JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **19**, No. 4, December 2006

SQUARE ROOTS OF HOMEOMORPHISMS

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Abstract. In this paper, we study the condition that a given homeomorphism has a square root and give an example of a wandering homeomorphism without square roots.

1. Introduction

Let X be a topological space and let $f : X \to X$ be a homeomorphism. A homeomorphism $g : X \to X$ is called a square root of f if $g \circ g = f$. Although the square root is not unique in general, we always denote g by $\sqrt{f}$.

For homeomorphisms of compact spaces, we can use the nonwandering sets to show the non-existence of square roots as follows: Let f be a homeomorphism. For any homeomorphism g, the nonwandering set $\Omega(f)$ of f and the nonwandering set $\Omega(gfg^{-1})$ of gfg^{-1} satisfies the relation $g(\Omega(f)) = \Omega(gfg^{-1})$. Since the square root of f commutes with f, $\Omega(f)$ is also invariant under the square root of f. By using this fact, we will construct a homeomorphism which has no square roots.

The above argument cannot be applied to study the square roots of homeomorphisms of non-compact spaces with empty nonwandering set. Thus we will introduce positive and negative limit sets which replaces the role of the nonwandering sets in the above argument. We recall some definitions from [3]. In this paper, X will always be a first countable hemicompact space unless stated otherwise. A topological space X is said to be hemicompact if there exist countably many compact subsets $M_1, M_2, \cdots$ of X such that for any compact subset M of X, there is a positive integer n such that $M \subset M_n$. Let f be a homeomorphism of X. For a compact subset M of X, we define its ω-limit(resp. α-limit) set by $\omega_f(M) = \bigcap_{n\geq 0} \overline{\bigcup_{m\geq n} f^m(M)}$ (*resp.* $\alpha_f(M) = \bigcap_{n\leq 0} \overline{\bigcup_{m\leq n} f^m(M)}$). For a homeomorphism f of X, we define the positive (resp. negative)

Received November 25, 2006.
2000 Mathematics Subject Classification: Primary 54H20.
Key words and phrases: limit set, ω-limit, α-limit, square roots.

limit set of f by $\omega(f) = \bigcup_{i=1}^{\infty} \omega_f(M_i)$ (resp. $\alpha(f) = \bigcup_{i=1}^{\infty} \alpha_f(M_i)$). By using these sets, we study the condition that a given homeomorphism of X has a square root and give an example of a wandering homeomorphism without square roots.

2. Limit sets for square roots

We show that the square root of f is not unique in general.

EXAMPLE 2.1. *Let functions f, g and h be defined by $f(x) = 4x + 3$, $g(x) = 2x + 1$ and $h(x) = -2x - 3$, respectively. Then f, g and h are homeomorphisms. Since*

$$\begin{aligned} g \circ g(x) &= 4x + 3 = f(x), \\ \text{and } \ h \circ h(x) &= 4x + 3 = f(x), \end{aligned}$$

functions g, h are square roots of f.

Let X be a compact space and f a homeomorphism of X. Then the nonwandering set $\Omega(f)$ of f is a nonempty set. For homeomorphisms of compact spaces, we can use the nonwandering sets to show the existence of square roots.

THEOREM 2.2. *Let $h : X \to X$ be a homeomorphism. Then we have $h(\Omega(f)) = \Omega(hfh^{-1})$.*

Proof. Let $x \in \Omega(f)$. Then there are nets (x_i) in X, (n_i) in $\mathbb{Z}^+$ such that

$$x_i \to x, n_i \to \infty, f^{n_i}(x_i) \to x.$$

Since $h(x_i) \to h(x)$ and

$$(hfh^{-1})^{n_i} h(x_i) = hf^{n_i}h^{-1}h(x_i) = hf^{n_i}(x_i) \to h(x),$$

we have $h(x) \in \Omega(hfh^{-1})$. Thus $h(\Omega(f)) \subset \Omega(hfh^{-1})$.

