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REDEI MATRIX IN FUNCTION FIELDS

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Abstract. Let K be a finite cyclic extension of $k = \mathbb{F}_q(T)$ of prime degree ℓ. Let $\widetilde{Cl}_{K,\ell}$ be the Sylow ℓ-subgroup of the ideal class group $\widetilde{Cl}_K$ of $\mathcal{O}_K$. The structure of $\widetilde{Cl}_{K,\ell}$ as $\mathbb{Z}_\ell[G]/\langle N_G\rangle$-module is determined the dimensions

$$\lambda_i := \dim_{\mathbb{F}_\ell}(\widetilde{Cl}_{K,\ell}^{(\sigma-1)^{i-1}}/\widetilde{Cl}_{K,\ell}^{(\sigma-1)^i})$$

for $i \geq 1$. In this paper we investigate the dimensions λ_1 and λ_2.

1. Introduction

Let k be the rational function field over the finite field $\mathbb{F}_q$ of q elements. Take a generator, say T, of k over $\mathbb{F}_q$. Then $k = \mathbb{F}_q(T)$. Let $\mathbb{A} = \mathbb{F}_q[T]$ and $\mathbb{A}^+$ be the set of all monic polynomials in $\mathbb{A}$. Let ∞ be the place of k associated to $(1/T)$ and k_∞ the completion of k at ∞. Set $\widetilde{C} := k_\infty(\sqrt[q-1]{-1/T})$, which is the maximal totally tamely ramified extension of k_∞. We denote by $\tilde{k}^{ab}$ the maximal abelian extension of k inside $\widetilde{C}$. Then $\tilde{k}^{ab} = \bigcup_{N\in\mathbb{A}^+} k_N$, where k_N is the cyclotomic function field of conductor N. Any finite abelian extension K of k inside $\tilde{k}^{ab}$ is contained in k_N for some $N \in \mathbb{A}^+$. By the conductor of K we mean the monic polynomial $N \in \mathbb{A}^+$ of the smallest degree such that K is contained in k_N. As in classical case, such finite abelian extensions of k can be described by Dirichlet characters of $\mathbb{A}$ ([1, §1]) and its narrow genus field can be easily obtained. Throughout the paper, by a finite abelian extension of k we always assume that it is contained in $\tilde{k}^{ab}$.

Fix a prime number ℓ. Consider a finite cyclic extension K/k of degree ℓ with Galois group G. Let $\mathcal{M}$ be a finite abelian ℓ-group with a natural G-action and annihilated by the norm $N_G = \sum_{g\in G} g$. Then it

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is a finite module over the discrete valuation ring $\mathbb{Z}_\ell[G]/\langle N_G\rangle$. Thus its Galois module structure is given by the dimensions

$$\lambda_i := \dim_{\mathbb{F}_\ell}\left(\mathcal{M}^{(\sigma-1)^{i-1}}/\mathcal{M}^{(\sigma-1)^i}\right) \text{ for } i \geq 1,$$

where the quotient is a $\mathbb{F}_\ell$-vector space in a natural way and σ is a generator of G. Let $\mathcal{O}_K$ be the integral closure of $\mathbb{A}$ in K. Recently Wittmann [5] has investigated the dimensions λ_1 and λ_2 when $\mathcal{M}$ is the Sylow ℓ-subgroup $\mathcal{C}l_{K,\ell}$ of the ideal class group $\mathcal{C}l_K$ of $\mathcal{O}_K$ and ℓ is different from the characteristic of k. The aim of this paper is to investigate the dimensions λ_1 and λ_2 when $\mathcal{M}$ is the Sylow ℓ-subgroup $\widetilde{\mathcal{C}l}_{K,\ell}$ of the narrow ideal class group $\widetilde{\mathcal{C}l}_K$ of $\mathcal{O}_K$ and ℓ is an arbitrary prime number. In the number field case the dimensions λ_i have been investigated first by Redei for $\ell = 2$ in [4], and then for arbitrary ℓ by Gras [3].

