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THE REGIONALLY REGULAR RELATION

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Abstract. In this paper four regular relations, R, L^*, M^* and Q^* in a transformation group (X,T) are defined and some of their properties are studied.

1. Introduction

Let (X,T) be a transformation group whose phase space X is compact Hausdorff. The proximal relation P and the regionally proximal relation Q of (X,T) were first defined by Ellis and Gottschalk([7]). The syndetically proximal relation L and the regionally syndetically proximal relation M of (X,T) were introduced and studied intensively by Clay([4]).

Auslander([2]) defined the regular minimal sets which are the universal minimal sets for certain properties. The class of regular minimal sets is shown to coincide with the minimal right ideals of the enveloping semigroups of the transformation groups. The author([9]) introduced the regular relation R of (X,T) on a basis of the notions of the regular minimal sets and the proximal relation P of (X,T).

In this paper, we will introduce the regionally regular relation and the syndetically regular relation of (X,T). And it will be given the necessary and sufficient conditions for the regionally proximal relation and the regionally regular relation to be transitive.

2. Preliminaries

Throughout this paper, (X,T) will denote the transformation group with compact Hausdorff phase space X. A closed nonempty subset A of

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X is called a *minimal subset* if for every $x \in A$, the orbit xT is a dense subset of A. If X is itself minimal, we say it is *minimal transformation group*.

The compact Hausdorff space X carries a natural uniformity $\mathcal{U}[X]$ whose indices are the neighborhoods of the diagonal in $X \times X$. Two points x and y of X are called *proximal* provided that for each index α of X, there exists a $t \in T$ such that $(xt, yt) \in \alpha$. The set of all proximal pairs is called the *proximal relation* and is denoted by $P(X)$ or simply P. If x and y are not proximal, x and y are said to be *distal.* If $P(X) = \Delta$, where Δ is the diagonal of $X \times X$, then (X, T) is called the *distal transformation group.*

A continuous map π from (X, T) to (Y, T) with $\pi(xt) = \pi(x)t$ $(x \in X)$ is called a *homomorphism.* If Y is minimal, π is always onto. Especially, if π is onto, π is called an *epimorphism.* A homomorphism π from (X, T) onto itself is called an *endomorphism* of (X, T), and an isomorphism $\pi : (X, T) \to (X, T)$ is called an *automorphism* of (X, T). We denote the group of automorphisms of X by $A(X)$.

The enveloping semigroup of (X, T) denoted $E(X)$, or E is defined to be the closure T in X^X, providing with its product topology. The enveloping semigroup $E(X)$ is thus a compact semigroup in X^X.

A subset A of a topological group T is called *syndetic* if there exists a compact subset K of T with $T = AK$. A transformation group (X, T) is called *uniformly almost periodic* if given any index $\alpha \in \mathcal{U}[X]$ there exists a syndetic subset $A \subset T$ such that $xA \subset x\alpha$ for all $x \in X$.

Definition 2.1. Let (X, T) be a transformation group. Two points x and y are called *syndetically proximal* if for every index $\alpha \in \mathcal{U}[X]$ there exists a syndetic subset $A \subset T$ such that $(xt, yt) \in \alpha$ for all $t \in A$. The set of all syndetically proximal pairs is called the *syndetically proximal relation* and is denoted by $L(X)$ or simply L.

Theorem 2.2. ([4]) *Let (X, T) be a transformation group. Then the syndetically proximal relation $L(X)$ is an invariant equivalence relation.*

Lemma 2.3. ([4]) *Let (X, T) be a transformation group. The followings hold.*

$$\text{(1) If } \ (x, y) \in L(X), \ \text{ then } \ \overline{(x, y)T} \subset L(X)$$

$$\text{(2) If } \ \overline{(x, y)T} \subset P(X), \ \text{ then } \ (x, y) \in L(X)$$

