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ON EQUIVALENT NORMS TO BLOCH NORM IN $\mathbb{C}^n$

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Abstract. For $f \in L^2(B, d\nu)$, $\| f \|_{BMO} = \widetilde{|f|^2}(z) - |\tilde{f}(z)|^2$. For f continuous on B, $\| f \|_{BO} = \sup\{w(f)(z) : z \in B\}$ where $w(f)(z) = \sup\{|f(z) - f(w)| : \beta(z, w) \le 1\}$. In this paper, we will show that if $f \in BMO$, then $\| f \|_{BO} \le M \| f \|_{BMO}$. We will also show that if $f \in BO$, then $\| f \|_{BMO} \le M \| f \|_{BO}^2$. A holomorphic function $f : B \to \mathbb{C}$ is called a Bloch function ($f \in \mathcal{B}$) if $\| f \|_{\mathcal{B}} = \sup_{z \in B} Qf(z) < \infty$. In this paper, we will show that if $f \in \mathcal{B}$, then $\| f \|_{BO} \le \| f \|_{\mathcal{B}}$. We will also show that if $f \in BMO$ and f is holomorphic, then $\| f \|_{\mathcal{B}}^2 \le M \| f \|_{BMO}$.

1. Introduction

Throughout this paper, $\mathbb{C}^n$ will be the Cartesian product of n copies of $\mathbb{C}$. For $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$ in $\mathbb{C}^n$, the inner product is defined by $< z, w >= \sum_{j=1}^n z_j \overline{w_j}$ and the norm by $|z|^2 =< z, z >$.

Let B be the open unit ball in the complex space $\mathbb{C}^n$. Let ν be the Lebesgue measure in $\mathbb{C}^n$ normalized by $\nu(B) = 1$. Let $L_a^2(B)$ be the Bergman space of holomorphic functions in $L^2(B, d\nu)$. Fix a point $z \in B$. Since the functional e_z given by $e_z(f) = f(z), f \in L_a^2(B)$, is continuous, there exists a function $K(\cdot, z) \in L_a^2(B)$ such that

$$f(z) = \int_B f(w)\overline{K(w, z)} d\nu(w)$$

by the Riesz representation theorem. $K(\cdot, z)$ is called the Bergman reproducing kernel in $L_a^2(B)$. If $z, w \in B$, then

$$K(z, w) = \frac{n!}{\pi^n} \frac{1}{(1 - \langle z, w \rangle)^{n+1}}$$

(See [8, Theorem 1.4.21]).

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The normalized(in $L_a^2(B)$) reproducing kernel is denoted by

$$k_z(\cdot) = K(z,z)^{-1/2} K(\cdot, z).$$

We define the Berezin transform of f in $L^2(B, d\nu)$ by

$$\tilde{f}(z) = \langle f k_z, k_z \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the usual $L^2(B, d\nu)$ inner product and $\| f \|^2 = \langle f, f \rangle$. Using the boundedness of the k_z, the Berezin transform extends to all $f \in L^1(B, d\nu)$ by the formula

$$\tilde{f}(z) = \frac{1}{K(z,z)} \int_B |K(z,w)|^2 f(w) d\nu(w).$$

Let $BC(B)$ denote the algebra of bounded continuous functions on B. For f in $L^2(B, d\nu)$, the quantity

$$\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2$$

is a continuous function on B. We denote $f \in BMO$ if $\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2$ is in $BC(B)$ as a function of z. For any subset S of B, we write

$$\| f \|_{BMO(S)} \equiv \sup\{\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2 : z \in S\}$$

and

$$\| f \|_{BMO} \equiv \| f \|_{BMO(B)} .$$

BMO in the Bergman metric was first exhibited in [2] and [3] where BMO was used to characterize the boundedness of Hankel operators on the Bergman spaces.

