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MULTIFRACTAL ANALYSIS OF A GENERAL CODING SPACE

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Abstract. We study Hausdorff and packing dimensions of subsets of a general coding space with a generalized ultra metric from a multifractal spectrum induced by a self-similar measure on a self-similar Cantor set using a function satisfying a Hölder condition.

1. Introduction

Recently we obtained some results([1, 3]) of relationship between members of a spectral class of a self-similar Cantor set([1, 3, 7]) using distribution sets([1, 3]) of a frequency sequence. We also found some relationship([4]) between subsets of a Cantor set and their corresponding subsets of a coding space using a function satisfying a Hölder condition. Nowadays most of the fractals have been dealt in the Euclidean space for the discoveries of their Hausdorff and packing dimensions([7]) in the Euclidean space. However Hausdorff and packing dimensions can also be considered in a non-Euclidean metric space. We consider such an example as a coding space with an ultra metric and they give also many informations of structures of the space. Recently we([4]) studied a relationship between subsets in a coding space with an ultra metric and subsets in a Cantor set with the Euclidean metric. Combining the results([1, 3, 4]), we get some information of multifractal spectra of a coding space of the Cantor set. We note that the bridge to connect the two subsets which are in a self-similar Cantor set and in a coding space is a natural code function([2]).

In this paper using the relationship([1, 3]) between members of spectral class of a self-similar Cantor set and their corresponding subsets in a coding space, we get the Hausdorff dimensions and packing dimensions of multifractal spectral members of a coding space.

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2. Preliminaries

We denote F a self-similar Cantor set, which is the attractor of the similarities $f_1(x) = ax$ and $f_2(x) = bx + (1-b)$ on $I = [0,1]$ with $a > 0$, $b > 0$ and $1-(a+b) > 0$. Let $I_{i_1,\cdots,i_k} = f_{i_1} \circ \cdots \circ f_{i_k}(I)$ where $i_j \in \{1,2\}$ and $1 \le j \le k$.

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{R}$ be the set of real numbers. We note that if $x \in F$, then there is $\sigma \in \{1,2\}^{\mathbb{N}}$ such that $\bigcap_{k=1}^{\infty} I_{\sigma|k} = \{x\}$ (Here $\sigma|k = i_1, i_2, \cdots, i_k$ where $\sigma = i_1, i_2, \cdots, i_k, i_{k+1}, \cdots$). If $x \in F$ and $x \in I_\sigma$ where $\sigma \in \{1,2\}^k$, $c_k(x)$ denotes I_σ and $|c_k(x)|$ denotes the diameter of $c_k(x)$ for each $k = 0,1,2,\cdots$. Let $p \in (0,1)$ and we denote γ_p a self-similar Borel probability measure on F satisfying $\gamma_p(I_1) = p$(cf. [7]). $\dim(E)$ denotes the Hausdorff dimension of E and $\mathrm{Dim}(E)$ denotes the packing dimension of E([7]). We note that $\dim(E) \le \mathrm{Dim}(E)$ for every set E([7]). We denote $n_1(x|k)$ the number of times the digit 1 occurs in the first k places of $x = \sigma$(cf. [1]).
For $q \in [0,1]$, we define lower(upper) distribution set $\underline{F}(r)(\overline{F}(r))$ containing the digit 1 in proportion q by

$$\underline{F}(q) = \{x \in F : \liminf_{k\to\infty} \frac{n_1(x|k)}{k} = q\},$$

$$\overline{F}(q) = \{x \in F : \limsup_{k\to\infty} \frac{n_1(x|k)}{k} = q\}.$$

We write $\underline{F}(q) \cap \overline{F}(q) = F(q)$ and call it a distribution set containing the digit 1 in proportion q.