Definition 2.4. Two points x and y of (X, T) are said to be *regionally proximal* if for each index $\alpha \in \mathcal{U}[X]$ and for each neighborhood U of x and each neighborhood V of y we can choose points $x_1 \in U$, $y_1 \in V$

and an element $t \in T$ in such a way that $(x_1t, y_1t) \in \alpha$. Equivalently, x and y are regionally proximal if there exist nets (x_n) and (y_n) in X, a net (t_n) in T such that $\lim x_n = x$, $\lim y_n = y$ and $\lim x_nt_n = \lim y_nt_n$. The set of all regionally proximal pairs of points is called the *regionally proximal relation* and is denoted by $Q(X)$ or Q.

It is well-known that (X,T) is uniformly almost periodic iff $Q(X) = \Delta$ iff given $\alpha \in \mathcal{U}[X]$, $\beta T \subset \alpha$ for some $\beta \in \mathcal{U}[X]$ (that is, T is equicontinuous)

DEFINITION 2.5. Let $M(X)$ or simply M denote the set of all pairs $(x,y) \in X \times X$ such that for every index $\alpha \in \mathcal{U}[X]$ and for every neighborhood U of x, every neighborhood V of y there exist point $x_1 \in U$ and $y_1 \in V$ and a syndetic subset $A \subset T$ such that $(x_1t, y_1t) \in \alpha$ for all $t \in A$. The set $M(X)$ is called the *regionally syndetically proximal relation.*

DEFINITION 2.6. A transformation group (X,T) is called *locally almost periodic* if for each neighborhood U of x there is a neighborhood V of x and a syndetic subset $A \subset T$ with $VA \subset U$. (X,T) is said to be *locally almost periodic* if it is locally almost periodic at each point $x \in X$.

THEOREM 2.7. ([3]) *Let (X,T) be a locally almost periodic transformation group. Then $P = Q = L = M$.*

THEOREM 2.8. ([4]) *Let (X,T) be a distal transformation group. Then $Q = M$.*

3. Regionally regular relations and syndetically regular relations

In this section, we introduce the regionally regular relations and syndetically regular relations as the generalized notions of regional proximalities and syndetical proximalities.

Let (X,T) be a transformation group. The points x and y of X are said to be *regular*([9]) if $h(x)$ and y are proximal for some automorphism h of X. The set of all regular pairs in X is called the *regular relation* and is denoted by $R(X)$ or R.

DEFINITION 3.1. Two points x and y of (X,T) are *regionally regular* if there exists an automorphism h of X such that $(h(x),y) \in Q(X)$. The set of all regionally regular pairs of points is called the *regionally* *regular relation* and is denoted by $Q^*(X)$ or Q^*. that is, $(x, y) \in Q^*(X)$ iff $(h(x), y) \in Q(X)$ for some $h \in A(X)$.

Definition 3.2. Two points x and y of (X, T) are said to be *syndetically regular* if $h(x)$ and y are syndetically proximal for some $h \in A(X)$. The set of all syndetically regular pairs is called the *syndetically regular relation* and is denoted by $L^*(X)$ or L^*. that is, $(x, y) \in L^*(X)$ iff $(h(x), y) \in L(X)$ for some $h \in A(X)$.

Definition 3.3. Let $M^*(X)$ or M^* denote the set of all pairs $(x, y) \in X \times X$ such that $h(x)$ and y are regionally syndetically proximal for some $h \in A(X)$. The set M^* is called the *regionally syndetically regular relation.* that is, $(x, y) \in M^*(X)$ iff $(h(x), y) \in M(X)$ for some $h \in A(X)$.

Remark 3.4. It is obvious that

(1) $P \subset R, \quad L \subset L^*, \quad M \subset M^*, \quad Q \subset Q^*$.

(2) $\Delta \subset L^* \subset M^* \subset Q^* \subset X \times X$.

(3) $\Delta \subset L^* \subset R \subset Q^* \subset X \times X$.

The relations R, L^*, M^*, Q^* are invariant, reflexive and symmetric.