β is the Bergman metric on B(See [1],[4] and [6]). For f continuous on B, we define

$$w(f)(z) = \sup\{|f(z) - f(w)| : \beta(z,w) \leq 1\}.$$

$w(f)(z)$ is called the oscillation of f at z in the Bergman metric. We say f is of bounded oscillation($f \in BO(B)$) if $w(f)(z)$ is in $BC(B)$(as a function of z). We write

$$\| f \|_{BO} = \sup\{w(f)(z) : z \in B\}.$$

In Section 2, we will show that if $f \in BMO$, then $\| f \|_{BO} \leq M_1 \| f \|_{BMO}$ for some constant M_1. We will also show that if $f \in BO$, then $\| f \|_{BMO} \leq M_2 \| f \|_{BO}^2$ for some constant M_2.

For $z \in B, \xi \in \mathbb{C}^n$, set

$$b_B{}^2(z,\xi) = \frac{n+1}{(1-|z|^2)^2}[\,(1-|z|^2)|\xi|^2 + |<z,\xi>|^2\,]\ .$$

Let $H(B)$ be the space of all holomorphic functions in B. If $f \in H(B)$, where $H(B)$ is the set of holomorphic functions on B, then the quantity Q_f is defined by

$$Q_f(z) = \sup_{|\xi|=1} \frac{|\nabla f(z)\cdot\xi|}{b_B(z,\xi)}, \quad z \in B, \quad \xi \in \mathbb{C}^n\ ,$$

where $\nabla f(z) = \left(\frac{\partial f(z)}{\partial z_1}, \cdots, \frac{\partial f(z)}{\partial z_n}\right)$ is the holomorphic gradient of f(See [1],[5],[6],[7] and [10]). A holomorphic function $f : B \to \mathbb{C}$ is called a Bloch function ($f \in \mathcal{B}$) if

$$\| f \|_{\mathcal{B}} = \sup_{z\in B} Q_f(z) < \infty\ .$$

Bloch functions on bounded homogeneous domains were first studied in [6]. In [10], Timoney showed that the linear space of all holomorphic functions $f : B \to \mathbb{C}$ which satisfy

$$\| f \|_{\mathcal{B}} = \sup_{z\in B}(1-|z|^2)\ \| \nabla f(z) \| < \ \infty$$

is equivalent to the space $\mathcal{B}$ of Bloch functions on B.

The theory for the space $\mathcal{B}$ of Bloch functions on B was extended to the weighted Bloch space in [4],[5] and [7]. In Section 3, we will show that if

$f \in \mathcal{B}$, then $\| f \|_{BO} \leq \| f \|_{\mathcal{B}}$ and that if $f \in BMO$ and f is holomorphic, then $\| f \|_{\mathcal{B}}^2 \leq M \| f \|_{BMO}$ for some constant M.

2. Relationship between $\| f \|_{BMO}$ and $\| f \|_{BO}^2$

Let $a \in B$ and P_a be the orthogonal projection of $\mathbb{C}^n$ onto the subspace generated by a, which is given by $P_0 = 0$, and

$$P_a z = \frac{< z, a >}{< a, a >} a, \quad if \quad a \neq 0.$$

Let $Q_a = I - P_a$. Define φ_a on B by

$$\varphi_a(z) = \frac{a - P_a z - \sqrt{1 - |a|^2} Q_a z}{1 - < z, a >}.$$

φ_a belongs to the group $Aut(B)$ of holomorphic automorphisms of B and satisfies $\varphi_a(0) = a, \varphi_a(a) = 0$ and $\varphi_a(\varphi_a(z)) = z$(See [10, Theorem2.2.2]).

Theorem 1. *Let $z \in B$. Let $J_C \varphi_z$ be the determinant of the complex Jacobian of φ_z. Then*

$$|J_C \varphi_z(w)|^2 = |k_z(w)|^2.$$

Proof. See [9, Theorem 2.2.6]. □

Theorem 2. *For $f \in L^2(B, d\nu)$,*

$$\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2 = \frac{1}{2} \int_B \int_B |(f \circ \varphi_z)(u) - (f \circ \varphi_z)(w)|^2 d\nu(u) d\nu(w).$$

Proof.

