_z-L_d{}^a{}_z\bar{\eta}_w)\eta_a+(h_z{}^y{}_d\bar{\phi}_{yw}-h_w{}^y{}_d\bar{\phi}_{yz})=0
\end{aligned}
$$

Transvecting (3.6.4) with η^d, we have

$$(3.7)\qquad {}^*\nabla_d\bar{\phi}_{wz}\eta^d+\nabla_w\bar{\eta}_z-\nabla_z\bar{\eta}_w+(h_z{}^y{}_d\bar{\phi}_{yw}-h_w{}^y{}_d\bar{\phi}_{yz})\eta^d=0$$

From the equations (3.6.2) and (3.7), we obtain

Theorem 3.2. *Let M is a fibred almost Kaehlerian manifold with totally geodesic fibre. Then the manifold F is almost cosymplectic if and only if* ${}^*\nabla_d\bar{\phi}_{wz}\eta^d=0$.

By use of (3.4), we can calculate

$$
\begin{aligned}
(3.8.1)\qquad &(\tilde{\nabla}_k J_{ji}+\tilde{\nabla}_j J_{ki})E^k{}_c E^j{}_d E^i{}_e=\nabla_c\phi_{de}+\nabla_d\phi_{ce}\\
&\qquad-(L_{ce}{}^y\eta_d+L_{de}{}^y\eta_c)\bar{\eta}_y
\end{aligned}
$$

$$(3.8.2)\qquad (\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ki}) E^k{}_c E^j{}_d C^i{}_z = -(\nabla_c \eta_d)\bar{\eta}_z - \eta^*_d \nabla_c \bar{\eta}_z - L_{caz}\phi_d{}^a - (\nabla_d \eta_c)\bar{\eta}_z - \eta^*_c \nabla_d \bar{\eta}_z - L_{daz}\phi_c{}^a$$

$$(3.8.3)\qquad (\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ki}) E^k{}_c C^j{}_z E^i{}_d = ({}^*\nabla_c \bar{\eta}_z)\eta_d + (\nabla_c \eta_d)\bar{\eta}_z + L_{cdy}\bar{\phi}_z{}^y - L_c{}^a{}_z \phi_{ad} +^{**} \nabla_z \phi_{cd} - \eta_c \bar{\xi}^y h_{zyd} + h_z{}^y{}_c \bar{\eta}_y \eta_d$$

$$(3.8.4)\qquad (\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ki}) E^k{}_c C^j{}_z C^i{}_w =^* \nabla_c \bar{\phi}_{zw} - L_{caw}\bar{\eta}_z \eta^a + L_c{}^a{}_z \eta_a \bar{\eta}_w - \bar{\eta}^{**}_w \nabla_z \eta_c - \eta_c \nabla_z \bar{\eta}_w + h_z{}^y{}_c \bar{\phi}_{yw} - h_{zwa}\phi_c{}^a + h_w{}^x{}_c H_{xz} - h_{wzb} A_c{}^b$$

$$(3.8.5)\qquad (\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ki}) C^k{}_z E^j{}_d E^i{}_e =^{**} \nabla_z \phi_{de} - \eta_d \bar{\xi}^y h_{zye} + h_z{}^y{}_d \bar{\eta}_y \eta_e + ({}^*\nabla_d \bar{\eta}_z)\eta_e + (\nabla_d \eta_e)\bar{\eta}_z + L_{dey}\bar{\phi}_z{}^y - L_d{}^a{}_z \phi_{ae}$$

$$(3.8.6)\qquad (\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ki}) C^k{}_z E^j{}_d C^i{}_w = -\bar{\eta}_w{}^{**}\nabla_z \eta_d - \eta_d \nabla_z \bar{\eta}_w + h_z{}^y{}_d \bar{\phi}_{yw} - h_{zwa}\phi_d{}^a +^* \nabla_d \bar{\phi}_{zw} - L_{daw}\bar{\eta}_z \xi^a + L_d{}^a{}_z \eta_a \bar{\eta}_w$$

$$(3.8.7)\qquad (\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ki}) C^k{}_z C^j{}_x E^i{}_c = (\nabla_z \bar{\eta}_x)\eta_c + \bar{\eta}^{**}_x \nabla_z \eta_c + h_{zyc}\bar{\phi}_x{}^y + (\nabla_x \bar{\eta}_z)\eta_c + \bar{\eta}^{**}_z \nabla_x \eta_c + h_{xyc}\bar{\phi}_z{}^y$$

$$(3.8.8)\qquad (\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ki}) C^k{}_z C^j{}_x C^i{}_y = \nabla_z \bar{\phi}_{xy} - h_{zyb}\xi^b \bar{\eta}_x + \nabla_x \bar{\phi}_{zy} - h_{xyb}\xi^b \bar{\eta}_z$$

