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GIRSANOV THEOREM FOR GAUSSIAN PROCESS WITH INDEPENDENT INCREMENTS

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Abstract. A characterization of Gaussian process with independent increments in terms of the support of covariance operator is established. We investigate the Girsanov formula for a Gaussian process with independent increments.

1. Introduction

Since the stochastic calculus for standard Brownian motion initiated by K. Itô, the stochastic calculi for several stochastic processes have been extensively developed with wide applications to physics, mathematical finance, engineering, biology etc, in [2, 4, 5, 9, 10, 11, 12] and the references cited therein. Recently, the stochastic calculus for Gaussian processes have been developed in [1, 3, 6, 7].

The main purpose of this paper is two folds. We first establish a characterization of Gaussian process with independent increments in terms of the support of covariance operator. Secondly, we study the Girsanov theorem for a Gaussian process with independent increments. We study a Gaussian process with its characterization and representation. Then we study stochastic calculus for the Gaussian process, and we investigate the Girsanov formula for the Gaussian process. As applications of the Girsanov theorem, the studies of financial model and nonlinear filtering problems for the Gaussian process are now in progress, see [3, 6, 8, 9, 10].

This paper is organized as follows: In Section 2 we study a characterization of a Gaussian process with independent increments. In Section

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3 we recall the stochastic integral and Itô formula for the Gaussian process. In Section 4 we establish the Girsanov theorem for the Gaussian process.

2. Gaussian process with independent increments

Let $H = L^2(\mathbf{R}_+, dt)$ and K a strictly positive selfadjoint operator with domain $\mathbf{D} \subset H$ containing all indicator functions, where $\mathbf{R}_+ = [0, \infty)$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let $B = \{B_{K,t}\}_{t\geq 0}$ be a Gaussian process with mean $\mathbf{E}[B_{K,t}] = \alpha(t)$ and covariance function

$$\mathbf{E}[(B_{K,t} - \alpha(t))(B_{K,s} - \alpha(s))] = \langle K\mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]}\rangle, \qquad s, t \geq 0, \tag{2.1}$$

where α is of bounded variation on closed intervals with $\alpha(0) = 0$.

Theorem 2.1. *For any $0 \leq s \leq t$, supp$(K\mathbf{1}_{[s,t]}) \subset [s, t]$ if and only if the Gaussian process $\{B_{K,t}\}_{t\geq 0}$ has independent increments.*

Proof. If $supp(Kf) \subset supp(f)$ for any $f \in \mathbf{D}$, then by (2.1) we have

$$\begin{aligned} &\mathbf{E}[(B_{K,t} - B_{K,s})(B_{K,v} - B_{K,u})] \\ &\quad = \mathbf{E}[(B_{K,t} - B_{K,s})]\mathbf{E}[(B_{K,v} - B_{K,u})], \quad 0 \leq s \leq t \leq u \leq v. \end{aligned} \tag{2.2}$$

Hence $\{B_{K,t}\}_{t\geq 0}$ has independent increments. Conversely, if $\{B_{K,t}\}_{t\geq 0}$ has independent increments, then (2.2) holds and so $\langle K\mathbf{1}_{[s,t]}, \mathbf{1}_{[u,v]}\rangle = 0$ for $0 \leq s \leq t \leq u \leq v$. The rest of the proof is straightforward. □

Theorem 2.2. *Let $T > 0$ and K a positive selfadjoint bounded operator on $L^2([0, T])$ such that supp$(Kf) \subset$ supp(f) for any $f \in L^2([0, T])$. Then there exists a strictly increasing absolutely continuous function β on $[0, T]$ for any $T > 0$ such that the multiplication operator $M_{\beta'} = K$, where $\beta'(t) = d\beta(t)/dt$ for $0 \leq t < T$.*