We assume that $\{1,2\}^{\mathbb{N}}$ is an ultra metric space with the ultra metric ρ satisfying, for $(x,y) \in \{(x,y)|0 < x,y < 1\}$, $\rho(\sigma,\sigma) = 0$ and if $\sigma \ne \tau$ then $\rho(\sigma,\tau) = x^{n_1(x|k)}y^{k-n_1(x|k)}$ where $\sigma = i_1 i_2 \cdots i_k i_{k+1} \cdots$ and $\tau = i_1 i_2 \cdots i_k j_{k+1} \cdots$ where $i_{k+1} \ne j_{k+1}$ for some $k = 0,1,2\cdots$. We will call $\{1,2\}^{\mathbb{N}}$ with such an ultra metric *a coding space with a generalized ultra metric* $\rho_{x,y}$.

If $a = b = r$ where $r \in (0,1)$, we will call the generalized ultra metric space by $\{1,2\}^{\mathbb{N}}$ with an ultra metric $\rho_{r,r}(\equiv \rho_r)$ *a coding space with a uniform ultra metric* ρ_r. In that case(([6]) ρ_r satisfies $\rho_r(\sigma,\sigma) = 0$ and if $\sigma \ne \tau$ then $\rho_r(\sigma,\tau) = r^{n_1(x|k)}r^{k-n_1(x|k)} = r^k$ where $\sigma = i_1 i_2 \cdots i_k i_{k+1} \cdots$ and $\tau = i_1 i_2 \cdots i_k j_{k+1} \cdots$ where $i_{k+1} \ne j_{k+1}$ for some $k = 0,1,2\cdots$. In [4] we considered $\{1,2\}^{\mathbb{N}}$ with a uniform ultra metric $\rho_{\frac{1}{2}}$.

In the coding space we can define a probability measure Γ_p induced by a natural set function defined on the class of its cylinders.

We define a natural code function $f : F \longrightarrow \{1,2\}^{\mathbb{N}}$ such that $f(x) = \sigma$ with $\{x\} = \bigcap_{k=0}^{\infty} I_{\sigma|k}$ where $\sigma \in \{1,2\}^{\mathbb{N}}$ and F is the self-similar Cantor set with contraction ratios a, b. If we define $\Gamma_p((f(x)|n) \times \{1,2\}^{\mathbb{N}}) = \gamma_p(I_{f(x)|n})$ for all $x \in F$, then Γ_p is easily extended to a Borel probability measure on $\{1,2\}^{\mathbb{N}}$.

Before going into our main theorems, we need some lemmas to be studied. Before going into our lemmas, we need some definitions to be considered. Let $\Delta_c = \{(a,b) | 0 < a, b < 1, a+b = c\}$ where $0 < c < 2$. We define $s_d(a,b)$ to be a real number s satisfying $a^s + b^s = d$ for each $(a,b) \in \bigcup_{0<c<2} \Delta_c$ for each $d \in (0,2)$. The definition is well-defined from the following Lemma.

LEMMA 2.1. *There is a unique positive number t satisfying $a^t + b^t = d$ where $0 < d < 2$ for any $(a,b) \in \Delta_c$ for each $0 < c < 2$. That is, a positive number $s_d(a,b)$ is well-defined for any $0 < d < 2$ and any $(a,b) \in \Delta_c$ for each $0 < c < 2$. Further $\{s_d(a,b) | (a,b) \in \Delta_c\} = (0, l_{c,d}]$ where $l_{c,d} = \frac{\log \frac{d}{2}}{\log \frac{c}{2}}$.*

Proof. It follows from that the function $p(t) = a^t + b^t$ is a strictly decreasing function for $t \in (0,\infty)$ having a range $(0,2)$. Further $l_{c,d}$ follows from the solution t of the equation $(\frac{c}{2})^t + (\frac{c}{2})^t = d$. □

LEMMA 2.2. *$\{(a^{s_d(a,b)}, b^{s_d(a,b)})\} = \{(a^t, b^t) | t > 0\} \cap \Delta_d$ for each $(a,b) \in \Delta_c$ where $0 < c < 2$.*

Proof. It follows from the uniqueness of the solution t of the equation $a^t + b^t = d$. □

