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ON CROSSING NUMBER OF KNOTS

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Abstract. The aim of this paper is to endow a monoid structure on the set S of all oriented knots(links) under the operation $\uplus$, called addition of knots. Moreover, we prove that there exists a homomorphism of monoids between $(S_d, \uplus)$ to $(\mathbf{N}, +)$, where S_d is a subset of S with an extra condition and $\mathbf{N}$ is the monoid of non negative integers under usual addition.

1. Introduction

By a knot is meant a smooth embedding of the circle S^1 in $\mathbf{R}^3$(or in the sphere S^3) as well as the image of this embedding. Two or more knots together are called a link. Let K_1 and K_2 be two oriented knots. By a composition (or connected sum operation), denoted by $\uplus$, of knots K_1 and K_2 we meant the oriented knot obtained by attaching the knot K_2 to the knot K_1 with respect to the orientation of both knots. This operation is well defined [5]. The number of crossings of a knot K, denoted by $N(K)$ is the minimal number of crossings needed for a diagram of the knot. There are several types of crossings in a knot K viz, regular and non-regular crossing. It is found that [2], $N(K_1 \uplus K_2) = N(K_1) + N(K_2)$, for K_1, K_2 are two alternating knots.

In this paper we study the additivity of crossing numbers of knots under the connected sum operation. For this we assume that a knot K has components consists of few distinct circles in $\mathbf{R}$, called Seifert circles. When the knot has only one component then $n(K) = 1$, where $n(K)$ is the number of components. We now projet a knot in a certain plane [1]. If D is the projection of a knot K, then its crossings can easily be visualized.

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Our paper has been distributed with four sections, the first one is the introduction and the second is preliminaries section where we have given some known definitions and results. In section 3, we show that $(S, \uplus)$ has a commutative monoid structure on the set S of all oriented knots under the operation $\uplus$, called addition of knots. Moreover, we prove that there exists a homomorphism of monoids between $(S_d, \uplus)$ to $(\mathbf{N}, +)$, where S_d is a subset of S and $\mathbf{N}$ is the monoid of all non negative integers under usual addition. Finally, we have discussed and investigated some relations on crossing numbers of different knots.

2. Preliminaries

The knots studied in knot theory are(almost) always considered to be closed loops. Knots that are equivalent to the unit circle are considered to be unknotted or trivial. The simplest non-trivial knot is the trefoil knot which comes in a left and a right handed form.

Two knots K_1 and K_2 are said to be topologically equivalent if there exists a level preserving ambient isotopy H connecting K_1 to K_2. For each regular projection D of K [1] we can count the number of crossing in it. If we take the minimum of this number over all possible regular projections of K or any knot K' i.e., equivalent to K, we obtain an invariant of all knots that are equivalent to K. This number is called the crossing number of the knot K and is denoted by $N(K)$. There are so many type of crossings, out of which two common examples are over and under crossing. In an alternating knot crossings alternate under-over-under-over-.... as one travels along the knot. We consider a knot projection D of a knot K with a given orientation. For convenience, let us assume that D is in the xy plane. At each crossing either it is of under or over type. We then remove crossings from D and reconnect them. This operation does not change the orientation of the other strands of D and the process can be repeated until all crossings are removed. This procedure is called Seifert's algorithm and the result of this algorithm is a set of disjoint topological circles. These circles are called as Seifert circles and were first introduced by H. Schubert. Since each Seifert circle is a plane curve, we can lift each Seifert circle to a plane with different z-coordinate so that no two of these Seifert circles share a plane with the same z-coordinate. The disks bounded by theses Seifert circles are called Seifert disks. If $n(K)$ be the number of components in K, then the boundary of this Seifert surface so constructed has exactly $n(K)$ components.

Now genus of this Seifert surface is calculated to be $g = [c + 2 - s - n(K)]/2$, where c is the number of crossings in D and s is the number of Seifert circles produced by the Seifert's algorithm. The genus of K, denoted by $g(K)$ is the minimum genus over the genera of all possible Seifert surfaces of knots that are equivalent to K.

An important observation is that for an alternating knot K with a reduced alternating projection of n crossings has crossing number $N(K) = n$. On the other hand it can be shown that if K_1 and K_2 are alternating knots then so is $K_1 \uplus K_2$. More precisely from the reduced alternating projection D_1 of K_1 and the reduced alternating projection of D_2 of K_2, we can construct a reduced alternating projection D of $K_1 \uplus K_2$ such that D has $N(K_1) + N(K_2)$ crossing. It is known that [8], if K_1 and K_2 are two alternating knots then $N(K_1 \uplus K_2) = N(K_1) + N(K_2)$.
Now we give some known definitions and results for our discussion.