Theorem 3.5. *Let (X, T) be a transformation group, and let $h : X \to X$ be an endomorphism. If $(x, y) \in L$, then $(h(x), h(y)) \in L$.*

Proof. Let $h : X \to X$ be an endomorphism, and let$(x, y) \in L$. Since X is a compact Hausdorff space and h is continuous, h is, in fact, uniformly continuous. So, for every index $\alpha \in \mathcal{U}[X]$, there is a $\beta \in \mathcal{U}[X]$ such that $(a, b) \in \beta$ implies $(h(a), h(b)) \in \alpha$. Since $(x, y) \in L$, for the given β, there exists a syndetic subset A of T such that $(xt, yt) = (x, y)t \in \beta$ for all $t \in A$. Hence, we obtain

$$(h(x), h(y))t = (h(x)t, h(y)t) = (h(xt), h(yt)) \in \alpha$$

for all $t \in A$. Therefore, $(h(x), h(y)) \in L$.

□

For a subset H of $A(X)$, we define

$$L^*_H = \{(x, y) \in X \times X \mid (h(x), y) \in L \text{ for some } h \in H\}.$$

Then, if H is the trivial group, then L^*_H coincides with L and if $H = A(X)$, then L^*_H coincides with L^*. For subgroups H, K of $A(X)$ with $H \subset K$, it is immediate that

$$L \subset L^*_H \subset L^*_K \subset L^*.$$

In [10], it is proved that L_H^* and L^* are invariant equivalence relations for every subgroup H of $A(X)$.

THEOREM 3.6. *Let (X,T) be a transformation group. The followings hold.*

$$\text{(1) If } (x,y) \in L^*, \quad \text{then } \overline{(x,y)T} \subset L^*.$$

$$\text{(2) If } \overline{(x,y)T} \subset R, \quad \text{then } (x,y) \in L^*.$$

Proof. (1) Let $(x,y) \in L^*$. Then $(h(x),y) \in L$ for some $h \in A(X)$. From Lemma 2.3, we have $\overline{(h(x),y)T} \subset L$. Since $\overline{(h(x),y)T} = (h(x),y)E(X)$,

$$(h(xp),yp) = (h(x)p,yp) = (h(x),y)p \in L,$$

for all $p \in E(X)$. that is, $(xp,yp) = (x,y)p \in L^*$ for all $p \in E(X)$. Therefore, we obtain $\overline{(x,y)T} \subset L^*$.

(2) Let $\overline{(x,y)T} \subset R$. Since $\overline{(x,y)T} = (x,y)E(X)$, $(xp,yp) = (x,y)p \in R$ for all $p \in E(X)$. There exists an $h \in A(X)$ such that

$$(h(xp),yp) = (h(x)p,yp) = (h(x),y)p \in P$$

for all $p \in E(X)$. From Lemma 2.3, we have $(h(x),y) \in L$. Therefore, $(x,y) \in L^*$.

□

The following theorem is an analogue of Theorem 3([4]).

THEOREM 3.7. *Let (X,T) be a transformation group. Then*

(1) $L^* = \{(x,y) \in X \times X \mid \overline{(x,y)T} \subset R\}$.

(2) $R = L^*$ *if and only if* $(x,y) \in R$ *implies* $\overline{(x,y)T} \subset R$.

Proof. (1) Let $(x,y) \in L^*$. Then $\overline{(x,y)T} \subset L^*$ by Theorem 3.6(1). Since $L^* \subset R$, it follows that $\overline{(x,y)T} \subset R$. Conversely, let $\overline{(x,y)T} \subset R$. Then $(x,y) \in L^*$ by Theorem 3.6(2).

