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UNION OF INTUITIONISTIC FUZZY SUBGROUPS

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Abstract

We study the conditions under which a given intuitionistic fuzzy subgroup of a given group can or can not be realized as a union of two proper intuitionistic fuzzy subgroups. Moreover, we provide a simple necessary and sufficient condition for the union of an arbitrary family of intuitionistic fuzzy subgroups to be an intuitionistic fuzzy subgroup. Also we formulate the concept of intuitionistic fuzzy subgroup generated by a given intuitionistic fuzzy set by level subgroups. Furthermore we give characterizations of intuitionistic fuzzy conjugate subgroups and intuitionistic fuzzy characteristic subgroups by their level subgroups. Also we investigate the level subgroups of the homomorphic image of a given intuitionistic fuzzy subgroup.

Key words : intuitionistic fuzzy subgroup, level subgroup, intuitionistic fuzzy conjugate subgroup, intuitionistic fuzzy characteristic subgroup

0. Introduction

In 1965, Zadeh[21] introduced the notion of a fuzzy set. Since then it has been a tremendous interest in the subject due to diverse applications ranging from engineering and computer science to social behavior studies. In particular, several researchers [3,7-9,19,20] applied the concept of a fuzzy set to group theroy.

In 1986, Atanassov[1] introduced the concept of intuitionistic fuzzy sets as the generalization of fuzzy sets. After that time, Çoker and his colleagues[5,6,10], Lee and Lee [18], and Hur and his colleagues [13] applied the notion of intuitionistic fuzzy sets to topology. In particular, Hur and his colleagues [15] applied one to topological group. Moreover, many researchers [2,4,11,12,14,16] applied the concept of intuitionistic fuzzy sets to algebra.

In this paper, we study the conditions under which a given intuitionistic fuzzy subgroup of a given group can or can not be realized as a union of two intuitionistic fuzzy proper subgroups. Moreover, we provide a simple necessary and sufficient condition for the union of an arbitrary family of intuitionistic fuzzy subgroups to be an intuitionistic fuzzy subgroup. Also we formulate the concept of intuitionistic fuzzy subgroup generated by a given intuitionistic fuzzy set by level subgroups. Furthermore we give characterizations of intuitionistic fuzzy conjugate subgroups and intuitionistic fuzzy characteristic subgroups by their level subgroups. Also we investigate the level subgroups of the homomorphic image of a given intuitionistic fuzzy subgroup.

1. Preliminaries

We will list some concepts and results needed in the later sections.

For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0, 1]$ as I.

Definition 1.1[1,5]. Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to I \times I$ is called an *intuitionistic fuzzy set* (in short, *IFS*) on X if $\mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$, where the mapping $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x \in X$ to A, respectively. In particular, $0_\sim$ and $1_\sim$

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denote the *intuitionistic fuzzy empty set* and the *intuitionistic fuzzy whole set* in X defined by $0_\sim(x) = (0,1)$ and $1_\sim(x) = (1,0)$ for each $x \in X$, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definitions 1.2[1]. Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs on X. Then

(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
(2) $A = B$ iff $A \subset B$ and $B \subset A$.
(3) $A^c = (\nu_A, \mu_A)$.
(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.
(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.
(6) $[\,]A = (\mu_A, 1-\mu_A)$, $<> A = (1-\nu_A, \nu_A)$.

Definition 1.3[5]. Let $\{A_i\}_{i\in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then

(a) $\bigcap A_i = (\bigwedge \mu_{A_i}, \bigvee \nu_{A_i})$.
(b) $\bigcup A_i = (\bigvee \mu_{A_i}, \bigwedge \nu_{A_i})$.

Definition 1.4[5]. Let X and Y be nonempty sets and let $f : X \to Y$ be a mapping. Let $A = (\mu_A, \nu_A)$ be an IFS in X and $B = (\mu_B, \nu_B)$ be an IFS in Y. Then

(a) the *preimage* of B under f, denoted by $f^{-1}(B)$, is the IFS in X defined by :

$$f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\nu_B)),$$

where $f^{-1}(\mu_B) = \mu_B \circ f$.

(b) the *image* of A under, denoted by $f(A)$, is the IFS in Y defined by :

$$f(A) = (f(\mu_A), f(\nu_A)),$$

where for each $y \in Y$,

$$f(\mu_A)(y) = \begin{cases} \bigvee_{x\in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{if } f^{-1}(y) = \emptyset, \end{cases}$$

$$f(\nu_A)(y) = \begin{cases} \bigwedge_{x\in f^{-1}(y)} \nu_A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 1 & \text{if } f^{-1}(y) = \emptyset. \end{cases}$$

Result 1.A[5, Corollary 2.10]. Let $A, A_i (i \in J)$ be IFSs in X, let $B, B_j (j \in K)$ be IFSs in Y and let $f : X \to Y$ be a mapping. Then

(1) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$.
(2) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$.
(3) $A \subset f^{-1}(f(A))$.
If f is injective, then $A = f^{-1}(f(A))$.
(4) $f(f^{-1}(B)) \subset B$.
If f is surjective, then $f(f^{-1}(B)) = B$.
(5) $f^{-1}(\bigcup B_j)) = \bigcup f^{-1}(B_j)$.
(6) $f^{-1}(\bigcap B_j)) = \bigcap f^{-1}(B_j)$.
(7) $f(\bigcup A_i)) = \bigcup f(A_i)$.
(8) $f(\bigcap A_i)) \subset \bigcap f(A_i)$.
If f is injective, then $f(\bigcap A_i)) = \bigcap f(A_i)$.
(9) $f(1_\sim) = 1_\sim$, if f is surjective and $f(0_\sim) = 0_\sim$.
(10) $f^{-1}(1_\sim) = 1_\sim$ and $f^{-1}(0_\sim) = 0_\sim$.
(11) $[f(A)]^c \subset f(A^c)$, if f is surjective.
(12) $f^{-1}(B^c) = [f^{-1}(B)]^c$.

Definition 1.5[12]. Let $(G, \cdot)$ be a groupoid and let $A \in$ IFS(X). Then A is called an *intuitionistic fuzzy subgroupoid* (in short, IFGP) of G if for any $x, y \in G$, $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$.