2. Narrow genus field and dimension λ_1

We fix a sign function $sgn : k_\infty^* \to \mathbb{F}_q^*$ by letting $sgn(1/T) = 1$. Let K be a finite abelian extension of k and $S_\infty(K)$ be the set of places of K lying above ∞. For each $v \in S_\infty(K)$, the completion K_v of K at v is a finite extension of k_∞ in $\widetilde{C}$. The sign map $sgn_v : L_v^* \to \mathbb{F}_q^*$ is defined by $sgn_v(x) = sgn(N_v(x))$, where N_v is the norm map from K_v to k_∞. An element $x \in K$ is called *totally positive* if $sgn_v(x) = 1$ for every $v \in S_\infty(K)$. Let $\mathcal{I}_K$ be the group of nonzero fractional ideals of $\mathcal{O}_K$ and $\widetilde{\mathcal{P}}_K$ its subgroup of principal ideals generated by totally positive elements. The *narrow ideal class group* $\widetilde{\mathcal{C}l}_K$ of $\mathcal{O}_K$ is defined to be the group $\mathcal{I}_K/\widetilde{\mathcal{P}}_K$. The *narrow Hilbert class field* $\widetilde{H}_K$ of K relative to $S_\infty(K)$ is defined as the maximal abelian extension of K in $\widetilde{C}$ which is unramified outside $S_\infty(K)$. It is well known that $Gal(\widetilde{H}_K/K)$ is isomorphic to $\widetilde{\mathcal{C}l}_K$ via Artin automorphism. The *narrow genus field* $\widetilde{G}_{K/k}$ is defined to be the maximal extension of K in $\widetilde{H}_K$ which is the composite of K and some abelian extension of k. The Galois group $Gal(\widetilde{G}_{K/k}/K)$ and the degree $[\widetilde{G}_{K/k} : K]$ are called the *narrow genus group* and *narrow genus number* of K/k, respectively. For more details on genus theory for function fields we refer to Bae and Koo's paper [2].

Now we consider a finite cyclic extension K/k of degree ℓ with Galois group G. Let σ be a generator of G. Then $Gal(\widetilde{G}_{K/k}/K)$ is isomorphic to $\widetilde{\mathcal{C}l}_K/\widetilde{\mathcal{C}l}_K^{\sigma-1}$ via Artin automorphism.

Lemma 2.1. *Let K/k be as above. Then* $\widetilde{\mathcal{C}l}_{K,\ell}/\widetilde{\mathcal{C}l}_{K,\ell}^{\sigma-1} \simeq \widetilde{\mathcal{C}l}_K/\widetilde{\mathcal{C}l}_K^{\sigma-1}$.

Proof. At first, we note that $\widetilde{\mathcal{C}l}_K^G = \widetilde{\mathcal{C}l}_{K,\ell}^G$. From the exact sequences $1 \to \widetilde{\mathcal{C}l}_K^G \to \widetilde{\mathcal{C}l}_K \to \widetilde{\mathcal{C}l}_K^{\sigma-1} \to 1$ and $1 \to \widetilde{\mathcal{C}l}_{K,\ell}^G \to \widetilde{\mathcal{C}l}_{K,\ell} \to \widetilde{\mathcal{C}l}_{K,\ell}^{\sigma-1} \to 1$, we have $|\widetilde{\mathcal{C}l}_K/\widetilde{\mathcal{C}l}_K^{\sigma-1}| = |\widetilde{\mathcal{C}l}_K^G| = |\widetilde{\mathcal{C}l}_{K,\ell}^G| = |\widetilde{\mathcal{C}l}_{K,\ell}/\widetilde{\mathcal{C}l}_{K,\ell}^{\sigma-1}|$. The inclusion $\widetilde{\mathcal{C}l}_{K,\ell} \hookrightarrow \widetilde{\mathcal{C}l}_K$ induces a homomorphism $\widetilde{\mathcal{C}l}_{K,\ell} \to \widetilde{\mathcal{C}l}_K/\widetilde{\mathcal{C}l}_K^{\sigma-1}$, whose kernel is $\widetilde{\mathcal{C}l}_{K,\ell} \cap \widetilde{\mathcal{C}l}_K^{\sigma-1} = \widetilde{\mathcal{C}l}_{K,\ell}^{\sigma-1}$. Thus we have an injective homomorphism $\widetilde{\mathcal{C}l}_{K,\ell}/\widetilde{\mathcal{C}l}_{K,\ell}^{\sigma-1} \to \widetilde{\mathcal{C}l}_K/\widetilde{\mathcal{C}l}_K^{\sigma-1}$, which must be an isomorphism because $|\widetilde{\mathcal{C}l}_{K,\ell}/\widetilde{\mathcal{C}l}_{K,\ell}^{\sigma-1}| = |\widetilde{\mathcal{C}l}_K/\widetilde{\mathcal{C}l}_K^{\sigma-1}|$. □