Now we suppose that the total space M is nearly Kaehlerian, that is $(\nabla_X J)Y + (\nabla_Y J)Y = 0$, equivalently in component, $\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ki} = 0$. Then the right hand side of the equations (3.8) vanishes identically, that is

$$(3.9.1)\qquad \nabla_c \phi_{de} + \nabla_d \phi_{ce} - (L_{ce}{}^y \eta_d + L_{de}{}^y \eta_c)\bar{\eta}_y = 0$$

$$(3.9.2)\qquad ({}^*\nabla_d \bar{\eta}_z)\eta_c + ({}^*\nabla_c \bar{\eta}_z)\eta_d + (\nabla_c \eta_d)\bar{\eta}_z + (\nabla_d \eta_c)\bar{\eta}_z + L_d{}^a{}_z \phi_{ca} + L_c{}^a{}_z \phi_{da} = 0$$

$$(3.9.3)\qquad {}^{**}\nabla_z \phi_{cd} + ({}^*\nabla_c \bar{\eta}_z)\eta_d + (\nabla_c \eta_d)\bar{\eta}_z + L_{cd}{}^y \bar{\phi}_{zy} - L_c{}^a{}_z \phi_{ad} + h_z{}^y{}_c \bar{\eta}_y \eta_d - h_z{}^y{}_d \bar{\eta}_y \eta_c = 0$$

$$(3.9.4)\qquad ({}^{**}\nabla_z \eta_c)\bar{\eta}_w -^* \nabla_c \bar{\phi}_{zw} + (\nabla_z \bar{\eta}_w)\eta_c + (L_c{}^a{}_w \bar{\eta}_z - L_c{}^a{}_z \bar{\eta}_w)\eta_a + h_{zw}{}^a \phi_{ca} - h_z{}^y{}_c \bar{\phi}_{yw} = 0$$

$$(3.9.5)\qquad ({}^{**}\nabla_x \eta_c)\bar{\eta}_z + ({}^{**}\nabla_z \eta_c)\eta_c + (\nabla_x \bar{\eta}_z)\eta_c + (\nabla_z \bar{\eta}_x)\eta_c + h_z{}^y{}_c \bar{\phi}_{xy} + h_x{}^y{}_c \bar{\phi}_{zy} = 0$$

$$(3.9.6)\qquad \nabla_z \bar{\phi}_{xy} + \nabla_x \bar{\phi}_{zy} - (h_{zy}{}^b \bar{\eta}_x + h_{xy}{}^b \bar{\eta}_z)\eta_b = 0$$

Thus we have

THEOREM 3.3. *Let M be a fibred nearly Kaehlerian manifold with totally geodesic fibre. Then each fibre F is a nearly cosymplectic. Moreover if $L \bigotimes \eta = 0$, then the base space $B$is nearly cosymplectic.*

References

[1] M. Ako, *Fibred spaces with almost complex structure*, Kodai Math.Sem.Rep. **24** (1972), 482–505.

[2] A. Besse, *Einstein manifold*, Springer-Verlag, Germany. (1987).

[3] S. Ishihara and M. Konishi, *Differential geometry of fibred spaces*, Publ. Study Group of Differential Geometry. **7** (1973)

[4] A. Morimoto, *On normal almost contact structures*, J.Math.Soc. Japan. **15** (1963), 420–436

[5] Y. Muto, *On some properties of a fibred Riemannian manifold*, Science reports Yokohama Nat. Univ. **1** (1952), 1–14

[6] T. Okubo, *Fibred spaces with almost Hermitian metrics whose base space admit almost contact metric structure*, Math.Ann. **183** (1969), 290–322

[7] B. O'Neill, *The fundamental equations of submersion*, Michigan Math. J. **13** (1966), 459–469

[8] B. L. Reinhart, *Foliated manifolds with bundle like metrics*, Ann. Math. **69** (1959), 119–132

[9] S. Tanno, *Almost complex structures in bundle spaces over almost contacts manifolds*, J. Math. Soc. Japan **17** (1965), 167–186

[10] Y. Tashiro and B. H. Kim, *Almost complex and almost contact structures in fibred Riemannian spaces*, Hiroshima Math. J. **18** (1988), 161–188

[11] K. Yano and S. Ishihara, *Differential geometry of fibred spaces*, Kodai Math. Sem. Rep. **19** (1967), 257–288

*

College of Liberal Arts
Kyung Hee University
Suwon 446-701, Republic of Korea
E-mail: jinhchoi@khu.ac.kr