LEMMA 2.3. *The function $S_d : \Delta_c \longrightarrow \Delta_d$ such that $S_d(a,b) = (a^{s_d(a,b)}, b^{s_d(a,b)})$ is a bijection.*

Proof. Let $(a,b) \in \Delta_c$. Then $a^{s_d(a,b)} + b^{s_d(a,b)} = d$ for some positive number $s_d(a,b)$. Suppose that $(a',b')(\neq (a,b)) \in \Delta_c$. Then $(a')^{s_d(a',b')} + (b')^{s_d(a',b')} = d$ for some positive number $s_d(a',b')$. Then we have

$$(a^{s_d(a,b)}, b^{s_d(a,b)}) \neq ((a')^{s_d(a',b')}, (b')^{s_d(a',b')}).$$

For, if we assume that

$$(a^{s_d(a,b)}, b^{s_d(a,b)}) = ((a')^{s_d(a',b')}, (b')^{s_d(a',b')}),$$

we get $\frac{\log a'}{\log a} = \frac{\log b'}{\log b}$. If we put $\frac{\log a'}{\log a} = \frac{\log b'}{\log b} = \alpha$, then we have $\alpha \neq 1$ since $(a', b') \neq (a, b)$. Assuming that $\alpha > 1$, we have a contradiction that $c = a' + b' = a^\alpha + b^\alpha < a + b = c$. Similarly we have a contradiction for $\alpha < 1$. Hence S_d is an injection. By the above Lemma, we get that S_d is a surjection. For, $\{(a^t, b^t) | t > 0\} = \{(x, x^\alpha) | 0 < x < 1\}$ since we have $b = a^\alpha$ for some $0 < \alpha < \infty$ noting that $0 < a, b < 1$. Further we easily see that such α varies in $(0, \infty)$ as (a, b) varies in Δ_c for each $c \in (0, 2)$. □

LEMMA 2.4. *$\{(a^{s_d(a,b)}, b^{s_d(a,b)}) | (a, b) \in \Delta_c\} = \Delta_d$ for each $0 < c, d < 2$.*

Proof. It follows from the above Lemma. □

3. Main results

The following lemma gives the scaling properties of Hausdorff and packing dimensions of an image of a function satisfying a bi-Hölder condition.

LEMMA 3.1. *Let E be a metric space with a metric ρ. Let $f : F \longrightarrow E$ be a function satisfying a Hölder condition*

$$c_1 |x - y|^\alpha \leq \rho(f(x), f(y)) \leq c_2 |x - y|^\alpha$$

for some constants c_1, c_2 and each $x, y \in F$. Then $\dim_H(f(F)) = \frac{1}{\alpha} \dim_H(F)$ and $\dim_p(f(F)) = \frac{1}{\alpha} \dim_p(F)$.

Proof. $\dim_H(f(F)) = \frac{1}{\alpha} \dim_H(F)$ follows from Proposition 2.3 of [7]. $\dim_p(f(F)) = \frac{1}{\alpha} \dim_p(F)$ follows from [4] or the similar arguments with the proof of Proposition 2.3 of [7]. □

The following theorem gives the close connection between the generalized ultra metric space and the self-similar Cantor set.

THEOREM 3.2. *Let $a + b = 1$ for positive real numbers a, b. Consider an arbitrary real number $r \in (0, 1)$. Then there there is $0 < s < 1$ such that $(ra)^s + (rb)^s = 1$. Let $f : F_{ra,rb} \longrightarrow \{1, 2\}^{\mathbb{N}}$ be a function such that $f(x) = \sigma$ with $\{x\} = \bigcap_{k=0}^{\infty} I_{\sigma|k}$ where $\sigma \in \{1, 2\}^{\mathbb{N}}$ and $F_{ra,rb}$ is a self-similar Cantor set with contraction ratios ra, rb. Then it satisfies a Hölder condition*

$$|x - y|^t \leq \rho_{(ra)^t,(rb)^t}(f(x), f(y)) \leq [\frac{1}{1 - (ra + rb)}]^t |x - y|^t$$

for each $x, y \in F_{ra,rb}$ *for each positive number* t*. Hence*

$$\dim(\{1,2\}^{\mathbb{N}}, \rho_{(ra)^t,(rb)^t}) = \mathrm{Dim}(\{1,2\}^{\mathbb{N}}, \rho_{(ra)^t,(rb)^t}) = \frac{s}{t}.$$