(2) Let $R = L^*$. Then $(x,y) \in R$ implies $(x,y) \in L^*$. From Theorem 3.6(1), we have $\overline{(x,y)T} \subset L^* \subset R$. Conversely, it suffices to show that $R \subset L^*$. Let $(x,y) \in R$. Then $\overline{(x,y)T} \subset R$ by hypothesis, and we obtain $(x,y) \in L^*$ by Theorem 3.6(2). □

COROLLARY 3.8. *Let (X,T) be a transformation group. If R is closed in $X \times X$, then $R = L^*$, and R is a closed invariant equivalence relation.*

Proof. Let R be closed in $X \times X$. Then $(x,y) \in R$ implies $\overline{(x,y)T} \subset R$. From Theorem 3.7(2), we have $R = L^*$. Since L^* is an invariant equivalence relation and $R = L^*$, R is an invariant equivalence relation.

□

Theorem 3.9. *Let (X,T) be a transformation group. The followings hold.*

(1) *If* $Q = M$, *then* $Q^* = M^*$.

(2) *If* $P = L$, *then* $R = L^*$.

(3) *If* $Q \subset P$, *then* $Q^* \subset R$.

(4) *If* (X,T) *is distal, then* $Q^* = M^*$.

Proof. (1),(2),(3) are obvious from the definitions of R, L^*, Q^* and M^*.

(4) In a distal transformation group (X,T), we always have $Q = M$ by Theorem 2.8. Therefore, $Q^* = M^*$ by (1). □

From Theorem 2.7 and Theorem 3.9 we have the following corollary.

Corollary 3.10. *Let (X,T) be a locally almost periodic transformation group. Then $R = Q^* = L^* = M^*$.*

Lemma 3.11. *Let (X,T) be a transformation group, and let $h : X \to X$ be a homomorphism. If $(x,y) \in Q$, then $(h(x), h(y)) \in Q$.*

Proof. Let $h : X \to X$ be a homomorphism, and let $(x,y) \in Q$. There exists nets (x_n), (y_n) in X, a net (t_n) in T such that $\lim x_n = x$, $\lim y_n = y$ and $\lim x_n t_n = \lim y_n t_n$. Consider nets $(h(x_n))$, $(h(y_n))$ in X. Then for the given net (t_n) in T, it follows that

$$\lim h(x_n) = h(x),\ \lim h(y_n) = h(y)$$

$$\lim h(x_n)t_n = \lim h(x_n t_n) = \lim h(y_n t_n) = \lim h(y_n)t_n.$$

Therefore, $(h(x), h(y)) \in Q$. □

For a fixed h in $A(X)$, we denote $Q_h^*(X)$ or Q_h^* to be the set of all $(x,y) \in X \times X$ satisfying the condition $(h(x), y) \in Q$. It is noted that

$$Q^* = \cup\{Q_h^* \mid h \in A(X)\}.$$

Remark 3.12. (1) If $1 : X \to X$ is the identity automorphism, then Q_1^* coincides with Q.

(2) Q_h^* is closed in $X \times X$ for all $h \in A(X)$. In fact, let $((x_n, y_n))$ be a net in Q_h^* such that $((x_n, y_n))$ converges to (x,y).Then $(h(x_n), y_n) \in Q$ for all n and $(h(x_n), y_n)$ converges to $(h(x), y)$. Since $Q = \cap\{\overline{\alpha T} \mid \alpha \in \mathcal{U}[X]\}$, Q is closed and it follows that $(h(x), y) \in Q$. that is $(x,y) \in Q_h^*$. Therefore Q_h^* is closed.

Remark 3.13. Let (X, T) be a minimal transformation group and let $Q = \Delta$, the diagonal of $X \times X$. Then the followings are verified easily.

(i) $Q_h^* \cap Q_k^* = \phi$ for $h \neq k$ in $A(X)$.

(ii) $Q_h^* \circ Q_k^* = Q_{hk}^*$ for h, k in $A(X)$.

(iii) $Q \circ Q_h^* = Q_h^* \circ Q = Q_h^*$ for all h in $A(X)$.