Recall that $\lambda_i := \dim_{\mathbb{F}_\ell}\big(\widetilde{\mathcal{C}l}_{K,\ell}^{(\sigma-1)^{i-1}}/\widetilde{\mathcal{C}l}_{K,\ell}^{(\sigma-1)^i}\big)$ for $i \geq 1$. It is known [2, Theorem 3.10] that $Gal(\widetilde{G}_{K/k}/K)$ is a finite elementary abelian ℓ-group of rank $t-1$, where t is the number of finite primes of k ramifying in K. Thus we have

Proposition 2.2. $\lambda_1 = t-1$.

Let χ be a Dirichlet character which is a generator of the character group X_G of G. If $F_\chi = \prod_{i=1}^t P_i^{e_i}$ is the prime factorization of the conductor F_χ of χ, then $\chi = \prod_{i=1}^t \chi_{P_i}$, where χ_{P_i} is a Dirichlet character of degree ℓ with the conductor $P_i^{e_i}$. We denote by $k(\chi_{P_i})$ the abelian extension of k corresponding to $\langle\chi_{P_i}\rangle$. Then we have

Proposition 2.3. $\widetilde{G}_{K/k} = \prod_{i=1}^t k(\chi_{P_i})$.

Proof. Let $\widetilde{K} = \prod_{i=1}^t k(\chi_{P_i})$, which is the finite abelian extension of k corresponding to $\widetilde{X} = \prod_{i=1}^t\langle\chi_{P_i}\rangle$. Since $X_G \subseteq \widetilde{X}$, we have $K \subseteq \widetilde{K}$ and so $\widetilde{H}_K \subseteq \widetilde{H}_{\widetilde{K}}$. Thus $\widetilde{G}_{K/k} \subseteq \widetilde{G}_{\widetilde{K}/k} = \widetilde{K}$. Here the equality follows from Theorem 3.9 in [2]. But $[\widetilde{G}_{K/k} : K] = \ell^{t-1} = [\widetilde{K} : K]$. Thus we have $\widetilde{G}_{K/k} = \widetilde{K}$. □

3. Redei matrix and dimension λ_2

We continue the notations in section 2. In this section we investigate the dimension λ_2.

Lemma 3.1. $\widetilde{\mathcal{C}l}_K^G = \mathcal{I}_K^G\widetilde{\mathcal{P}}_K/\widetilde{\mathcal{P}}_K$.

Proof. Let $\alpha \in \widetilde{Cl}_K^G$. Then α is represented by a fractional ideal $\mathfrak{a}$ such that $\mathfrak{a}^\sigma = (x)\mathfrak{a}$ for some totally positive element $x \in K$ with $N_{K/k}(x) = 1$. Thus $x = y/y^\sigma$ for some $y \in K^*$ by Hilbert's Theorem 90. We may assume that y is totally positive. Since $((y)\mathfrak{a})^\sigma = (y)\mathfrak{a}$, we have $(y)\mathfrak{a} \in \mathcal{I}_K^G$. Thus $\alpha \in \mathcal{I}_K^G\widetilde{\mathcal{P}}_K/\widetilde{\mathcal{P}}_K$. Therefore $\widetilde{Cl}_K^G \subseteq \mathcal{I}_K^G\widetilde{\mathcal{P}}_K/\widetilde{\mathcal{P}}_K$. The converse is obvious. □