Proof. Let $\beta(t) = \langle K\mathbf{1}_{[0,t]}, \mathbf{1}_{[0,t]}\rangle$ for $0 \leq t \leq T$. Then $\beta(t)$ is a strictly increasing function, in fact, for any $0 = t_0 < t_1 < \cdots < t_n \leq T$ we have

$$\begin{aligned} \beta(t_i) - \beta(t_{i-1}) &= \langle K\mathbf{1}_{[0,t_i]}, \mathbf{1}_{[0,t_i]}\rangle - \langle K\mathbf{1}_{[0,t_{i-1}]}, \mathbf{1}_{[0,t_{i-1}]}\rangle \\ &= \int_{t_{i-1}}^{t_i} K\mathbf{1}_{[t_{i-1},t_i]}(u)du. \end{aligned}$$

Therefore, β is absolutely continuous on $[0, T]$. Since $M_{\mathbf{1}_A}K = KM_{\mathbf{1}_A}$ for any measurable set $A \subset [0, T]$, $KM_f = M_f K$ for any bounded measurable function f. On the other hand, the algebra of all multiplication operators of bounded measurable functions is maximal abelian and so $K = M_g$ for some $g \in L^\infty([0,T])$. Therefore we have $\beta(t) = \langle M_g \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,t]}\rangle = \int_0^t g(s)ds$ which implies that $g = \beta'$. $\square$

From now on, we assume that $supp(K\mathbf{1}_{[s,t]}) \subset [s,t]$ for any $0 \le s \le t$ and put $\beta(t) = \langle K\mathbf{1}_{[0,t]}, \mathbf{1}_{[0,t]}\rangle$ for $t \ge 0$, and $\{B_{K,t}\}_{t\ge 0}$ has a continuous version.

For each $t \ge 0$ we put $A_{K,t} = B_{K,t} - \alpha(t)$. Then $\{A_{K,t}\}_{t\ge 0}$ is a Gaussian process such that $A_{K,t}$ has the normal distribution with mean 0 and variance $\beta(t)$.

3. Stochastic integral and Itô formula

Let $\{B_{K,t}\}_{t\ge 0}$ be a Gaussian process. For each $t \ge 0$, let $\mathcal{B}_t$ be the σ–algebra generated by $\{B_{K,s} : 0 \le s \le t\}$ and then we write $B_{K,t} = (B_{K,t}, \mathcal{B}_t)$ for $t \ge 0$.

Definition 3.1. Let $T > 0$ and $\mathfrak{M}_{\mathrm{G}} = \mathfrak{M}_{\mathrm{G}}[0,T]$ the class of functions $f : [0,T] \times \Omega \longrightarrow \mathbf{R}$ such that

(i) the map $(t,\omega) \mapsto f(t,\omega)$ is $\mathcal{B}([0,T]) \times \mathcal{F}$–measurable,
(ii) for each $0 \le t \le T$, $f(t,\cdot)$ is $\mathcal{B}_t$–measurable,
(iii) $\mathbf{E}\left[\int_0^T f(t,\omega)^2 d\left(\beta(t) + |\alpha|(t)\right)\right] < \infty$.

A function $\phi \in \mathfrak{M}_{\mathrm{G}}$ is called an *elementary function* if it has the form:

$$\phi(t,\omega) = \sum_j e_j(\omega)\mathbf{1}_{[t_j,t_{j+1})}(t), \tag{3.1}$$

where e_j is $\mathcal{B}_{t_j}$-measurable. For the elementary function ϕ given as in (3.1), we define

$$\int_0^T \phi(t,\omega)dB_{K,t}(\omega) = \sum_j e_j(\omega)\left(B_{K,t_{j+1}} - B_{K,t_j}\right). \tag{3.2}$$