Proof. Considering a fundamental interval of $F_{ra,rb}$, we easily get

$$|x-y| \leq \rho_{ra,rb}(f(x), f(y)) \leq [\frac{1}{1-(ra+rb)}]|x-y|$$

for each $x, y \in F_{ra,rb}$. Hence we have for each positive real number t

$$|x-y|^t \leq \rho_{(ra)^t,(rb)^t}(f(x), f(y)) \leq [\frac{1}{1-(ra+rb)}]^t|x-y|^t$$

for each $x, y \in F_{ra,rb}$. By Lemma 2.1, we get result from $\dim(F_{ra,rb}) = \mathrm{Dim}(F_{ra,rb}) = s$ since $(ra)^s + (rb)^s = 1$. □

Corollary 3.3. *([4]) Let* $f : F \longrightarrow \{1,2\}^{\mathbb{N}}$ *be a function such that* $f(x) = \sigma$ *with* $\{x\} = \bigcap_{k=0}^{\infty} I_{\sigma|k}$ *where* $\sigma \in \{1,2\}^{\mathbb{N}}$ *and* F *is the classical Cantor ternary set. Then it satisfies a Hölder condition*

$$|x-y|^{\frac{\log 2}{\log 3}} \leq \rho_{\frac{1}{2}}(f(x), f(y)) \leq 2|x-y|^{\frac{\log 2}{\log 3}}$$

for each $x, y \in F$*. Hence* $\dim(\{1,2\}^{\mathbb{N}}, \rho_{\frac{1}{2}}) = \mathrm{Dim}(\{1,2\}^{\mathbb{N}}, \rho_{\frac{1}{2}}) = 1.$

Proof. We consider $a = b = \frac{1}{2}$ and $r = \frac{2}{3}$ in the above Lemma. Then $s = \frac{\log 2}{\log 3}$. Considering $t = \frac{\log 2}{\log 3}$, we have our result. □

Example 3.4. Let $a = \frac{1}{3}$ and $b = \frac{2}{3}$. Consider $r = \frac{3}{4}$. Then the solution s of the equation $(\frac{1}{4})^s + (\frac{1}{2})^s = 1$ is $\frac{\log(\frac{\sqrt{5}-1}{2})}{\log(\frac{1}{2})}$. Hence for $s = \frac{\log(\frac{\sqrt{5}-1}{2})}{\log(\frac{1}{2})}$ we have

$$|x-y|^s \leq \rho_{\frac{3-\sqrt{5}}{2},\frac{\sqrt{5}-1}{2}}(f(x), f(y)) \leq [\frac{1}{1-(ra+rb)}]^s|x-y|^s$$

for each $x, y \in F_{\frac{1}{4},\frac{1}{2}}$. Hence we have

$$\dim(\{1,2\}^{\mathbb{N}}, \rho_{\frac{3-\sqrt{5}}{2},\frac{\sqrt{5}-1}{2}}) = \mathrm{Dim}(\{1,2\}^{\mathbb{N}}, \rho_{\frac{3-\sqrt{5}}{2},\frac{\sqrt{5}-1}{2}}) = 1.$$

Remark 3.5. From the above Example, we find $\frac{3-\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2}$ from $a = \frac{1}{3}$ and $b = \frac{2}{3}$ and $r = \frac{3}{4}$.