Let $x \in \Omega(hfh^{-1})$. Then there are nets (x_i) in X, (n_i) in $\mathbb{Z}^+$ such that

$$x_i \to x, n_i \to \infty, (hfh^{-1})^{n_i}(x_i) \to x.$$

Since $h^{-1}(x_i) \to h^{-1}(x)$ and

$$f^{n_i}h^{-1}(x_i) = h^{-1}hf^{n_i}h^{-1}(x_i) = h^{-1}(hfh^{-1})^{n_i}(x_i) \to h^{-1}(x),$$

we have $h^{-1}(x) \in \Omega(f)$. Since $x = hh^{-1}(x) \in h(\Omega(f))$, we have $\Omega(hfh^{-1}) \subset h(\Omega(f))$. Thus $h(\Omega(f)) = \Omega(hfh^{-1})$. $\square$

COROLLARY 2.3. *Let f be a homeomorphism of X. If $g = \sqrt{f}$, then $g(\Omega(f)) = \Omega(f)$.*

Proof. Since $g \circ f = g \circ (g \circ g) = (g \circ g) \circ g = f \circ g$, by Theorem 2.2 we have $g(\Omega(f)) = \Omega(gfg^{-1}) = \Omega(fgg^{-1}) = \Omega(f)$. $\square$

By using Corollary 2.3, we can construct a homeomorphism which has no square roots.

EXAMPLE 2.4. *Let $X = \{(x, y) \in \mathbb{R}^2 \mid |x| + y \leq 1, y \geq 0\}$. Define a function $f : X \to X$ by $f(0, 1) = (0, 1)$, and*

$$f(x,y) = \begin{cases} \left(\frac{1-\sqrt{y}}{(1-y)^2}x^2, \sqrt{y}\right), & \text{if } x \leq 0, 0 \leq y < 1 \\ \left(\frac{\sqrt{y}-1}{(1-y)^2}x^2, \sqrt{y}\right), & \text{if } x \geq 0, 0 \leq y < 1. \end{cases}$$

We prove that f is a homeomorphism. First we show that f is continuous. Let $y - 1 \leq x \leq 0$. Since $0 \leq x^2 \leq (1 - y)^2$, we have

$$0 \leq \frac{1-\sqrt{y}}{(1-y)^2}x^2 \leq 1 - \sqrt{y}.$$

Since $1 - \sqrt{y} \to 0$ as $y \to 1$, we have

$$\frac{1-\sqrt{y}}{(1-y)^2}x^2 \to 0 \text{ as } y \to 1.$$

Let $0 \leq x \leq 1 - y$. Since $0 \leq x^2 \leq (1 - y)^2$, we have

$$\sqrt{y} - 1 \leq \frac{\sqrt{y}-1}{(1-y)^2}x^2 \leq 0.$$

Since $\sqrt{y} - 1 \to 0$ as $y \to 1$, we have

$$\frac{\sqrt{y}-1}{(1-y)^2}x^2 \to 0 \text{ as } y \to 1.$$

Hence f is continuous at $(0, 1)$.

Also, define a function $h : X \to X$ by $h(0, 1) = (0, 1)$, and

$$h(x,y) = \begin{cases} \left(\frac{1-y^2}{\sqrt{1-y}}\sqrt{-x},\ y^2\right), & \text{if } x \leq 0,\ 0 \leq y < 1 \\ \left(\frac{y^2-1}{\sqrt{1-y}}\sqrt{x},\ y^2\right), & \text{if } x \geq 0,\ 0 \leq y < 1. \end{cases}$$

Then it is easy to show that h is continuous at $(0, 1)$. Since $h \circ f(x) = x$ and $f \circ h(x) = x$, we have $h = f^{-1}$. Thus f is a homeomorphism.

Let $a = (0, 1)$, $b = (-1, 0)$, $c = (0, 0)$ and $d = (1, 0)$. Then we have $f(a) = a$, $f(b) = d$, $f(c) = c$, $f(d) = b$ and $\Omega(f) = \{a, b, c, d\}$. Let $g = \sqrt{f}$. By Corollary 2.3, we have $g(\{a, b, c, d\}) = \{a, b, c, d\}$. If

$g(b) = b$, then $d = f(b) = g^2(b) = g(b) = b$. If $g(b) = d$, then $d = f(b) = g^2(b) = g(d)$. If $g(d) = b$, then $b = f(d) = g^2(d) = g(b)$. If $g(d) = d$, then $b = f(d) = g^2(d) = g(d) = d$. These four cases contradict. Thus we have $g(b) = a$, $g(d) = c$, or $g(b) = c$, $g(d) = a$.

Let $g(b) = a$ and $g(d) = c$. Then we have $d = f(b) = g^2(b) = g(a)$, $b = f(d) = g^2(d) = g(c)$. This contradicts the fact that $a = f(a) = g^2(a) = g(d) = c$.