Definition 1.6[11]. Let G be a group and let $A \in$ IFS(G). Then A is called an *intuitionistic fuzzy subgroup* (in short, IFG) of G if it satisfies the following conditions :

(i) A is an IFGP of G.
(ii) $\mu_A(x^{-1}) \geq \mu_A(x)$ and $\nu_A(x^{-1}) \leq \nu_A(x)$ for each $x \in G$.

We will denote the set of all IFGs of G as IFGG.

Result 1.B[11, Proposition 2.13]. Let $f : G \to G'$ be a group homomorphism. If $B \in$ IFG(G'), then $f^{-1}(B) \in$ IFG(G).

Result 1.C[16, Proposition 2.20]. Let $f : G \to G'$ be a group homomorphism. If $A \in$ IFG(G), then $f(A) \in$ IFG(G').

Definition 1.7[12]. Let A be an IFS in a set X and let $\lambda, \mu \in I$ with $\lambda + \mu \leq 1$. Then the set $A^{(\lambda,\mu)} = \{x \in X : \mu_A(x) \geq \lambda \text{ and } \nu_A(x) \leq \mu\}$ is called a (λ,μ)*-level subset* of A.

Result 1.D[11, Proposition 2.18 and Proposition 2.19]. Let A be an IFS in a group G. Then A is an IFG of G if and only if $A^{(\lambda,\mu)}$ is a subgroup of G for each $(\lambda,\mu) \in$ ImA.

Definition 1.8[11,17]. Let A be an IFG of a group G and let $(\lambda,\mu) \in$ Im(A). Then the subgroup $A^{(\lambda,\mu)}$ is called a (λ,μ)*-level subgroup* of A.

Definition 1.9[12,16]. Let A be an IFS in a groupoid G. Then A is said to have *the sup property* if for any $T \in P(G)$, there exists a $t_0 \in T$ such that $A(t_0) = \bigcup_{t\in T} A(t)$, i.e., $A(t_0) = (\bigvee_{t\in T} \mu_A(t), \bigwedge_{t\in T} \nu_A(t))$, where $P(G)$ denotes the power set of G.

2. Union of intuitionistic fuzzy subgroups

It is well known in classical group theory that a group cannot be realized as a union of two proper subgroups. Hur and his colleagues in [16] tried to establish the intuitionistic fuzzy analog of the above result in the form of the following :

Result 2.A [11, Proposition 2.10]. A group G cannot be the union of two proper intuitionistic fuzzy subgroups.

A generalization of the above problem can be formulated as follows :

Is it possible for a intuitionistic fuzzy subgroup to be realized as union of two proper intuitionistic fuzzy proper subgroups such that none is contained in the other?

In this section, we shall demonstrate that the answer of the above question depends on the image set of the intuitionistic fuzzy subgroup under consideration.

It is interesting to note that if the image set of the given intuitionistic fuzzy subgroup contains at least two nonzero numbers in $I \times I$, then the intuitionistic fuzzy subgroup can always be realized as a union of two proper intuitionistic fuzzy subgroups, such that none is contained in the other ; a result contrary to the corresponding result of group theory. However, in case the image set consists of

$\{(0,1),(t,s)\}$,

where$(t,s) \in (0,1] \times [0,1)$such that $t+s \leq 1$,

then the result of the group theory stated above can be successfully extended to the intuitionistic fuzzy setting.

Definition 2.1. Let G be a group. An IFG A of G is said to be *proper* if A is not constant on G, i.e., ImA has at least two elements.

Lemma 2.2. Let G be a group and let $A \in$ IFG(G). If for any $x, y \in G$, $\mu_A(x) < \mu_A(y)$ and $\nu_A(x) > \nu_A(y)$, then $A(xy) = A(x) = A(yx)$.

Proof. Since $A \in$ IFG(G),

$$\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) = \mu_A(x)$$

and

$$\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y) = \nu_A(x).$$

On the other hood,

$$\mu_A(x) = \mu_A(xyy^{-1}) \geq \mu_A(xy) \wedge \mu_A(y^{-1}) = \mu_A(xy) \wedge \mu_A(y)$$

and

$$\nu_A(x) = \nu_A(xyy^{-1}) \leq \nu_A(xy) \vee \nu_A(y^{-1}) = \nu_A(xy) \vee \nu_A(y).$$

Since $\mu_A(x) < \mu_A(y)$ and $\nu_A(x) > \nu_A(y)$, $\mu_A(x) \geq \mu_A(xy)$ and $\nu_A(x) \leq \nu_A(xy)$.

Thus $A(xy) = A(x)$. Similarly, we have $A(xy) = A(y)$. This completes the proof.

In the following example, we show that the union of two IFGs need not be an IFG.

Example 2.3. Let G be the Klein's four group :

$$G = \{e, a, b, ab\},$$

where $a^2 = e = b^2$ and $ab = ba$. Let $(t_i, s_i), 0 \leq i \leq 5$, be the numbers lying in $I \times I$ such that

$$t_i + s_i \leq 1, t_0 > t_1 > \cdots > t_5$$

and

$$s_0 < s_1 < \cdots < s_5.$$

Define two complex mappings $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) : G \to I \times I$ as follows :

$A(e) = (t_1, s_1), A(a) = (t_3, s_3)$ and $A(b) = A(ab) = (t_4, s_4)$,

$B(e) = (t_0, s_0), B(a) = (t_5, s_5), B(b) = (t_2, s_2)$ and $B(ab) = (t_5, s_5)$.

Then clearly, $A, B \in$ IFG(G). Moreover, we can easily see that $A \cup B \notin$ IFG(G).

Now we state without proof the following result. In fact, the proof can be obtained along the lines of Result 2.A.

Proposition 2.4. If C is a constant (improper) IFG of a group G, then C cannot be realized as $A \cup B$, where A and B are proper IFGs of G with $A \neq C$ and $B \neq C$.

We illustrate through an example that a proper IFG can be realized as a union of two proper IFGs A and B such that $C \neq A, C \neq B$ and $A \neq B$.

Example 2.5. Let G be any non-trivial group. We define a complex mapping $C = (\mu_C, \nu_C) : G \to I \times I$ as follows :

$C(e) = (t_0, s_0)$ and $C(x) = (t_1, s_1)$ for each $e \neq x \in G$,

where $(t_0, s_0), (t_1, s_1) \in (0,1] \times [0,1), t_0 > t_1$ and $s_0 < s_1, t_i + s_i \leq 1 (i = 0, 1)$.