The integral defined as in (3.2) is called the *stochastic integral* of ϕ with respect to the Gaussian process $\{B_{K,t}\}_{t\ge 0}$. We first note that if $f \in \mathfrak{M}_{\mathrm{G}}$, we can choose elementary functions $\phi_n \in \mathfrak{M}_{\mathrm{G}}$ such that

$$\mathbf{E}\left[\int_0^T |f - \phi_n|^2 d(\beta + |\alpha|)(t)\right] \longrightarrow 0 \quad \text{as} \quad n \to \infty, \tag{3.3}$$

for the proof, we refer to [6]. Therefore, for each $f \in \mathfrak{M}_{\mathrm{G}}$, the *stochastic integral* with respect to the process $\{B_{K,t}\}_{t\geq 0}$ is defined by

$$\int_0^T f(t,\omega)dB_{K,t}(\omega) = \lim_{n\to\infty}\int_0^T \phi_n(t,\omega)dB_{K,t}(\omega),$$

where $\{\phi_n\}$ is a sequence of elementary functions in $\mathfrak{M}_{\mathrm{G}}$ given as in (3.3) and the limit exists in $L^2(\Omega)$. Then we can prove the following elementary inequality: for any $f \in \mathfrak{M}_{\mathrm{G}}$ we have

$$\mathbf{E}\left[\left(\int_0^T f(t,\omega)dB_{K,t}(\omega)\right)^2\right] \leq 2\mathbf{E}\left[\int_0^T f(t,\omega)^2 d\beta(t)\right] + 2V_0^T(\alpha)\mathbf{E}\left[\int_0^T f(t,\omega)^2 d|\alpha|(t)\right],$$

where $V_0^T(\alpha)$ is the total variation of α over $[0,T]$, see [6].

A *diffusion process* for a Gaussian process $\{B_{K,t}\}_{t\geq 0}$ is a stochastic process X_t given by

$$dX_t = u(t,\omega)dt + v(t,\omega)dB_{K,t}, \qquad t\geq 0,$$

where $v\in \mathfrak{M}_{\mathrm{G}}[0,T]$ and u is $\mathcal{B}_t$-adapted with $\mathbf{E}\left[\int_0^T u(s,\omega)^2 ds\right] < \infty$.

THEOREM 3.2 ([6]). *Let X_t be a diffusion process given by*

$$dX_t = udt + vdB_{K,t}.$$

Let $g \in C^2(\mathbf{R}_+ \times \mathbf{R})$. Then $Y_t = g(t, X_t)$ is again a diffusion process and satisfies the following stochastic differential equation:

$$dY_t = \frac{\partial g}{\partial t}(t,X_t)dt + \frac{\partial g}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 g(t,X_t)}{\partial x^2}(dX_t)^2,$$

where $(dX_t)^2 = (dX_t)(dX_t)$ is computed according to the rules:

$$dt\cdot dt = dt\cdot dB_{K,t} = dB_{K,t}\cdot dt = 0, \qquad dB_{K,t}\cdot dB_{K,t} = \beta(t).$$

4. The Girsanov theorem

In this section we study the Girsanov Theorem for a Gaussian process with independent increments. We start with a theorem for the existence and uniqueness of (strong) solution of a stochastic differential equation. For the proof, we refer to [6].

THEOREM 4.1. *Let* $T > 0$ *and* $b(\cdot,\cdot) : [0,T] \times \mathbf{R} \to \mathbf{R}$, $\sigma(\cdot,\cdot) : [0,T] \times \mathbf{R} \to \mathbf{R}$ *be measurable functions satisfying*

$$|b(t,x)| + |\sigma(t,x)| \le C(t)(1 + |x|), \quad x \in \mathbf{R},\ t \in [0,T];$$

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le C(t)|x - y|, \quad x, y \in \mathbf{R},\ t \in [0,T]$$

for some bounded measurable function C. *Let* Z *be a random variable which is independent of the* σ*-algebra* $\mathcal{B}_\infty$ *generated by* $\{B_{K,s} : s \ge 0\}$ *and such that* $\mathbf{E}[|Z|^2] < \infty$. *Then the stochastic differential equation*

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_{K,t}, \qquad X_0 = Z, \quad 0 \le t \le T$$

has a unique t*-continuous solution* X_t *with the property that* X_t *is adapted to the filtration* $\mathcal{B}_t^Z$ *generated by* Z *and* $\mathcal{B}_t$ *and*

$$\mathbf{E}\left[\int_0^T |X_t|^2 \, d(\rho + \beta + |\alpha|)(t)\right] < \infty,$$

where ρ *is the identity function on* $\mathbf{R}_+$.