Theorem 2.1. [4] *Let K_1, K_2 be any two knots. Then the relation $g(K_1 \uplus K_2) = g(K_1) + g(K_2)$ holds.*

Definition 2.2. [4] Let K be a knot. An overpass of K is a subarc of K that goes over at least one crossing but never goes under a crossing. A maximal overpass is an overpass that could not be made larger.

Definition 2.3. [9]The bridge index of D, where D is a knot projection of a knot K, is the number of maximal passes in D and the bridge index of K, denoted by bg(K) is the least bridge index of all regular projection of any knot that is equivalent to K.

Now we have the following theorem

Theorem 2.4. [2] *For any two knots K_1 and K_2, we have $bg(K_1 \uplus K_2) = bg(K_1) + bg(K_2)$.*

Definition 2.5. [5] Let K be a link. The braid number of a link K, denoted by br(K), is the least number of strings needed in a braid representation of K.

Now from the above definition, we have the following result due to Birman and Menasco.

Theorem 2.6. [5] *For any two knots K_1 and K_2, we have $br(K_1 \uplus K_2) = br(K_1) + br(K_2) - 1$.*

It is found that $bg(K) \leq br(K)$ almost every time in general, since a closed braid with n strings always has n bridges or less. For our discussion we state the following theorem given by S.Yamada.

THEOREM 2.7. [3] *Let K be a link and s(K) be the minimum number of Seifert circles over all possible regular projections of links that are equivalent to K. Then s(K) = br(K).*

Now we have two more theorems for our need.

THEOREM 2.8. [6] *The bridge index of a Torus knot T(p,q) is defined by the minimum of (p,q).*

THEOREM 2.9. [7] *Let T(p,q) be a (p,q) torus knot. Then g(T) = (p-1)(q-1)/2 and* $N(T) = min\{(p-1)q, (q-1)p\}$.

3. Crossing Number of Knots

In this section we show that there exists a homomorphism from the set $(S_d, \uplus)$ to $(\mathbf{N}, +)$, where S_d is a subset of S with an extra property. To do this we need the following definition followed by some lemmas and theorems.

DEFINITION 3.1. The decision of a link K, denoted by d_K is defined by $d_K = N(K) + 2 - br(K) - 2g(K) - n(K)$, where n(K) is the number of components of the link K, br(K) is the braid number of the link K.

THEOREM 3.2. *For any link K,* $d_K \geq 0$.

Proof. Let D be a minimum projection of a link K. Let s be the number Seifert circles in D and let g be the genus of the Seifert surface obtained by applying the Seifert's algorithm, where $g = [2 + c - s - n(K)]/2$, where c is the number of crossing in diagram D. Since D is a minimum projection, so $c = N(K)$. Then $N(K) + 2 = 2g + s + n(K)$. By the theorem 2.7, we can have $g \geq g(K), s \geq s(K) = br(K)$. So, $N(K) + 2 \geq 2g(K) + s(K) + n(K) \geq 2g(K) + br(K) + n(K)$. Then, $d = N(K) + 2 - br(K) - n(K) - 2g(K)$. From this we get, $d_K \geq 0$. □

From the above results and discussion we have the following Proposition.

PROPOSITION 3.3. *For any link K, if* $br(K) = s(K)$, *where* $br(K), s(K)$ *denote the braid number and the number of Seifert circles in K respectively, then* $d_K = 0$.

LEMMA 3.4. *The set S forms a monoid under the connected sum operation* $\uplus$, *where S is the set of all oriented knots.*

Proof. The set of all oriented knots with the connected sum operation of knots(links) forms a monoid, which can be illustrated by the following way.

i) If we consider two knots[4], K_1 and K_2 from S. Now applying the connected sum operation on K_1 and K_2, we find that $K_1 \uplus K_2$ is also an oriented knot which belong to the set S. Hence, closure property holds.

ii) Applying the connected sum operation on oriented knots the associativity of the addition follows trivially.

iii) Here the unknot is zero for this addition. So unknot is the zero element of S.

Consequently, $(S, \uplus)$ forms a monoid. □

COROLLARY 3.5. *The set S forms a commutative monoid under* $\uplus$.

The commutativeness follows from [5].

Let S_d be the set of all oriented knots in which the *decision* of each knot is zero. Then S_d must be a subset of S and we will show that it forms a commutative monoid under the connected sum operation of knots. To prove that we need the following lemma.

LEMMA 3.6. *Let* K_1 *and* K_2 *be two knots such that* $d_{K_1} = d_{K_2} = 0$. *Then* $N(K_1 \uplus K_2) = N(K_1) + N(K_2)$.