Then clearly C is a proper IFG of G. We can see that $C = A \cup B$, where A and B are proper IFGs of G defined as follows :

$A(e) = (t'_0, s'_0)$ and $A(x) = (t_1, s_1)$ for each $e \neq x \in G$,

$B(e) = (t_0, s_0)$ and $B(x) = (t'_1, s'_1)$ for each $e \neq x \in G$,

where $t_0 > t'_0 > t_1 > t'_1, (t'_0, s'_0), (t'_1, s'_1) \in I \times I$ and $t'_i + s'_i \leq 1 (i = 0, 1)$. But, $A \neq C, B \neq C$ and $A \neq B$.

Proposition 2.6. Let G be a group and let C be a proper IFG of G such that Im$C = \{(0,1),(t,s)\}$, where $(t,s) \in (0,1] \times [0,1)$. If $C = A \cup B$, where A and B are IFGs of G, then either $A \subset B$ or $B \subset A$.

Proof. Assume that $A \not\subset B$ and $B \not\subset A$. Then there exist $x_1, x_2 \in G$ such that

$$\mu_A(x_1) > \mu_B(x_1), \nu_A(x_1) < \nu_B(x_1),$$
and
$$\mu_A(x_2) < \mu_B(x_2), \nu_A(x_2) > \nu_B(x_2).$$
Since $C = A \cup B$,
$$\mu_C(x_1) = \mu_A(x_1) \vee \mu_B(x_1) = \mu_A(x_1) > \mu_B(x_1) \geq 0,$$
$$\nu_C(x_1) = \nu_A(x_1) \wedge \nu_B(x_1) = \nu_A(x_1) < \nu_B(x_1) \leq 1.$$
and
$$\mu_C(x_2) = \mu_A(x_2) \vee \mu_B(x_2) = \mu_B(x_2) > \mu_A(x_2) \geq 0,$$
$$\nu_C(x_2) = \nu_A(x_2) \wedge \nu_B(x_2) = \nu_B(x_2) < \nu_B(x_2) \leq 1.$$
Since $\text{Im}C = \{(0,1),(t,s)\}$,
$$C(x_1) = A(x_1) = (t,s) = B(x_2) = C(x_2) = C(x_1x_2).$$
By Lemma 4.2 and the fact that
$$\mu_A(x_2) < t = \mu_A(x_1), \nu_A(x_2) > s = \nu_A(x_1)$$
and
$$\mu_B(x_1) < t = \mu_A(x_2), \nu_B(x_1) > s = \nu_B(x_2),$$
We obtain that $A(x_1x_2) = A(x_2)$ and $B(x_1x_2) = B(x_1)$. So
$$\mu_C(x_1x_2) = \mu_A(x_1x_2) \vee \mu_B(x_1x_2) = \mu_A(x_2) \vee \mu_B(x_1) < t$$
and
$$\nu_C(x_1x_2) = \nu_A(x_1x_2) \wedge \nu_B(x_1x_2) = \nu_A(x_2) \wedge \nu_B(x_1) > s.$$
This is a contradiction. Hence either $A \subset B$ or $B \subset A$.

Proposition 2.7. Let C be a proper IFG of a group G with $3 \leq |\text{Im}C| < \infty$. Then C can always be realized as a union of two proper IFGs A of G with $A \neq C, B \neq C$ and $A \neq B$.

Proof. Let $\text{Im}C = \{(t_0,s_0),(t_1,s_1),\cdots,(t_n,s_n)\}$ with $t_0 > t_1 > \cdots > t_n, s_0 < s_1 < \cdots < s_n$ and $2 \leq n < \infty$. Let
$$C^{(t_0,s_0)} \subset C^{(t_1,s_1)} \subset \cdots \subset C^{(t_n,s_n)} = G$$
be the chain of level subgroups of C in G. Let $(\lambda_1,\mu_1),(\lambda_2,\mu_2) \in (0,1)\times(0,1)$ such that $\lambda_i + \mu_i \leq 1 (i = 1,2)$ and
$$1 \geq t_0 > \lambda_1 > t_1 > \lambda_2 > t_2 > \cdots > t_n,$$
$$0 \leq s_0 < \mu_1 < s_1 < \mu_2 < s_2 < \cdots < s_n.$$
We define two complex mappings $A, B : G \to I \times I$ as follows, respectively : for each $x \in G$,

$$A(x) = \begin{cases} (t_1,s_1) & \text{if } x \in C^{(t_0,s_0)} \\ & \text{(i.e., } C(x) = (t_0,s_0)), \\ C(x) & \text{if } x \notin C^{(t_0,s_0)} \\ & \text{(i.e., } \mu_C(x) < t_0 \text{ and } \nu_C(x) > s_0), \end{cases}$$

and

$$B(x) = \begin{cases} (t_2,s_2) & \text{if } C(x) = (t_1,s_1), \\ C(x) & \text{if } C(x) \neq (t_1,s_1). \end{cases}$$

Then we can easily see that A and B are proper IFGs of G such that $A \neq C, B \neq C, A \neq B$ and $C = A \cup B$. This completes the proof.

Now we state without proof the following result.

Corollary 2.8. If C is an IFG of a group G with $\text{Im}C = \{(1,0),(t,s)\}$, where $(t,s) \in (0,1)\times(0,1)$ with $t + s \leq 1$, then C can always be realized as a union of two proper intuitionistic fuzzy subgroups A and B of G such that $A \neq C, B \neq C$ and $A \neq B$.

Definition 2.9. Let A and B be IFGs of a group G. Then A and B are said to be *equivalent* if they have the same family of level subgroups. Otherwise A and B are said to be *non-equivalent.*

It follows that the union of two proper intuitionistic fuzzy subgroups is an IFG.