THEOREM 4.2. *Let* $\{X_t\}_{t\ge0}$ *be a continuous stochastic process on the probability space* $(\Omega, \mathcal{F}, P)$. *Then the following are equivalent:*

(i) $\{X_t\}_{t\ge0}$ *is a Gaussian process with independent increments, and mean* 0 *and variance* $\beta(t)$.

(ii) $\{X_t\}_{t\ge0}$ *is a martingale with respect to the filtration* $\{\mathcal{B}_t\}_{t\ge0}$ *of which the quadratic variation on* $[0,t]$ *is* $\beta(t)$.

Proof. The proof of (i) $\Rightarrow$ (ii) is straightforward. To prove the converse we assume that (ii) is satisfied. The proof is a simple modification of the proof of Theorem 6.1 in [5]. For each $t \ge 0$, we put $Z_t = e^{iuX_t}$, $u \in \mathbf{R}$. Then by applying Itô formula (see Theorem 5.1 in [5]) we obtain that

$$Z_t = Z_s + \int_s^t iuZ_v dX_v - \frac{u^2}{2}\int_s^t Z_v d\beta(v), \qquad 0 < s \le t. \tag{4.1}$$

On the other hand, it is obvious that

$$\mathbf{E}\left[\int_s^t iuZ_v dX_v | \mathcal{B}_s\right] = 0 \quad P-\text{a.s.}$$

From (4.1) for each $A \in \mathcal{B}_s$ we have

$$\mathbf{E}[e^{iu(X_t - X_s)}\mathbf{1}_A] = P(A) - \frac{u^2}{2}\int_s^t \mathbf{E}[e^{iu(X_v - X_s)}\mathbf{1}_A] d\beta(v)$$

which implies that

$$\mathbf{E}[e^{iu(X_t - X_s)}\mathbf{1}_A] = P(A)\exp\{-\frac{u^2}{2}(\beta(t) - \beta(s))\}, \qquad A \in \mathcal{B}_s.$$

Hence we prove that for $0 \leq s < t$

$$\mathbf{E}[\exp\{iu(X_t - X_s)\}|\mathcal{B}_s] = \exp\{-\frac{u^2}{2}(\beta(t) - \beta(s))\} \quad P - \text{a.s.}$$

We conclude that $X_t - X_s$ is independent of $\mathcal{B}_s$ and that normally distributed with mean 0 and variance $\beta(t) - \beta(s)$. □

Lemma 4.3 ([10]). *Let μ and ν be two probability measures on a measurable space $(\Omega, \mathcal{F})$ such that $d\nu(\omega) = Z(\omega)d\mu(\omega)$ for some $Z \in L^1(\Omega, \mu)$. Let X be a random variable on $(\Omega, \mathcal{F})$ such that*

$$\mathbf{E}_\nu[|X|] = \int_\Omega |X(\omega)|Z(\omega)d\mu(\omega) < \infty.$$

Let $\mathcal{H}$ be a σ-algebra with $\mathcal{H} \subset \mathcal{F}$. Then

$$\mathbf{E}_\nu[X|\mathcal{H}] \cdot \mathbf{E}_\mu[Z|\mathcal{H}] = \mathbf{E}_\mu[ZX|\mathcal{H}] \quad P - \text{a.s.}$$

From now on, we assume that the mean function α of the Gaussian process $\{B_{K,t}\}_{t\geq 0}$ is absolutely continuous.