Proof. It follows from [8], $N(K_1 \uplus K_2) \leq N(K_1) + N(K_2)$.
So we need to show that $N(K_1 \uplus K_2) \geq N(K_1)+N(K_2)$. Here $n(K_1 \uplus K_2) = n(K_1) + n(K_2) - 1$. Again, $N(K_1) = 2g(K_1) + br(K_1) + n(K_1) - 2$ and $N(K_2) = 2g(K_2) + br(K_2) + n(K_2) - 2$.

$$\begin{aligned} N(K_1 \biguplus K_2) &\geq 2g(K_1 \biguplus K_2) + br(K_1 \biguplus K_2) + n(K_1 \biguplus K_2) - 2 \\ &= 2(g(K_1) + g(K_2)) + (br(K_1) + br(K_2) - 1) + (n(K_1) + n(K_2) - 1) - 2 \\ &= (2g(K_1) + br(K_1) + n(K_1) - 2) + (2g(K_2) + br(K_2) + n(K_2) - 2) \\ &= N(K_1) + N(K_2) \end{aligned}$$

□

LEMMA 3.7. *If* K_1 *and* K_2 *are two links such that* $d_{K_1} = d_{K_2} = 0$, *where* d_{K_1} *and* d_{K_2} *are decisions of the links* K_1 *and* K_2 *respectively, then the decision of* $K_1 \uplus K_2$ *is zero.*

Proof. Here, the decision of $K_1 \uplus K_2$ is given by

$$\begin{aligned} d_K &= N(K_1 \uplus K_2) - 2g(K_1 \uplus K_2) - br(K_1 \uplus K_2) - n(K_1 \uplus K_2) + 2 \\ &= N(K_1) + N(K_2) - 2(g(K_1) + g(K_2)) - \\ &\quad (br(K_1) + br(K_2) - 1) - (n(K_1) + n(K_2) - 1) + 2 \\ &= (N(K_1) - 2g(K_1) - br(K_1) - n(K_1) + 2) \\ &\quad + (N(K_2) - 2g(K_2) - br(K_2) - n(K_2) + 2) \\ &= d_{K_1} + d_{K_2} \\ &= 0. \end{aligned}$$

□

LEMMA 3.8. *Let T(p,q) be torus knot. Then* $d_T = 0$.

Proof. It follows from the theorem 2.8 and theorem 2.9. □

The immediate results follow from the Lemma 3.6 and the above Lemma 3.8 are given below.

REMARK 3.9. If K_1 and K_2 are two torus knots, then $N(K_1 \uplus K_2) = N(K_1) + N(K_2)$.

REMARK 3.10. If $T_1, T_2, ..., T_n$ are $n(\geq 2)$ torus knots(links), then $N(T_1 \uplus T_2 \uplus .... \uplus T_n) = N(T_1) + N(T_2) + ... + N(T_n)$.

Using the above lemmas, let us prove the following two main theorems.

THEOREM 3.11. *The set* S_d *forms a commutative monoid under the connected sum operation of knots.*

Proof. The set S_d is non empty, since the torus knot $T(p,q) \in S_d$, by Lemma 3.8. Now S_d is a subset of S, so the closure property and associativity follow from the Lemma 3.4 and the Lemma 3.7. Thus S_d forms a semigroup under the connected sum operation $\uplus$.

Here the unknot is zero for this addition, since the decision of the unknot is zero. So unknot is the zero element of S_d.
Consequently, $(S_d, \uplus)$ forms a monoid. Since S_d is a subset of S, so the commutativeness follows trivially, by Corollary 3.5. □

Now the immediate result follows:

PROPOSITION 3.12. S_d *is a commutative sub-monoid of* S *under* $\uplus$.

Proof. As S_d is a commutative monoid under the connected sum operation and S_d is a proper subset of S , so S_d is a commutative sub-monoid of S[by Corollary 3.5 and Theorem 3.11].

□

THEOREM 3.13. *Let $K_1, K_2 \in S_d$,where K_1, K_2 are two knots. Then there exists a homomorphism* $N : (S_d, \biguplus) \rightarrow (\mathbf{N}, +)$ *defined by*

$$N(K_1 \biguplus K_2) = N(K_1) + N(K_2).$$

The proof of the theorem follows from our previous results and discussion.

The family of alternating knots does not contained properly in S_d, to show this we consider the alternating knot 6_2. After counting its decision we found that $d(6_2) = 1 \neq 0$. Thus 6_2 does not belong to S_d. On the other hand, we show that, even the decision of a non alternating knot is zero. To show this fact we consider the knot 9_{38}, a knot with 9 crossing. This knot is a non alternating knot having decision zero. Thus we may state that.

REMARK 3.14. Decision of a knot is type independent and is not topological invariant under the connected sum operations of knots.

4. Conclusion

Thus we have proved that there exists a homomorphism of monoids between $(S_d, \uplus)$ to $(\mathbf{N}, +)$, where S_d is a subset of S with an extra condition and $\mathbf{N}$ is the monoid of non negative integers under usual addition.The results proved so far have been enable us for further study of crossing number of knots (links) in a different way.

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