Example 2.10. Let G be a cyclic group of prime power order p^n, where p is a prime and n is an integer such that $n \geq 1$. Then G has a sequence of subgroups $G_i's$ of order $p^i, i = 0,1,\cdots,n$. We define two complex mappings
$$A : G \to I \times I,$$
and
$$B : G \to I \times I.$$
as follows, respectively : for each $x \in G$,
$$A(e) = (1,0),$$
$$A(x) = (\tfrac{1}{2m}, 1 - \tfrac{1}{2m}) \text{ if } x \in G_{2m} \setminus G_{2m-2},$$
and
$$B(e) = (1,0),$$

$$B(x) = \begin{cases} (\frac{2}{3},\frac{1}{3}) \\ \quad \text{if } x \in G_1 \setminus G_0, \\ (\frac{1}{2m+1}, 1 - \frac{1}{2m+1}) \\ \quad \text{if } x \in G_{2m+1} \setminus G_{2m-1}. \end{cases}$$

Let $x \in G_2$ such that $x \notin G_1$. Then $x \notin G_0$ and $x \in G_3$. Thus $A(x) = (\frac{1}{2},\frac{1}{2})$ and $B(x) = (\frac{1}{3},\frac{2}{3})$. So $\mu_A(x) > \mu_B(x)$ and $\nu_A(x) < \nu_B(x)$. Now let $y \in G_1$ such that $y \notin G_2$. Then $y \in G_2$. Thus $A(y) = (\frac{1}{2},\frac{1}{2})$ and $B(y) = (\frac{2}{3},\frac{1}{3})$. So $\mu_A(y) < \mu_B(y)$ and $\nu_A(y) > \nu_B(y)$. Hence neither $A \not\subset B$ nor $B \not\subset A$. Moreover, it is easily seen that $A, B \in$ IFG(G) and A and B are non-equivalent. Consider the union $A \cup B$. Then $A \cup B$ is given by : for each $x \in G$,
$$(A \cup B)(e) = (1,0),$$
$$(A \cup B)(x) = (\tfrac{2}{3},\tfrac{1}{3}) \text{ if } x \in G_1 \setminus G_0,$$
$$(A \cup B)(x) = (\tfrac{1}{2},\tfrac{1}{2}) \text{ if } x \in G_2 \setminus G_1,$$
$$(A \cup B)(x) = (\tfrac{1}{3},\tfrac{2}{3}) \text{ if } x \in G_3 \setminus G_2,$$
$$(A \cup B)(x) = (\tfrac{1}{n}, 1 - \tfrac{1}{n}) \text{ if } x \in G_n \setminus G_{n-1}.$$
It is clear that $A \cup B \in$ IFG(G). Hence we have two non-equivalent intuitionistic fuzzy subgroups such that their

union is an IFG.

Definition 2.11[2]. Let X and Y be sets, let $f : X \to Y$ be a mapping and let $A \in$ IFS(X). Then A is said to be *IF-invariant* if $f(x) = f(y)$ implies $A(x) = A(y)$ for any $x, y \in X$.

Result 2.B [2,Proposition 6.6]. Let X and Y be sets, let $f : X \to Y$ be a mapping and let $A \in$ IFS(X). If A is f-invariant, then $f^{-1}(f(A)) = A$.

Proposition 2.12. Let $f : G \to G'$ be a group homomorphism and let $\{A_\alpha\}_{\alpha\in\Gamma} \subset$ IFG(G).

(1) If $\bigcup_{\alpha\in\Gamma} A_\alpha \in$ IFG(G), then $\bigcup_{\alpha\in\Gamma} f(A_\alpha) \in$ IFG(G').

(2) If $\bigcup_{\alpha\in\Gamma} f(A_\alpha) \in$ IFG$(G)'$ and each A_α is IF-invariant, then $\bigcup_{\alpha\in\Gamma} A_\alpha \in$ IFG(G).

Proof. (1) Suppose $\bigcup_{\alpha\in\Gamma} A_\alpha \in$ IFG(G). Then, by Result 1.C, $f(\bigcup_{\alpha\in\Gamma} A_\alpha) \in$ IFG(G'). By Result 1.A(7), $f(\bigcup_{\alpha\in\Gamma} A_\alpha) = \bigcup_{\alpha\in\Gamma} f(A_\alpha)$. Hence $\bigcup_{\alpha\in\Gamma} f(A_\alpha) \in$ IFG(G').

(2) Suppose $\bigcup_{\alpha\in\Gamma} f(A_\alpha) \in$ IFG(G'). Then, by Result 1.B,$f^{-1}(\bigcup_{\alpha\in\Gamma} f(A_\alpha)) \in$ IFG(G). By Result 1.A(5), $f^{-1}(\bigcup_{\alpha\in\Gamma} A_\alpha) = \bigcup_{\alpha\in\Gamma} f^{-1}(f(A_\alpha))$. Since each A_α is IF-invariant, by Result 2.B, $f^{-1}(F(A_\alpha)) = A_\alpha$. So $f^{-1}(\bigcup_{\alpha\in\Gamma} f(A_\alpha) = \bigcup_{\alpha\in\Gamma} A_\alpha$. Hence $\bigcup_{\alpha\in\Gamma} A_\alpha \in$ IFG(G).

Proposition 2.13. Let $f : G \to G'$ be a group epimorphism and let $\{B_\alpha\}_{\alpha\in\Gamma} \subset$ IFG(G'). Then $\bigcup_{\alpha\in\Gamma} B_\alpha \in$ IFG(G') if and only if $\bigcup_{\alpha\in\Gamma} f^{-1}(B_\alpha) \in$ IFG(G).

Proof. ($\Rightarrow$): Suppose $\bigcup_{\alpha\in\Gamma} B_\alpha \in$ IFG(G'). Then, by Result 1.B, $f^{-1}(\bigcup_{\alpha\in\Gamma} B_\alpha) \in$ IFG(G). By Result 1.A(5), $f^{-1}(\bigcup_{\alpha\in\Gamma} B_\alpha) = \bigcup_{\alpha\in\Gamma} f^{-1}(B_\alpha)$. Hence $\bigcup_{\alpha\in\Gamma} f^{-1}(B_\alpha) \in$ IFG(G).

($\Leftarrow$): Suppose $\bigcup_{\alpha\in\Gamma} f^{-1}(B_\alpha) \in$ IFG(G). Then, by Result 1.C, $f(\bigcup_{\alpha\in\Gamma} f^{-1}(B_\alpha)) \in$ IFG(G'). Since f is surjective by Results 1.A(7) and 1.A(4),

$$f(\bigcup_{\alpha\in\Gamma} f^{-1}(B_\alpha)) = \bigcup_{\alpha\in\Gamma} f(f^{-1}(B_\alpha)) = \bigcup_{\alpha\in\Gamma} B_\alpha.$$

Hence $\bigcup_{\alpha\in\Gamma} B_\alpha \in$ IFG(G'). This completes the proof.