On the other hand, let $g(b) = c$ and $g(d) = a$. Then we have $d = f(b) = g^2(b) = g(c)$ and $b = f(d) = g^2(d) = g(a)$. Also, this contradicts the fact that $a = f(a) = g^2(a) = g(b) = c$. Hence there are no square roots of f.

To prove Theorem 2.7, we need two lemmas.

Lemma 2.5. [3] *Let f and h be homeomorphisms of X. For any compact subset M of X, the following hold:*

$$h(\omega_f(M)) = \omega_{hfh^{-1}}(h(M))$$

and

$$h(\alpha_f(M)) = \alpha_{hfh^{-1}}(h(M)).$$

Lemma 2.6. [3] *(1) $x \in \omega_f(M)$ if and only if there are sequences (x_i) in M and (n_i) in $\mathbb{Z}^+$ such that $n_i \to \infty$ and $f^{n_i}(x_i) \to x$.*

(2) $x \in \alpha_f(M)$ if and only if there are sequences (x_i) in M and (n_i) in $\mathbb{Z}^-$ such that $n_i \to -\infty$ and $f^{n_i}(x_i) \to x$.

Theorem 2.7. *Let f and g be homeomorphisms of X. If $g = \sqrt{f}$, then $g(\omega(f)) = \omega(f)$ and $g(\alpha(f)) = \alpha(f)$.*

Proof. First we show that $g(\omega(f)) = \omega(f)$. Let $\{M_i\}$ be a countable collection of compact subsets of X satisfying the condition of hemicompactness. Since g is a homeomorphism, the set $g(M_i)$ is compact. For any compact subset M of X, since $g^{-1}(M)$ is compact, there exists an integer i such that $g^{-1}(M) \subset M_i$ and so we obtain $M \subset g(M_i)$. By Lemma 2.5, we have

$$\begin{aligned} g(\omega(f)) &= g\Big(\bigcup_{i=1}^{\infty} \omega_f(M_i)\Big) = \bigcup_{i=1}^{\infty} g(\omega_f(M_i)) = \bigcup_{i=1}^{\infty} \omega_{gfg^{-1}}(g(M_i)) \\ &= \bigcup_{i=1}^{\infty} \omega_f(g(M_i)) = \omega(f). \end{aligned}$$

We can show $g(\alpha(f)) = \alpha(f)$ in the same way. This completes the proof. $\square$

By using Theorem 2.7, we can show the non-existence of a square root in the non-compact space.

Example 2.8. *Let* $X = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}$. *We take the singular foliation* F *illustrated in Figure 1. In the region between the two straight lines with thorns, the homeomorphism* f *preserves the leaves of* F. *On the upper (resp. lower) straight line with thorns,* f *maps each thorn to the next thorn on the right (resp. left) side. By modifying* f *along the straight lines with thorns, we can construct a homeomorphism of* X *preserving the leaves of* F. *Then* $\omega(f)$ *is the lower straight line with thorns and* $\alpha(f)$ *is the upper one. If* f *has a square root* g, *then* $\omega(f)$ *must be invariant under* g *by Theorem 2.7. Furthermore,* g *maps the adjacent branch points of* $\omega(f)$ *on themselves because* g *is a homeomorphism. Let* $n \equiv (n, -1)$. *Then* $\omega(f) = \{n | n \in \mathbb{Z}\}$. *Let* $g(0) = 0$. *Since* $-1 = f(0) = g^2(0) = g(0) = 0$, *this contradicts. Next let* $g(0) = n$. *Then we have*

$$\begin{aligned}
&g(n) = g(g(0)) = f(0) = -1,\\
&g(-1) = g(g(n)) = f(n) = n - 1,\\
&g(n-1) = g(g(-1)) = f(-1) = -2,\\
&g(-2) = g(g(n-1)) = f(n-1) = n - 2, \cdots,\\
&g(n) = -1 = n - (n+1) = g(-(n+1)).
\end{aligned}$$

This contradicts the fact that g *is injective. Hence there are no square roots of* f.

Finally we give a relationship between the positive (negative) limit set of f and that of $\sqrt{f}$.

Theorem 2.9. *Let* f *and* g *be homeomorphisms of* X. *If* $g = \sqrt{f}$, *then* $\omega(f) = \omega(g)$ *and* $\alpha(f) = \alpha(g)$.