$$
\begin{aligned}
&2(\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2) \\
&= \widetilde{|f|^2}(z) - \int_B f(u)|k_z(u)|^2 d\nu(u) \int_B \overline{f(w)}|k_z(w)|^2 d\nu(w) \\
&\qquad - \int_B \overline{f(u)}|k_z(u)|^2 d\nu(u) \int_B f(w)|k_z(w)|^2 d\nu(w) + \widetilde{|f|^2}(z) \\
&= \int_B \int_B (|f(u)|^2 - f(u)\overline{f(w)} - \overline{f(u)}f(w) \\
&\qquad + |f(w)|^2)|k_z(u)|^2|k_z(w)|^2 d\nu(u)d\nu(w) \\
&= \int_B \int_B |f(u) - f(w)|^2|k_z(u)|^2|k_z(w)|^2 d\nu(u)d\nu(w).
\end{aligned}
$$

By Theorem 1,

$$
\begin{aligned}
&\int_B \int_B |f(u) - f(w)|^2|k_z(u)|^2|k_z(w)|^2 d\nu(u)d\nu(w) \\
&= \int_B \int_B |f(u)\overline{f(u)} - f(u)\overline{f(w)} \\
&\qquad - \overline{f(u)}f(w) - f(w)\overline{f(w)}|^2|k_z(u)|^2|k_z(w)|^2 d\nu(u)d\nu(w) \\
&= \int_B \int_B \{f(\varphi_z(u))\overline{f(\varphi_z(u))} - f(\varphi_z(u))\overline{f(w)} \\
&\qquad - \overline{f(\varphi_z(u))}f(\varphi_z(w)) - f(\varphi_z(w))\overline{f(\varphi_z(w))}\} d\nu(u)d\nu(w) \\
&= \int_B \int_B |(f \circ \varphi_z)(u) - (f \circ \varphi_z)(w)|^2 d\nu(u)d\nu(w).
\end{aligned}
$$

□

Theorem 3. *The function $\beta(0, \cdot)$ is in $L^p(B, d\nu)$ for all $p > 0$.*

Proof. See [3, Theorem E]. □

Lemma 4. *For fixed $r > 0$ and continuous f such that*

$$\sup\{|f(z) - f(w)| : \beta(z, w) \leq r\} = c(r, f),$$

we have

$$|f(z) - f(w)| \leq c(f, r)[1 + r^{-1}\beta(z, w)]$$

for all $z, w \in B$.

Proof. See [3, Lemma 12]. □

Theorem 5. $\| f \|_{BMO} \leq M \| f \|_{BO}^2$ *for some constant* M.

Proof. By the proof of Theorem 2,

$$\begin{aligned}&\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2 \\ &= \frac{1}{2}\int_B \int_B |f(u) - f(w)|^2 |k_z(u)|^2 |k_z(w)|^2 d\nu(w) d\nu(w).\end{aligned}$$

Since, by Lemma 4,

$$|f(u) - f(w)| = \| f \|_{BO} (\beta(u, w) + 1),$$

$$\begin{aligned}&\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2 \\ &= \frac{1}{2} \| f \|_{BO}^2 \int_B \int_B (\beta(u, w) + 1)^2 |k_z(u)|^2 |k_z(w)|^2 d\nu(w) d\nu(w) \\ &= \frac{1}{2} \| f \|_{BO}^2 \int_B \int_B (\beta(\varphi_z(u), \varphi_z(w)) + 1)^2 d\nu(w) d\nu(w) \\ &\leq \frac{1}{2} \| f \|_{BO}^2 \int_B \int_B (\beta(u, w) + 1)^2 d\nu(w) d\nu(w) \\ &\leq M \| f \|_{BO}^2\end{aligned}$$

where the last inequality follows from Theorem 3. This implies that

$$\| f \|_{BMO} = \sup\{\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2 : z \in B\} \leq M \| f \|_{BO}^2 .$$

□

Theorem 6. *If* $f \in BMO$, *then*

$$|\tilde{f}(a) - \tilde{f}(b)| \leq M \| f \|_{BMO} \, \beta(a, b)$$

for some constant M.

Proof. See [3, Corollary 1]. □

Theorem 7. *If $f \in BMO$ is holomorphic, then*

$$\| f \|_{BO} \leq M \| f \|_{BMO} .$$

for some constant M.