Theorem 3.6. *Each pair* (x, y) *of real numbers in the simplex* $\Delta = \{(x,y)|x, y > 0, x+y = 1\}$ *has a proper* $(a, b) \in \Delta$ *with* $0 < r < 1$ *to give* $(ra)^s + (rb)^s = 1$ *and* $(ra)^s = x, (rb)^s = y$*.*

Proof. Consider a pair (x, y) of real numbers in the simplex $\Delta = \{(x, y)|x, y > 0, x + y = 1\}$. For each $a \in (0, 1)$ $s_a(r)$ is a continuous function whose range is $(0, 1)$ for $r \in (0, 1)$ where $(ra)^{s_a(r)} + (r(1 - a))^{s_a(r)} = 1$. Then for each $a \in (0, 1)$ and a fixed $s \in (0, 1)$ there exists $r \in (0, 1)$ such that $(ra)^s + (r(1 - a))^s = 1$ by the intermediate value theorem. Noting $x = \frac{1}{1+(\frac{1-a}{a})^s}$, we find $a \in (0, 1)$ satisfying $x = \frac{1}{1+(\frac{1-a}{a})^s}$ for the above fixed $s \in (0, 1)$ using $r \in (0, 1)$ properly. □

The proof of the above theorem adopts a direct method whereas that of the following theorem does not.

THEOREM 3.7. *Each pair (x, y) of real numbers with $0 < x, y < 1$ has a proper (a, b) in the simplex $\Delta = \{(a, b)|a, b > 0, a + b = 1\}$ with $0 < r < 1$ to give $(ra)^s + (rb)^s = 1$ and $(ra)^t = x, (rb)^t = y$. Hence it gives* $\dim(\{1, 2\}^{\mathbb{N}}, \rho_{x,y}) = \mathrm{Dim}(\{1, 2\}^{\mathbb{N}}, \rho_{x,y}) = \frac{s}{t}$.

Proof. From the above Theorem, for each $(z, w) \in \Delta$ we have $(a, b) \in \Delta$ and $r \in (0, 1)$ with $(ra)^s + (rb)^s = 1$ to give $((ra)^s, (rb)^s) = (z, w)$. Then the curves $C_{z,w} = \{((ra)^t, (rb)^t)|t > 0\}$ fill up the set $\{(x, y)|0 < x, y < 1\}$. That is

$$\bigcup_{(z,w)\in\Delta} C_{z,w} = \{(x, y)|0 < x, y < 1\}.$$

Putting $c = 1$ and considering all $0 < d < 2$ in Lemma 2.4, we have the above fact. Theorem 3.2 gives the Hausdorff and packing dimension of $\{1, 2\}^{\mathbb{N}}, \rho_{x,y}$. □

REMARK 3.8. If $(x, y) \to (1, 1)$ where $(x, y) \in \{(x, y)|0 < x, y < 1\}$, then we see that $\dim(\{1, 2\}^{\mathbb{N}}, \rho_{x,y}) = \mathrm{Dim}(\{1, 2\}^{\mathbb{N}}, \rho_{x,y}) \to \infty$. For, if $(x, y) \to (1, 1)$ where $(x, y) \in \{(x, y)|0 < x, y < 1\}$ then t should approach to 0 where $(ra)^t = x, (rb)^t = y$ for some $a, b \in \Delta$. Hence the Hausdorff and packing dimension $\frac{s}{t} \to \infty$ for a fixed positive value s which is derived from fixed a, b.

COROLLARY 3.9. *If $G \subset \{1, 2\}^{\mathbb{N}}, \rho_{(ra)^t,(rb)^t}$, then* $\dim(G) = \frac{\dim(f^{-1}(G))}{t}$, *where $f : F_{ra,rb} \longrightarrow \{1, 2\}^{\mathbb{N}}$.*

Proof. We note that f is a bijection. It follows from Lemma 3.1 and Theorem 3.2. □

COROLLARY 3.10. *If $G \subset \{1, 2\}^{\mathbb{N}}, \rho_{(ra)^t,(rb)^t}$, then*

$$\mathrm{Dim}(G) = \frac{\mathrm{Dim}(f^{-1}(G))}{t},$$

where $f : F_{ra,rb} \longrightarrow \{1,2\}^{\mathbb{N}}$.