Theorem 4.4. *Let $\{Y_t\}_{0\leq t\leq T}$ be a diffusion process of the form:*

$$dY_t = v(t)dt + dB_{K,t}, \qquad 0 \leq t \leq T, \quad Y_0 = 0$$

for a $\mathcal{B}_t$-adapted process $\{v(t)\}_{0\leq t\leq T}$, where $T \leq \infty$ is a given constant and $\{B_{K,t}\}_{t\geq 0}$ is a Gaussian process with mean function $\alpha(t)$ and the variance function $\beta(t)$. Assume that

$$\mathbf{E}\left[\exp\left(\frac{1}{2}\int_0^T \frac{(v(s)+\alpha'(s))^2}{\beta'(s)}ds\right)\right] < \infty, \tag{4.2}$$

where $\mathbf{E} = \mathbf{E}_P$ is the expectation with respect to P. Put

$$M_t = \exp\left(-\int_0^t \frac{v(s)+\alpha'(s)}{\beta'(s)}dA_{K,s} - \frac{1}{2}\int_0^t \frac{(v(s)+\alpha'(s))^2}{\beta'(s)}ds\right) \tag{4.3}$$

for $0 \leq t \leq T$, where $dA_{K,t} = dB_{K,t} - \alpha'(t)dt$. Then $\{Y_t\}_{0\leq t\leq T}$ is a Gaussian process on $(\Omega, \mathcal{B}_T, Q)$ with mean 0 and variance $\beta(t)$, where the probability measure Q is defined by

$$dQ(\omega) = M_T(\omega)dP(\omega). \tag{4.4}$$

In fact, $Q(\Omega) = \mathbf{E}_P[M_T] = 1$ since $\{M_t\}_{0\leq t\leq T}$ is a martingale.

Proof. For the proof we use similar arguments used in the proof of Theorem 8.6.3 in [10]. For each $0 \le t \le T$ we put $K_t = M_t Y_t$. Then by using Itô formula we have

$$\begin{aligned} dK_t &= M_t dY_t + Y_t dM_t + dY_t dM_t \\ &= M_t \left[1 - Y_t \left(\frac{v(t) + \alpha'(t)}{\beta'(t)} \right) \right] dA_{K,t}. \end{aligned}$$

Therefore, K_t is a martingale with respect to P and so, by Lemma 4.3, for $0 \le s < t \le T$ we have

$$\mathbf{E}_Q[Y_t|\mathcal{B}_s] = \frac{\mathbf{E}[M_t Y_t|\mathcal{B}_s]}{\mathbf{E}[M_t|\mathcal{B}_s]} = \frac{\mathbf{E}[K_t|\mathcal{B}_s]}{M_s} = \frac{K_s}{M_s} = Y_s$$

which implies that $\{Y_t\}_{0\le t\le T}$ is a martingale with respect to the probability Q. Therefore, $\mathbf{E}_Q[Y_t] = \mathbf{E}_Q[Y_0] = 0$ for any $0 \le t \le T$. On the other hand, since the process $\{Y_t\}_{t\ge 0}$ has independent increments, by applying Itô formula we have

$$\mathbf{E}\left[Y_t^2\right] = 2\mathbf{E}\left[\int_0^t Y_s dY_s + \beta(t)\right] = \beta(t), \qquad 0 \le t \le T.$$

Therefore, $\{Y_t\}_{0\le t\le T}$ is a martingale with respect to the filtration $\{\mathcal{B}_t\}$ and the probability measure Q of which the quadratic variation on $[0,t]$ is $\beta(t)$. Hence by Theorem 4.2, $\{Y_t\}_{0\le t\le T}$ is a Gaussian process on $(\Omega, \mathcal{B}_T, Q)$ with mean 0 and variance $\beta(t)$. □

Remark 4.5. Note that (4.2) is a sufficient condition to be that $\{M_t\}$ is a martingale and so, in general, Theorem 4.4 is true with the assumption that $\{M_t\}$ is a martingale.