Let $P_1, P_2, \ldots, P_t$ be the finite primes of k ramifying in K. Let $\mathfrak{p}_i$ be the prime ideal of K lying above P_i $(1 \le i \le t)$. Then we have

COROLLARY 3.2. *$\widetilde{Cl}_K^G = \langle \alpha_1, \alpha_2, \ldots, \alpha_t \rangle$, where α_i is the class in $\widetilde{Cl}_K^G$ represented by $\mathfrak{p}_i$ $(1 \le i \le t)$.*

Proof. For any finite prime P of k, let e_P be the ramification index of P in K. Then $\mathcal{I}_K^G$ is a free abelian group with a basis $\{(P\mathcal{O}_K)^{1/e_P} : P \text{ is a finite prime of } k\}$. Since $P_i\mathcal{O}_K = \mathfrak{p}_i^\ell$ with $e_{P_i} = \ell$ $(1 \le i \le t)$ and $P\mathcal{O}_K \in \widetilde{\mathcal{P}}_K$ with $e_P = 1$ for $P \ne P_i$, $\widetilde{Cl}_K^G$ is generated by $\{\alpha_i : 1 \le i \le t\}$ by Lemma 3.1. □

LEMMA 3.3. *Let $\phi : \widetilde{Cl}_{K,\ell}^G \to \widetilde{Cl}_{K,\ell}/\widetilde{Cl}_{K,\ell}^{\sigma-1}$ be the natural homomorphism induces by the inclusion $\widetilde{Cl}_{K,\ell}^G \hookrightarrow \widetilde{Cl}_{K,\ell}$. Then we have $\lambda_2 = \dim_{\mathbb{F}_\ell}(\mathrm{Ker}(\phi))$.*

Proof. At first, we note that $\mathrm{Ker}(\phi) = \widetilde{Cl}_{K,\ell}^G \cap \widetilde{Cl}_{K,\ell}^{\sigma-1}$. From the exact sequence $1 \to \widetilde{Cl}_{K,\ell}^G \to \widetilde{Cl}_{K,\ell} \to \widetilde{Cl}_{K,\ell}^{\sigma-1} \to 1$, we get the following exact sequence

$$(1) \qquad 1 \to \widetilde{Cl}_{K,\ell}^G\widetilde{Cl}_{K,\ell}^{\sigma-1}/\widetilde{Cl}_{K,\ell}^{\sigma-1} \to \widetilde{Cl}_{K,\ell}/\widetilde{Cl}_{K,\ell}^{\sigma-1} \to \widetilde{Cl}_{K,\ell}^{\sigma-1}/\widetilde{Cl}_{K,\ell}^{(\sigma-1)^2} \to 1.$$

Since $\widetilde{Cl}_{K,\ell}^G\widetilde{Cl}_{K,\ell}^{\sigma-1}/\widetilde{Cl}_{K,\ell}^{\sigma-1} \simeq \widetilde{Cl}_{K,\ell}^G/\widetilde{Cl}_{K,\ell}^G \cap \widetilde{Cl}_{K,\ell}^{\sigma-1}$, (1) induces the following exact sequence

$$1 \to \widetilde{Cl}_{K,\ell}^G \cap \widetilde{Cl}_{K,\ell}^{\sigma-1} \to \widetilde{Cl}_{K,\ell}^G \to \widetilde{Cl}_{K,\ell}/\widetilde{Cl}_{K,\ell}^{\sigma-1} \to \widetilde{Cl}_{K,\ell}^{\sigma-1}/\widetilde{Cl}_{K,\ell}^{(\sigma-1)^2} \to 1,$$

from where we get $\lambda_2 = \dim_{\mathbb{F}_\ell}(\widetilde{Cl}_{K,\ell}^G \cap \widetilde{Cl}_{K,\ell}^{\sigma-1})$. □

Let $\{\mathbf{e}_i : 1 \le i \le t\}$ be the standard basis of the $\mathbb{F}_\ell$-vector space $\mathbb{F}_\ell^t$. We define $\rho : \mathbb{F}_\ell^t \to \widetilde{Cl}_K^G$ by $\rho(\mathbf{e}_i) = \alpha_i$ for $1 \le i \le t$. Then we have