Proof. First we prove that $\omega(f) = \omega(g)$. By hemicompactness of X, there exists a countable collection $\{M_i\}$ of compact subsets of X such that for any compact subset M of X, there is a positive integer n such that $M \subset M_n$. Let $x \in \omega(f)$. Then there is an integer i such that $x \in \omega_f(M_i)$. By Lemma 2.6, there are sequences (x_n) in M_i and (m_n) in $\mathbb{Z}^+$ such that $m_n \to \infty$ and $f^{m_n}(x_n) \to x$. Since $2m_n \to \infty$, we have $g^{2m_n}(x_n) = f^{m_n}(x_n) \to x$. It follows that $x \in \omega_g(M_i) \subset \omega(g)$. Thus we

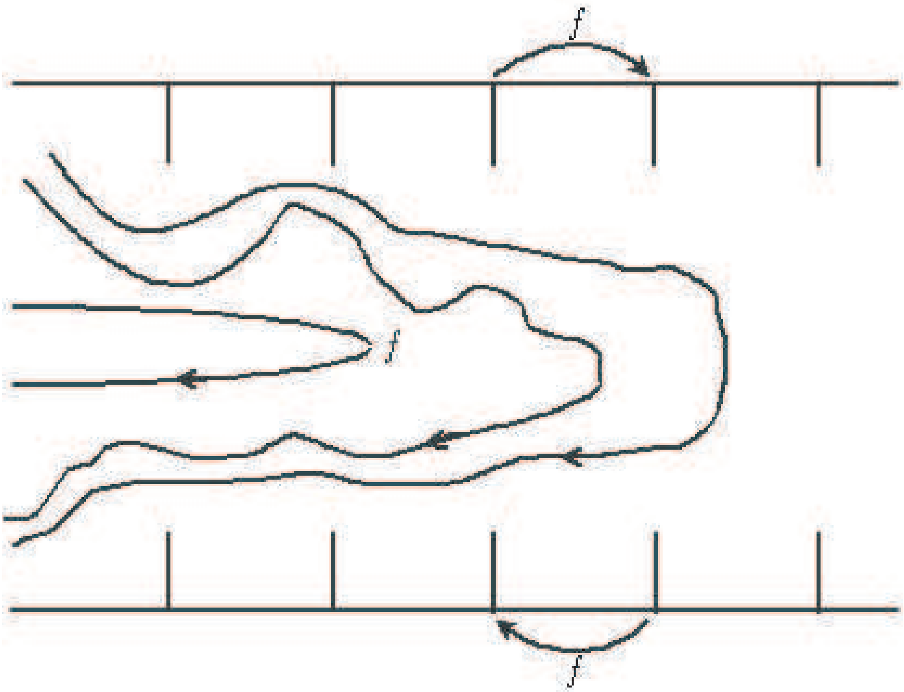

Figure 1

obtain $\omega(f) \subset \omega(g)$. Next let $x \in \omega(g)$. Then there exists an integer i such that $x \in \omega_g(M_i)$. By Lemma 2.6, there are sequences (x_n) in M_i and (m_n) in $\mathbb{Z}^+$ such that $m_n \to \infty$ and $g^{m_n}(x_n) \to x$. If infinitely many m_n are even, we may assume without loss of generality that all m_n is even. Since $\frac{m_n}{2} \to \infty$, we have

$$f^{\frac{m_n}{2}}(x_n) = (g^2)^{\frac{m_n}{2}}(x_n) = g^{m_n}(x_n) \to x.$$

Thus $x \in \omega_f(M_i) \subset \omega(f)$. Similarly, if infinitely many m_n are odd, we can suppose without loss of generality that all m_n is odd. Since $\frac{m_n-1}{2} \to \infty$ and $g(x_n) \in g(M_i)$, we have

$$f^{\frac{m_n-1}{2}} g(x_n) = (g^2)^{\frac{m_n-1}{2}} g(x_n) = g^{m_n}(x_n) \to x.$$

Thus we have

$$x \in \omega_f(g(M_i)) \subset \bigcup_{i=1}^{\infty} \omega_f(g(M_i)) = \omega(f),$$

and so $\omega(g) \subset \omega(f)$. Hence we have $\omega(f) = \omega(g)$.

A similar argument shows that $\alpha(f) = \alpha(g)$. This completes the proof. $\square$

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