3. Intuitionistic fuzzy subgroup generated by an intuitionistic fuzzy set

Result 3.A[17, Proposition 2.4]. Let G be a group and let the following be any chain of subgroups

$$C_0 \subset C_1 \subset \cdots \subset C_r = G.$$

Then there exists an intuitionistic fuzzy subgroup of G whose level subgroups are precisely the members of this chain.

Proposition 3.1. Let A be an IFS in a group G with $|\text{Im}A| < \infty$. Define subgroups G_i of G inductively as follows :

$$G_0 = \langle\{x \in G : A(x) = (\bigvee_{z\in G} \mu_A(z), \bigwedge_{z\in G} \nu_A(z))\}\rangle,$$
$$G_i = \langle G_{i-1} \cup \{x \in G : A(x) = (\bigvee_{z\in G\setminus G_{i-1}} \mu_A(z), \bigwedge_{z\in G\setminus G_{i-1}} \nu_A(z))\}\rangle$$

for each $i = 1, \cdots, k$, where $k \leq |\text{Im}A|, G_k = G$ and $\langle H\rangle$ denotes the subgroup generated by a subset H of G. Then $A^* \in$ IFG(S), where $A^* : G \to I \times I$ is a complex mapping defined as follows : for each $x \in G$,

$$A^*(x) = \begin{cases} (\bigvee_{z\in G} \mu_A(z), \bigwedge_{z\in G} \nu_A(z)) \\ \quad \text{if } x \in G_0, \\ (\bigvee_{z\in G\setminus G_{i-1}} \mu_A(z), \bigwedge_{z\in G\setminus G_{i-1}} \nu_A(z)) \\ \quad \text{if } \ x \in G_i \setminus G_{i-1} (1 \leq i \leq k). \end{cases}$$

In this case, A^* is called the IFG *generated by* A in G.

Proof. By the definition of A^*, it is clear that $A \subset A^*$. Moreover, the $G_i's$ form a chain of subgroups ending at G :

$$G_0 \subset G_1 \subset \cdots \subset G_k = G. \qquad (*)$$

By Result 3.A, it follows that A^* is an IFG of G whose level subgroups are precisely the members of the chain $(*)$. Now we shall prove that A^* is the IFG generated by A. Let $B \in$ IFG(G) such that $A \subset B$. Then, by definition of A^*,

$$\mu_{A^*}(e) = \bigvee_{z\in G} \mu_A(z) \leq \bigvee_{z\in G} \mu_B(z) \leq \mu_B(e)$$

and

$$\nu_{A^*}(e) = \bigwedge_{z\in G} \nu_A(z) \geq \bigwedge_{z\in G} \nu_B(z) \geq \nu_B(e).$$

Thus $\mu_{A^*}(e) \leq \mu_B(e)$ and $\nu_{A^*}(e) \geq \nu_B(e)$.

Let $K_0 = \{x \in G : A(x) = (\bigvee_{z\in G} \mu_A(z), \bigwedge_{z\in G} \nu_A(z))\}$, let $\{B^{(t_i,s_i)}\}$ be the chain of level subgroups of B and let $e \neq x \in K_0$. Then $\bigvee_{z\in G} \mu_A(z) = \mu_A(x) \leq \mu_B(x)$ and $\bigwedge_{z\in G} \nu_A(z) = \nu_A(x) \geq \nu_B(x)$. Thus $\bigvee_{z\in G} \mu_A(z) \leq \bigwedge_{x\in K_0} \mu_B(x)$ and $\bigwedge_{z\in G} \nu_A(z) \geq \bigvee_{x\in K_0} \nu_B(x)$. Let $(t_i, s_i) = (\bigwedge_{x\in K_0} \mu_B(x), \bigvee_{x\in K_0} \nu_B(x))$. Then $\mu_B(x) \geq t_i$ and $\nu_B(x) \leq s_i$ for each $x \in K_0$. Thus $K_0 \subset B^{(t_i,s_i)}$. Since $G_0 = \langle K_0\rangle, G_0 \subset B^{(t_i,s_i)}$. So $\mu_B(x) \geq t_i$ and $\nu_B(x) \leq s_i$ for each $x \in G_0$. Let $x \in G_0$. Then

$$\mu_{A^*}(x) = \bigvee_{z\in G} \mu_A(z) \leq \bigwedge_{x\in K_0} \mu_B(x) = t_i \leq \mu_B(x)$$

and

$$\nu_{A^*}(x) = \bigwedge_{z\in G} \nu_A(z) \geq \bigvee_{x\in K_0} \nu_B(x) = s_i \geq \nu_B(x).$$

Let $x \in G_1 \setminus G_0$. Then $\mu_{A^*}(x) = (\bigvee_{z\in G\setminus G_0} \mu_A(z), \bigwedge_{z\in G\setminus G_0} \nu_A(z))$ and $G_1 = \langle K_1\rangle$, where $K_1 = G_0 \cup \{x \in G : \mu_A(x) = (\bigvee_{z\in G\setminus G_0} \mu_A(z), \bigwedge_{z\in G\setminus G_0} \nu_A(z)\}$. We claim that $G_1 \subset B^{(t_{i_1},s_{i_1})}$, where $(t_{i_1}, s_{i_1}) =$

$(\bigwedge_{x\in K_1\setminus G_0}\mu_B(x), \bigvee_{x\in K_1\setminus G_0}\nu_B(x))$. Let $x \in K_1 \setminus G_0$. Then $\mu_A(x) = \bigvee_{z\in G\setminus G_0}\mu_A(z)$ and $\nu_A(x) = \bigwedge_{z\in G\setminus G_0}\nu_A(z)$. Since $A \subset B$,

$\bigvee_{z\in G\setminus G_0}\mu_A(z) \le \bigwedge_{x\in K_1\setminus G_0}\mu_B(x) = t_{i_1} \le \mu_B(x)$

and

$\bigwedge_{z\in G\setminus G_0}\nu_A(z) \ge \bigvee_{x\in K_1\setminus G_0}\nu_B(x) = s_{i_1} \ge \nu_B(x)$.