LEMMA 3.4. *ρ is a surjective homomorphism with $\dim_{\mathbb{F}_\ell}(\mathrm{Ker}(\rho)) = 1$.*

Proof. By Corollary 3.2, ρ is a surjective homomorphism. Since $\dim_{\mathbb{F}_\ell}(\widetilde{\mathcal{C}l}_K^G) = \dim_{\mathbb{F}_\ell}(\widetilde{\mathcal{C}l}_K/\widetilde{\mathcal{C}l}_K^{\sigma-1}) = t-1$, we have $\dim_{\mathbb{F}_\ell}(\mathrm{Ker}(\rho)) = t - \dim_{\mathbb{F}_\ell}(\widetilde{\mathcal{C}l}_K^G) = 1$. □

Since χ and χ_{P_i} are characters of degree ℓ, we may regard them as characters with values in $\mathbb{F}_\ell$. Then $\chi = \sum_{i=1}^t \chi_{P_i}$. Let $R = (a_{ij})_{1\le i,j\le t}$ be the Redei matrix of K defined by $a_{ij} = \chi_{P_i}(P_j)$ if $i \neq j$ and $\sum_{i=1}^t a_{ij} = 0$ in $\mathbb{F}_\ell$.

THEOREM 3.5. $\lambda_2 = t - 1 - \mathrm{rank}_{\mathbb{F}_\ell}(R)$.

Proof. Each character χ_{P_i} induces an isomorphism $Gal(K(\chi_{P_i})/k) \overset{\sim}{\to} \mathbb{F}_\ell$. We denote it also by χ_{P_i}. They can be combined into an isomorphism $\bigoplus_{i=1}^t \chi_{P_i} : Gal(\widetilde{G}_{K/k}/k) \overset{\sim}{\to} \mathbb{F}_\ell^t$. The Redei map $R : \mathbb{F}_\ell^t \to \mathbb{F}_\ell^t$ is defined as the composite map

$$R : \mathbb{F}_\ell^t \overset{\rho}{\to} \widetilde{\mathcal{C}l}_K^G \overset{\phi}{\to} \widetilde{\mathcal{C}l}_K/\widetilde{\mathcal{C}l}_K^{\sigma-1} \overset{\sim}{\to} Gal(\widetilde{G}_{K/k}/K) \hookrightarrow Gal(\widetilde{G}_{K/k}/k) \overset{\bigoplus \chi_{P_i}}{\longrightarrow} \mathbb{F}_\ell^t.$$

Since $\dim_{\mathbb{F}_\ell}(\mathrm{Ker}(\rho)) = 1$, we have

$$\lambda_2 = \dim_{\mathbb{F}_\ell}(\mathrm{Ker}(R)) - 1 = t - 1 - \mathrm{rank}_{\mathbb{F}_\ell}(R).$$

The image of a basis vector $\mathbf{e}_j$ is the Artin symbol of $\mathfrak{p}_j$ in $Gal(\widetilde{G}_{K/k}/K)$. If $i \neq j$, the restriction of this symbol to $Gal(k(\chi_{P_i})/k)$ is the Artin symbol of P_j, and this is mapped to $a_{ij} = \chi_{P_i}(P_j)$ by χ_{P_i}. For any $\mu \in Gal(\widetilde{G}_{K/k}/K)$, let μ_i be the restriction of μ to $Gal(K(\chi_{P_i})/k)$. Then $\bigoplus_{i=1}^t \chi_{P_i}(\mu) = (\chi_{P_i}(\mu_i)) \in \mathbb{F}_\ell^t$. Since μ is the identity on K, we have $\sum_{i=1}^t \chi_{P_i}(\mu_i) = \chi(\mu) = 0$. Thus $\bigoplus_{i=1}^t \chi_{P_i}$ maps $Gal(\widetilde{G}_{K/k}/K)$ into the hyperplane $\{(a_i)_i \in \mathbb{F}_\ell^t : \sum_{i=1}^t a_i = 0\}$. The desired identity $\sum_{i=1}^t a_{ij} = 0$ follows immediately. □

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