(iv) $(Q_h^*)^{-1} = Q_{h^{-1}}^*$ for all h in $A(X)$.

From Remark 3.13 we have the following theorem.

Theorem 3.14. *Let (X, T) be an uniformly almost periodic minimal transformation group. Then $\{Q_h^* \mid h \in A(X)\}$ is a partition of Q^*. Moreover, $\{Q_h^* \mid h \in A(X)\}$ forms a group.*

The following theorem gives us a necessary and sufficient condition for the regionally proximal relation Q and the regionally regular relation Q^* to be equivalence relations.

Theorem 3.15. *Let (X, T) be a transformation group. The following statements hold.*

(1) Q is an equivalence relation if and only if $Q_h^* \circ Q_k^* = Q_{hk}^*$ for all $h, k \in A(X)$.

(2) Q^* is an equivalence relation if and only if for h and k in $A(X)$, there exists a g in $A(X)$ such that $Q_h^* \circ Q_k^* \subset Q_g^*$.

Proof. (1) Necessity. Let Q be an equivalence relation and let $(x, z) \in Q_h^* \circ Q_k^*$. There exists a $y \in X$ such that $(x, y) \in Q_k^*$ and $(y, z) \in Q_h^*$. that is, $(k(x), y) \in Q$ and $(h(y), z) \in Q$. We also have $(hk(x), h(y)) \in Q$ by Lemma 3.11. Since Q is transitive, $(hk(x), z) \in Q$. Therefore $(x, z) \in Q_{hk}^*$.

On the other hand, let $(x, z) \in Q_{hk}^*$. Then $(hk(x), z) \in Q$. Put $k(x) = y$. Then we have $(h(y), z) \in Q$ and $(k(x), y) = (y, y) \in Q$. That is, $(y, z) \in Q_h^*$ and $(x, y) \in Q_k^*$. Therefore, $(x, z) \in Q_h^* \circ Q_k^*$.

Sufficiency. Suppose that $Q_h^* \circ Q_k^* = Q_{hk}^*$ for all $h, k \in A(X)$. To show that Q is an equivalence relation, we need only to show that Q is transitive. Let $(x, y) \in Q = Q_1^*$, and $(y, z) \in Q = Q_1^*$, where $1 : X \to X$ is the identity automorphism. By hypothesis, $(x, z) \in Q_1^* \circ Q_1^* = Q_1^* = Q$, which shows that Q is transitive.

(2) Necessity. Suppose that Q^* is an equivalence relation and let $(x, z) \in Q_h^* \circ Q_k^*$. There is an $y \in X$ such that $(x, y) \in Q_k^*$ and $(y, z) \in Q_h^*$. Since $Q_k^* \subset Q^*$, $Q_h^* \subset Q^*$ and Q^* is transitive, we have $(x, z) \in Q^*$. Therefore, $(x, z) \in Q_g^*$ for some $g \in A(X)$.

Sufficiency. Suppose that for h and k in $A(X)$, $Q^*_h \circ Q^*_k \subset Q^*_g$ for some $g \in A(X)$. It suffices to show that Q^* is transitive. Let $(x, y) \in Q^*$ and $(y, z) \in Q^*$. Then $(x, y) \in Q^*_k$ and $(y, z) \in Q^*_h$ for some h, k in $A(X)$. Since $Q^*_h \circ Q^*_k \subset Q^*_g$, we have

$$(x, z) \in Q^*_h \circ Q^*_k \subset Q^*_g \subset Q^*.$$

Therefore, Q^* is transitive. The proof is completed. □

The following corollary is immediate from the Theorem 3.15.

Corollary 3.16. *Let (X, T) be a transformation group. If the regionally proximal relation Q is an equivalence relation, then so is the regionally regular relation Q^*.*

From Corollary 3.16, we have the following corollary obviously.

Corollary 3.17. *Let (X, T) be an uniformly almost periodic minimal transformation group. Then the regionally regular relation Q^* is an invariant equivalence relation.*

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