Thus $x \in B^{(t_{i_1},s_{i_1})}$. So $K_1 \setminus G_0 \subset B^{(t_{i_1},s_{i_1})}$. Also $G_0 \subset B^{(t_i,s_i)} \subset B^{(t_{i_1},s_{i_1})}$. Hence $G_1 = \langle K_1\rangle \subset B^{(t_{i_1},s_{i_1})}$. Thus $\mu_B(x) \ge t_{i_1}$ and $\nu_B(x) \le s_{i_1}$ for each $x \in G_1$. Let $x \in G_1 \setminus G_0$. Then

$\mu_{A^*}(x) = \bigvee_{z\in G\setminus G_0}\mu_A(z) \le t_{i_1} \le \mu_B(x)$

and

$\nu_{A^*}(x) = \bigwedge_{z\in G\setminus G_0}\nu_A(z) \ge s_{i_1} \ge \nu_B(x)$.

Proceeding as above, we can also see that

$\mu_{A^*}(x) \le \mu_B(x)$ and $\nu_{A^*}(x) \ge \nu_B(x)$ for each $x \in G_i \setminus G_{i-1}, 2 \le i \le k$.

Consequentely, $A^* \subset B$. Hence A^* is the IFG generated by A. This completes the proof.

The following is an example showing that the cardinality of the intuitionistic fuzzy set A may not be equal to the cardinality of the image of the intuitionistic fuzzy subgroup A^* generated by A.

Example 3.2. Let $G = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$ be the octic group, where $a^4 = e = b^2$ and $ba = a^{-1}b$. Let $(t_i, s_i) \in I \times I, 0 \le i \le 6$, be such that $t_i + s_i \le 1, t_0 > t_1 > \cdots > t_6$ and $s_0 < s_1 < \cdots < s_6$. We define a complex mapping $A : G \to I \times I$ as follows :

$A(e) = (t_0, s_0), A(a^2) = (t_1, s_1), A(a) = (t_2, s_2),$
$A(a^3) = (t_3, s_3), A(b) = (t_4, s_4),$
$A(ab) = A(a^2b) = (t_5, s_5), A(a^3b) = (t_6, s_6).$

Then clearly $A \in$ IFS(G) but $A \notin$ IFG(G). Moreover,

$G_0 = \langle e\rangle = \{e\}, G_1 = \langle G_0, a^2\rangle = \{e, a^2\},$
$G_2 = \langle G_1, a\rangle = \{e, a, a^2, a^3\}$ and
$G_3 = \langle G_2, b\rangle = G.$

By definition of A^*, we have that

$A^*(e) = (t_0, s_0), A^*(a^2) = (t_1, s_1),$
$A^*(a) = A^*(a^3) = (t_2, s_2),$

and

$A^*(b) = A^*(ab) = A^*(a^2b) = A^*(a^2b) = (t_4, s_4).$

It is clear that A^* is the IFG generated by A and $|\text{Im}A| > |\text{Im}A^*|$.

Definition 3.3. Let G be a group, let $A_1, A_2, A \in$ IFG(G) and let $g \in G$.

(1) A_1 is said to be *conjugate* to A_2 if there exists an $a \in G$ such that $A_1(x) = A_2(a^{-1}xa)$ for each $x \in G$.

(2) We define a complex mapping $A_g^* : G \to I \times I$ as follows :

$A_g^*(x) = A(g^{-1}xg)$ for each $x \in G$.

Then A_g^* is called the *intuitionistic fuzzy conjugate subgroup* of G determined by A and $g \in G$.

It is clear that $A_g^* \in$ IFG(G).

Proposition 3.4. Let G be any group and let $A \in$ IFG(G) such that Im$A = \{(t_0, s_0), (t_1, s_1), \cdots, (t_n, s_n)\}$, $t_0 > t_1 > \cdots > t_n$ and $s_0 < s_1 < \cdots < s_n$. If the chain of level subgroup of A is given by :

$$A^{(t_0,s_0)} \subset A^{(t_1,s_1)} \subset \cdots \subset A^{(t_n,s_n)} = G,$$

then the chain of level subgroups of A_x^* is given by :

$$xA^{(t_0,s_0)}x^{-1} \subset xA^{(t_1,s_1)}x^{-1} \subset \cdots \subset xA^{(t_n,s_n)}x^{-1} = G,$$

where A_x^* is the intuitionistic fuzzy conjugate subgroup of G determined by A and $x \in G$.

Proof. Let $x \in G$. We define a mapping $\varphi_x : G \to G$ as follows :

$\varphi_x(a) = x^{-1}ax$ for each $a \in G$.

Then clearly $\varphi_x \in$ Aut$G$, where Aut$G$ denotes the set of all automorphisms on G. Moreover, $A_x^* = A \circ \varphi_x = \varphi_x^{-1}(A)$. Then Im$A_x^*$ = ImA. On the other hand,

$$\begin{aligned} a \in A_x^{*(t_i,s_i)} &\Leftrightarrow \mu_{A_x^*}(a) \ge t_i \text{and} \nu_{A_x^*}(a) \le s_i \\ &\Leftrightarrow \mu_A(x^{-1}ax) \ge t_i \text{and} \nu_A(x^{-1}ax) \le s_i \\ &\Leftrightarrow x^{-1}ax \in A^{(t_i,s_i)} \\ &\Leftrightarrow \varphi_x(a) \in A^{(t_i,s_i)} \\ &\Leftrightarrow a \in \varphi_x^{-1}(A^{(t_i,s_i)}) \\ &\Leftrightarrow a \in \varphi_{x^{-1}}(A^{(t_i,s_i)}) \\ &\Leftrightarrow a \in xA^{(t_i,s_i)}x^{-1}. \end{aligned}$$

Hence $A_x^{*(t_i,s_i)} = xA^{(t_i,s_i)}x^{-1}$. This completes the proof.

Remark 3.5. The proof of Proposition 3.4 is similar as the proof of Theorem 4.2 in [8]. The following is the converse of Proposition 3.4.