Theorem 4.6. *Let $\{Y_t\}_{0\le t\le T}$ be a diffusion process of the form:*

$$dY_t = u(t)dt + \theta(t)dB_{K,t}, \qquad 0 \le t \le T \tag{4.5}$$

for a $\mathcal{B}_t$-adapted process $\{u(t)\}_{0\le t\le T}$. Suppose there exist processes $\{v(t)\}_{0\le t\le T} \in \mathfrak{M}_{\mathrm{G}}$ and $\{\eta(t)\}_{0\le t\le T} \in \mathfrak{M}_{\mathrm{G}}$ such that

$$\theta(t)v(t) = u(t) - \eta(t), \qquad 0 \le t \le T. \tag{4.6}$$

Assume that (4.2) *holds. Let Q be the measure defined as in* (4.4). *Then*

$$\widetilde{B}_{K,t} = \int_0^t v(s)ds + B_{K,t}, \qquad 0 \le t \le T \tag{4.7}$$

is a Gaussian process on $(\Omega, \mathcal{B}_T, Q)$ with mean 0 and variance $\beta(t)$, and the process $\{Y_t\}_{0\le t\le T}$ has the following stochastic integral representation:

$$dY_t = \eta(t)dt + \theta(t)d\widetilde{B}_{K,t}, \qquad 0 \le t \le T.$$

Proof. It is obvious from Theorem 4.4 that $\{\widetilde{B}_{K,t}\}_{0\le t\le T}$ is a Gaussian process on $(\Omega, \mathcal{B}_T, Q)$ with mean 0 and variance $\beta(t)$. On the other hand, by (4.5), (4.7) and (4.6) we have

$$dY_t = u(t)dt + \theta(t)(d\widetilde{B}_{K,t} - v(t)dt) = \eta(t)dt + \theta(t)d\widetilde{B}_{K,t}.$$

Hence we complete the proof. □

THEOREM 4.7. *Let $\{X_t\}_{0\le t\le T}$ and $\{Y_t\}_{0\le t\le T}$ be diffusion processes of the forms:*

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_t)dB_{K,t}, \\ dY_t &= [\gamma(t) + b(Y_t)]dt + \sigma(Y_t)dB_{K,t},\ 0 \le t \le T,\ X_0 = Y_0 = x \in \mathbf{R}, \end{aligned}$$

where the functions b and σ satisfy the conditions of Theorem 4.1 and $\gamma(t) \in \mathfrak{M}_{\mathrm{G}}$. Suppose there exists a stochastic process $\{v(t)\}_{0\le t\le T}$ such that

$$\sigma(Y_t)\left(v(t) + \alpha'(t)\right) = \gamma(t), \qquad 0 \le t \le T.$$

Assume that (4.2) holds. Then the process $\{Y_t\}_{0\le t\le T}$ has the following stochastic integral representation:

$$dY_t = b(Y_t)dt + \sigma(Y_t)\left(d\widetilde{B}_{K,t} + d\alpha(t)\right), \qquad 0 \le t \le T, \tag{4.8}$$

where $\{\widetilde{B}_{K,t}\}_{0\le t\le T}$ is given as in (4.7).

Proof. By applying Theorem 4.6 to the case $\theta(t) = \sigma(Y_t)$, $u(t) = \gamma(t) + b(Y_t)$ and $\eta(t) = b(Y_t) + \sigma(Y_t)\alpha'(t)$, we have the representation (4.8). □

REMARK 4.8. Let $\{X_t\}_{0\le t\le T}$ and $\{Y_t\}_{0\le t\le T}$ be the diffusion processes given as in Theorem 4.7, and Q a probability measure defined as in (4.4). For each $0 \le t \le T$, put

$$B'_{K,t} = \alpha(t) + \widetilde{B}_{K,t}$$

for the Gaussian process $\{\widetilde{B}_{K,t}\}$ given as in (4.7). Then $\{B'_{K,t}\}$ is a Gaussian process on $(\Omega, \mathcal{B}_T, Q)$ with mean $\alpha(t)$ and variance $\beta(t)$, and (4.8) becomes $dY_t = b(Y_t)dt + \sigma(Y_t)dB'_{K,t}$, $0 \le t \le T$. Therefore, by the uniqueness of a solution of a stochastic differential equation in Theorem 4.1, the Q-law of $\{Y_t\}_{0\le t\le T}$ is the same as the P-law of $\{X_t\}_{0\le t\le T}$.

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