Proof. We note that f is a bijection. It follows from Lemma 3.1 and Theorem 3.2. □

PROPOSITION 3.11. *In* $F_{ra,rb}$, *for a distribution set* $F(q)$ *with a lower distribution set* $\underline{F}(q)$ *where* $q \in [0,1]$,

$$\dim(\underline{F}(q)) = \dim(F(q)) = \mathrm{Dim}(F(q)) = \frac{q\log q + (1-q)\log(1-q)}{q\log ra + (1-q)\log rb}.$$

Proof. We note that

$$\dim(\underline{F}(q)) = \dim(F(q)) = \mathrm{Dim}(F(q)) = \frac{q\log q + (1-q)\log(1-q)}{q\log a + (1-q)\log b}$$

([1, 3]) for a self-similar Cantor set with contraction ratios a, b. It follows from the above with contraction ratios ra, rb. □

COROLLARY 3.12. *In* $F_{ra,rb}$, *for each* $q \in [0,1]$ *and* $f : F_{ra,rb} \longrightarrow \{1,2\}^{\mathbb{N}}, \rho_{(ra)^t,(rb)^t}$,

$$\begin{aligned}\dim(f(\underline{F}(q))) &= \dim(f(F(q))) = \mathrm{Dim}(f(F(q))) \\ &= \frac{q\log q + (1-q)\log(1-q)}{tq\log ra + t(1-q)\log rb}.\end{aligned}$$

Proof. It follows from the above Proposition and Corollaries. □

REMARK 3.13. In the above Corollary, we see that in $\{1,2\}^{\mathbb{N}}, \rho_{(ra)^t,(rb)^t}$,

$$\dim(f(F((ra)^s))) = \mathrm{Dim}(f(F((ra)^s))) = \frac{s}{t}$$

for s satisfying $(ra)^s + (rb)^s = 1$. We note that $\gamma_{(ra)^s}(F((ra)^s)) = 1 > 0$ by the strong law of large numbers. Hence $\Gamma_{(ra)^s}(f(F((ra)^s))) = 1$.

REMARK 3.14. Let s satisfy $(ra)^s + (rb)^s = 1$. We clearly see that $\Gamma_{(ra)^s}(f(F(q))) = 0$ for all $q (\neq (ra)^s) \in [0,1]$. We note that $\{f(F(q)) : q \in [0,1]\}$ forms a multifractal spectrum of a coding space $\{1,2\}^{\mathbb{N}}$ with a non-Euclidean metric $\rho_{(ra)^t,(rb)^t}$ giving for its members

$$\dim(f(F(q))) = \mathrm{Dim}(f(F(q))) = \frac{q\log q + (1-q)\log(1-q)}{tq\log ra + t(1-q)\log rb}.$$

EXAMPLE 3.15. Let s satisfy $(ra)^s + (rb)^s = 1$. Let

$$E = \cup_{q(\neq (ra)^s)\in[0,1]} f(F(q)).$$

We see that $\Gamma_{(ra)^s}(E) = 0$ since $\Gamma_{(ra)^s}(f(F((ra)^s))) = 1$ and $\Gamma_{(ra)^s}(\{1,2\}^{\mathbb{N}}) = 1$. We note that $\dim(E) = \mathrm{Dim}(E) = \frac{s}{t}$ without the condition that $\Gamma_{(ra)^s}(E) > 0$. It follows from that $\dim(E) \geq \sup_{q(\neq (ra)^s)\in[0,1]} \dim(f(F(q)))$ by monotonicity and

$$\sup_{q(\neq (ra)^s)\in[0,1]} \dim(f(F(q))) = \sup_{q(\neq (ra)^s)\in[0,1]} \frac{q\log q + (1-q)\log(1-q)}{tq\log ra + t(1-q)\log rb} = \frac{s}{t}.$$

Similarly it holds for packing case.

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