Proposition 3.6. Let G be a group and let $A \in$ IFG(G) such that Im$A = \{(t_0, s_0), (t_1, t_1), \cdots, (t_n, s_n)\}, t_0 > t_1 > \cdots > t_n$ and $s_0 < s_1 < \cdots < s_n$. If the chain of level subgroups of A in G is

$$A^{(t_0,s_0)} \subset A^{(t_1,s_1)} \subset \cdots \subset A^{(t_n,s_n)} = G,$$

then there exists a $B \in$ IFG(G) such that the chain of level subgroups of B is given by :

$$xA^{(t_0,s_0)}x^{-1} \subset xA^{(t_1,s_1)}x^{-1} \subset \cdots \subset xA^{(t_n,s_n)}x^{-1} = G,$$

where $x \in G$ and $B = A_x^*$.

Proof. We define a complex mapping $B : G \to I \times I$ as follows : for each $g \in G$,

$$B(g) = \begin{cases} (t_0, s_0) \text{ if } g \in xA^{(t_0,s_0)}x^{-1}, \\ (t_i, s_i) \text{ if } g \in xA^{(t_i,s_i)}x^{-1} \setminus xA^{(t_{i-1},s_{i-1})}x^{-1}, \\ \qquad i = 1, \cdots, n. \end{cases}$$

Then clearly $B \in \text{IFS}(G)$. Let $i \in \{1, 2, \cdots, n\}$ and let $g \in G$. Then

$$\begin{aligned} B(g) = (t_i, s_i) &\Leftrightarrow g \in xA^{(t_i,s_i)}x^{-1} \setminus xA^{(t_{i-1},s_{i-1})}x^{-1} \\ &\Leftrightarrow x^{-1}gx \in A^{(t_i,s_i)} \setminus A^{(t_{i-1},s_{i-1})} \\ &\Leftrightarrow A(x^{-1}gx) = (t_i, s_i) \\ &\Leftrightarrow A_x^*(g) = (t_i, s_i). \end{aligned}$$

On the other hood,

$$\begin{aligned} B(g) = (t_0, s_0) &\Leftrightarrow g \in xA^{(t_0,s_0)}x^{-1} \\ &\Leftrightarrow x^{-1}gx \in A^{(t_0,s_0)}0 \\ &\Leftrightarrow A(x^{-1}gx) = (t_0, s_0) \\ &\Leftrightarrow A_x^*(g) = (t_0, s_0). \end{aligned}$$

So $B = A_x^*$.

Now we define a mapping $\varphi_x : G \to G$ as follows:
$\varphi_x(g) = x^{-1}gx$ and for each $g \in G$.
Then clearly $\varphi_x \in \text{Aut}G$. Since $A \in \text{IFG}(G)$, by Result 1.B, $A_x^* = \varphi_x^{-1}(A) \in \text{IFG}(G)$. So $B = A_x^* \in \text{IFG}(G)$. Moreover, it is clear that
$\text{Im}A_x^* = \text{Im}A$ and $A_x^{*(t_i,s_i)} = xA^{(t_i,s_i)}x^{-1}$.
Hence the chain of level subgroup of B is given by :
$xA^{(t_0,s_0)}x^{-1} \subset xA^{(t_1,s_1)}x^{-1} \subset \cdots \subset xA^{(t_n,s_n)}x^{-1} = G.$
This completes the proof.

A subgroup H of a group G is called a *characteristic subgroup* of G if $f(H) = H$ for each $f \in \text{Aut}(G)$.

Definition 3.7. Let A be an IFG of a group G. Then A is called an *intuitionistic fuzzy characteristic subgroup* of G if $\varphi^{-1}(A) = A$ for each $\varphi \in \text{Aut}G$.

Proposition 3.8. Let G be a finite group and let A be an intuitionistic fuzzy characteristic subgroup of G. Then each level subgroup of A is a characteristic subgroup of G.

Proof. Since G is a finite group, $|\text{Im}A| < \infty$. Let $(\lambda, \mu) \in \text{Im}A$ and let $\varphi \in \text{Aut}G$. Since A is an intuitionistic fuzzy characteristic subgroup of G, $\varphi^{-1}(A) = A$. Let $x \in A^{(\lambda,\mu)}$. Then
$\mu_A(\varphi(x)) = \mu_{\varphi^{-1}(A)}(x) = \mu_A(x) \geq \lambda.$
and
$\nu_A(\varphi(x)) = \nu_{\varphi^{-1}(A)}(x) = \nu_A(x) \leq \mu.$
Thus $\varphi(x) \in A^{(\lambda,\mu)}$. So $\varphi(A^{(\lambda,\mu)}) \subset A^{(\lambda,\mu)}$. Hence $A^{(\lambda,\mu)}$ is a characteristic subgroup of G.

The following is the converse of Proposition 3.8.

Proposition 3.9. Let G be a finite group and let $A \in \text{IFG}(G)$. If each level subgroup of A is a characteristic subgroup of G, then A is an intuitionistic fuzzy characteristic subgroup of G.

Proof. Since G is a finite, $|\text{Im}A| < \infty$. Let $\text{Im}A = \{(t_0, s_0), (t_1, s_1), \cdots, (t_n, s_n)\}$ such that $t_0 > t_1 > \cdots > t_n$ and $s_0 < s_1 < \cdots < s_n$. Then, by the hypothesis, $A^{(t_i,s_i)}$ is a characteristic subgroup of G for each $i = 0, \cdots, n$. Let $\varphi \in \text{Aut}G$. Then clearly, $\text{Im}\varphi^{-1}(A) = \text{Im}A$. On the other hood, for each $i = 0, \cdots, n$,

$$\begin{aligned} x \in\ (\varphi^{-1}(A))^{(t_i,s_i)} & \\ \Leftrightarrow\ & \mu_{\varphi^{-1}(A)}(x) = \mu_A(\varphi(x)) \geq t_i \\ & \text{and } \nu_{\varphi^{-1}(A)}(x) = \nu_A(\varphi(x)) \leq s_i \\ \Leftrightarrow\ & \varphi(x) \in A^{(t_i,s_i)} \\ \Leftrightarrow\ & x \in \varphi^{-1}(A^{(t_i,s_i)}) \\ \Leftrightarrow\ & x \in A^{(t_i,s_i)}. \end{aligned}$$

Thus $\varphi^{-1}(A) = A$. Hence A is an intuitionistic fuzzy characteristic subgroup of G.