Proof. Since f is holomorphic, $\tilde{f} = f$. By Theorem 6,

$$\| f \|_{BO} \leq M \| f \|_{BMO} .$$

□

3. Equivalent norms to $\| f \|_{\mathcal{B}}$

Theorem 8. *For $f \in L^2_a(B)$, if γ is a geodesic of length $\beta(z, w)$ joining z to w, then*

$$|f(z) - f(w)| \leq \{\sup_{a \in \gamma} Q_f(a)\}\beta(z, w).$$

Proof. See [10, I]. □

Theorem 9. *If $f \in \mathcal{B}$, then*

$$\| f \|_{BO} \leq \| f \|_{\mathcal{B}} .$$

Proof. This follows from Theorem 8. □

Lemma 10. *If f is in $L^2_a(B)$,*

$$|\nabla(f \circ \varphi_a)(0)|^2 \leq n(n+1)^2 \| f(a) - f \circ \varphi_a \|^2_{L^2} .$$

Proof. Let $g \in L^2_a(B)$. Then

$$\begin{aligned} g(z) &= \int_B g(w) K(z, w) d\nu(w) \\ &= \frac{n!}{\pi^n} \int_B \frac{g(w)}{(1 - \langle z, w \rangle)^{n+1}} d\nu(w). \end{aligned}$$

$$\frac{\partial g}{\partial z_i}(z) = (n+1)\int_B \frac{\overline{w_i} g(w)}{(1-\langle z,w\rangle)^{n+2}} d\nu(w).$$

$$\left|\frac{\partial g}{\partial z_i}(0)\right| \le (n+1)\int_B |g(w)| d\nu(w).$$

Replacing g by $g - g(0)$ yields

$$\begin{aligned}\left|\frac{\partial g}{\partial z_i}(0)\right| &\le (n+1)\int_B |g(w)-g(0)| d\nu(w)\\ &\le (n+1)\left(\int_B d\nu(w)\right)^{1/2}\left(\int_B |g(w)-g(0)|^2 d\nu(w)\right)^{1/2}\\ &= (n+1)\left(\int_B |g(w)-g(0)|^2 d\nu(w)\right)^{1/2}.\end{aligned}$$

$$|\nabla g(0)|^2 = \sum_{i=1}^n \left|\frac{\partial g}{\partial z_i}(0)\right|^2 \le n(n+1)^2 \int_B |g(w)-g(0)|^2 d\nu(w).$$

Replacing g by $f \circ \varphi_a$, we have

$$|\nabla(f\circ\varphi_a)(0)|^2 \le n(n+1)^2 \parallel f(a) - f\circ\varphi_a \parallel_{L^2}^2 .$$

□

Theorem 11. *For all φ_a in $Aut(B)$,*

$$Q_f(\varphi_a(z)) = Q_{f\circ\varphi_a}(z).$$

Proof. See [10, Remark 4.4].

□

Theorem 12. *If $f \in BMO$ is holomorphic,*

$$\parallel f \parallel_{\mathcal{B}}^2 \le M \parallel f \parallel_{BMO} .$$

Proof. By Theorem 11,

$$Q_f(a) = Q_f(\varphi_a(0)) = Q_{f\circ\varphi_a}(0).$$

From the definition of $Q_f(z)$ and Lemma 10,

$$Q_f(a) \le A|\nabla(f\circ\varphi_a)(0)| \le B \parallel f(a) - f\circ\varphi_a \parallel_{L^2}$$

for some constants A and B. Since

$$\begin{aligned}
&\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2\\
&= \widetilde{|f|^2}(z) - f(z)\overline{f(z)} - \overline{f(z)}f(z) - |\tilde{f}(z)|^2\\
&= \int_B |(f\circ\varphi_z)(w)|^2 - (f\circ\varphi_z)(w)\overline{f(z)}\\
&\qquad - \overline{(f\circ\varphi_z)(w)}f(z) + |f(z)|^2)d\nu(w)\\
&= \int_B |f\circ\varphi_z(w) - f(z)|^2 d\nu(w),
\end{aligned}$$

$$Q_f(z)^2 \le \widetilde{|f|^2}(z) - |\tilde{f}(z)|^2.$$

This implies that

$$\parallel f \parallel_{\mathcal{B}}^2 = \sup_{z\in B} Q_f(z)^2 \le \sup\{\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2 : z \in B\} = M \parallel f \parallel_{BMO}.$$

□

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