Result 3.B[16, Proposition 2.17]. Let A be an IFG of a group G and let $x \in G$. Then $A(x) = (\lambda, \mu)$ if and only if $x \in A^{(\lambda,\mu)}$ and $x \notin A^{(t,s)}$ for each $(t, s) \in I \times I$ such that $t + s \leq 1, t > \lambda$ and $s < \mu$.

The following is the generalization of Proposition 3.8 and 3.9.

Theorem 3.10. Let G be a group and let $A \in \text{IFG}(G)$. Then A is an intuitionistic fuzzy characteristic subgroup of G if and only if each level subgroup of A is a characteristic subgroup of G.

Proof. $(\Rightarrow)$: Suppose A is an intuitionistic fuzzy characteristic subgroup of G. Let $(\lambda, \mu) \in \text{Im}A$, let $\varphi \in \text{Aut}G$ and let $x \in A^{(\lambda,\mu)}$. Then, by the hypothesis, $\mu_A(\varphi(x)) = \mu_A(x) \geq \lambda$ and $\nu_A(\varphi(x)) = \nu_A(x) \leq \mu$. Thus $\varphi(x) \in A^{(\lambda,\mu)}$. So $\varphi(A^{(\lambda,\mu)}) \subset A^{(\lambda,\mu)}$. Now let $x \in A^{(\lambda,\mu)}$ and let $g \in G$ such that $\varphi(g) = x$. Then $\mu_A(g) = \mu_A(\varphi(g)) = \mu_A(x) \geq \lambda$ and $\nu_A(g) = \nu_A(\varphi(g)) = \nu_A(x) \leq \mu$. Thus $g \in A^{(\lambda,\mu)}$. So $x \in \varphi(A^{(\lambda,\mu)})$. Then $A^{(\lambda,\mu)} \subset \varphi(A^{(\lambda,\mu)})$. Thus $\varphi(A^{(\lambda,\mu)}) = A^{(\lambda,\mu)}$. Hence $A^{(\lambda,\mu)}$ is a characteristic subgroup of G for each $(\lambda, \mu) \in \text{Im}A$.

$(\Leftarrow)$: Suppose the necessary condition holds. Let $x \in G$, let $\varphi \in \text{Aut}G$ and let $A(x) = (\lambda, \mu)$. Then, by Result 3.B, $x \in A^{(\lambda,\mu)}$ but $x \notin A^{(t,s)}$ for all $t > \lambda$ and $s < \mu$ such that $t+s \leq 1$. By the hypothesis, $\varphi(A^{(\lambda,\mu)}) = A^{(\lambda,\mu)}$. Thus $\varphi(x) \in A^{(\lambda,\mu)}$. So $\mu_A(\varphi(x)) \geq \lambda$ and $\nu_A(\varphi(x)) \leq \mu$. Let $A(\varphi(x)) = [\varphi^{-1}(A)](x) = (t, s)$. If possible, let $t > \lambda$ and $s < \mu$. Then $\varphi(x) \in A^{(t,s)} = \varphi(A^{(t,s)})$. Since φ is injective, $x \in A^{(t,s)}$. This contradicts the fact that $x \notin A^{(t,s)}$. So $A(\varphi(x)) = (\lambda, \mu) = A(x)$, i.e., $\varphi^{-1}(A) = A$. Hence A is an intuitionistic fuzzy characteristic fuzzy subgroup of G. This completes the proof.

Theorem 3.11. Let G be a finite group and let $f : G \to G'$ be a group epimorphism. Let $A \in \text{IFG}(G)$ such that $\text{Im}A = \{(t_0, s_0), (t_1, s_1), \cdots, t_n, s_n)\}, t_0 > t_1 > \cdots > t_n$ and $s_0 < s_1 < \cdots < s_n$. If the chain of level subgroups of A is

$$A^{(t_0,s_0)} \subset A^{(t_1,s_1)} \subset \cdots \subset A^{(t_n,s_n)} = G,$$

then the chain of level subgroups of $f(A)$ is

$$f(A^{(t_0,s_0)}) \subset f(A^{(t_1,s_1)}) \subset \cdots \subset f(A^{(t_n,s_n)}) = G'.$$

Proof. By Result 1.C,$f(A) \in \text{IFG}(G')$. It is clear that $\text{Im}f(A) \subset \text{Im}A$. Let $(t_i, s_i) \in \text{Im}f(A)$ and let $y \in (f(A))^{(t_i,s_i)}$. Then

$$\mu_{f(A)}(y) = \bigvee_{z \in f^{-1}(y)} \mu_A(z) \geq t_i$$

and

$$\nu_{f(A)}(y) = \bigwedge_{z \in f^{-1}(y)} \nu_A(z) \leq s_i.$$

Since G is a finite group, it is clear that A has sup property. Then there exists a $z_0 \in G$ such that $f(z_0) = y$ and $\bigvee_{z \in f^{-1}(y)} \mu_A(z) = \mu_A(z_0) \geq t_i, \bigwedge_{z \in f^{-1}(y)} \nu_A(z) = \nu_A(z_0) \leq s_i$. Then $z_0 \in A^{(t_i,s_i)}$. Thus $y = f(z_0) \in f(A^{(t_i,s_i)})$. So $(f(A))^{(t_i,s_i)} \subset f(A^{(t_i,s_i)})$. Now let $f(x) \in f(A^{(t_i,s_i)})$. Then $x \in A^{(t_i,s_i)}$. Thus $\mu_A(x) \geq t_i$ and $\nu_A(x) \leq s_i$. So

$$\mu_{f(A)}(f(A)) = \bigvee_{z \in f^{-1}(y)} \mu_A(z) \geq t_i$$

and

$$\nu_{f(A)}(f(A)) = \bigwedge_{z \in f^{-1}(y)} \nu_A(z) \leq s_i.$$

Hence $f(x) \in (f(A))^{(t_i,s_i)}$, i.e., $f(A^{(t_i,s_i)}) \subset (f(A))^{(t_i,s_i)}$. Therefore $f(A^{(t_i,s_i)}) = (f(A))^{(t_i,s_i)}$. This completes the proof.

Lastly, in view of the study of notion of level subgroups of an intuitionistic fuzzy subgroup, we recast Proposition 2.14 in [11] as follows.

Proposition 3.12. Let G be a finite cyclic group of prime order. Then $A \in \text{IFG}(G)$ if and only if the chain of level subgroups of A consists of only trivial subgroups of G.

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