

Some generalized weak vector quasivariational-like inequalities for fuzzy mappings

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Abstract

Some Stampacchia type of generalized weak vector quasivariational-like inequalities for fuzzy mappings was introduced and the existence of solutions to them under non-compact assumption was considered using the particular form of the generalized Ky Fan's section theorem due to Park [15]. As a corollary, Stampacchia type of generalized vector quasivariational-like inequalities for fuzzy mappings was studied under compact assumption using Ky Fan's section theorem [7].

Key words : Fuzzy mapping, weak vector quasivariational-like inequality, generalized Ky Fan's section theorem, topologically open fuzzy set-valued, weakly open fuzzy set-valued

1. Introduction

Since Chang and Zhu [5] firstly introduced a variational inequality problem for fuzzy mappings, many authors [2, 3, 8, 10-13] considered vector variational inequality problems for fuzzy mappings. In particular, Chang et al. studied vector quasivariational inequalities [2, 3] and vector variational-like inequalities [1] for fuzzy mappings and Lee et al. [10] obtained a fuzzy extension of Siddiqi et al.'s results [16] for vector variational-like inequalities. In [8], the authors considered the existence of solutions to generalized fuzzy vector quasivariational-like inequalities (**F-VQVLI**) under the compact assumption. And then, in [9], the author considered the weak case of the results in [8] under the non-compact assumption.

In this paper, we recall a Stampacchia type of generalized weak vector quasivariational-like inequalities for fuzzy mappings (**F-WVQVLI**) and re-consider the existence of solutions to them under the non-compact assumption and to (**F-VQVLI**) under the compact assumption considered in [8, 9].

Throughout this paper, X denotes a Hausdorff topological vector space, Y is a topological vector space, Z is an ordered topological vector space, $\mathfrak{S}(Y)$ is the collection of all fuzzy sets in Y and $L(X, Z)$ is the set of all linear continuous mappings from X to Z . Let K be a nonempty convex subset of X , D be a nonempty subset of Y and $\{C(x) : x \in K\}$ be a family of solid convex cones in Z , that is, for any $x \in K$, $\text{int } C(x) \neq \emptyset$ and $C(x) \neq Z$. Let $F : K \rightarrow \mathfrak{S}(D)$ and $G : K \rightarrow \mathfrak{S}(K)$ be fuzzy mappings, $M : K \times D \rightarrow 2^{L(X, Z)}$ and $H : K \times K \rightarrow 2^Z$

be multivalued mappings, $\eta : X \times X \rightarrow X$ be a mapping, $\beta : X \rightarrow (0, 1]$ be a function and γ be a constant in $(0, 1]$. $(F_x)_{\beta(x)} := \{d \in D : F_x(d) \geq \beta(x)\}$ is a cut set and $[F_x]_{\beta(x)} := \{d \in D : F_x(d) > \beta(x)\}$ is a strong cut set.

An ordering \leq with respect to the cone C in Z is defined as $y \leq_{\text{int } C} x$ if and only if $x - y \notin -\text{int } C$ for $x, y \in Z$.

We consider the existence of solutions to the following Stampacchia type of generalized weak vector quasivariational-like inequalities for fuzzy mappings:

(F-WVQVLI) Find $\bar{x} \in K$ such that there exists $\bar{s} \in (F_{\bar{x}})_{\beta(\bar{x})}$ satisfying the following inequality:

$$\max \langle M(\bar{x}, \bar{s}), \eta(x, z) \rangle + u \notin -\text{int } C(\bar{x})$$

for any $x \in K$, $z \in (G_{\bar{x}})_{\gamma}$ and $u \in H(x, \bar{x})$, where

$$\max \langle M(\bar{x}, \bar{s}), \eta(x, z) \rangle = \max_{s \in M(\bar{x}, \bar{s})} \langle s, \eta(x, z) \rangle$$

and $\langle s, \eta(x, z) \rangle$ denotes the evaluation of continuous linear operator s from X into Z at $\eta(x, z)$.

In addition, we obtain the existence of solutions to the following Stampacchia type of generalized vector quasivariational-like inequalities for fuzzy mappings:

(F-VQVLI) Find $\bar{x} \in K$ such that, for any $x \in K$, there exists $\bar{s} \in (F_{\bar{x}})_{\beta(\bar{x})}$ satisfying the following inequality:

$$\max \langle M(\bar{x}, \bar{s}), \eta(x, z) \rangle + u \notin -\text{int } C(\bar{x})$$

for any $z \in (G_{\bar{x}})_{\gamma}$ and $u \in H(x, \bar{x})$.

Replacing $\mathfrak{S}(K)$ and $\mathfrak{S}(D)$ with 2^K and 2^D , respectively, in **(F-WVQVLI)** and **(F-VQVLI)**, we obtain the following Stampacchia type of generalized weak vector quasivariational-like inequalities and vector quasivariational-like inequalities for multivalued mappings:

(WVQVLI) Find $\bar{x} \in K$ and $\bar{s} \in F(\bar{x})$ such that

$$\max\langle M(\bar{x}, \bar{s}), \eta(x, z) \rangle + u \notin -\text{int } C(\bar{x})$$

for any $x \in K$, $z \in G(\bar{x})$ and $u \in H(x, \bar{x})$.

(VQVLI) Find $\bar{x} \in K$ such that, for any $x \in K$, that exists $\bar{s} \in F(\bar{x})$ satisfying the following inequality:

$$\max\langle M(\bar{x}, \bar{s}), \eta(x, z) \rangle + u \notin -\text{int } C(\bar{x})$$

for any $z \in G(\bar{x})$ and $u \in H(x, \bar{x})$.

Deleting a topological vector space Y and a fuzzy mapping F , and then replacing Z with an ordered topological vector space Y , $H : K \times K \rightarrow 2^Z$ with $H : K \times K \rightarrow Y$, and $M : K \times D \rightarrow 2^{L(X, Z)}$ with $S : K \rightarrow 2^{L(X, Y)}$ in **(F-WVQVLI)** or **(F-VQVLI)**, respectively, we obtain the following vector variational-like inequality for fuzzy mappings:

(F-VVLI) Find $\bar{x} \in K$ satisfying the following inequality:

$$\max\langle S(\bar{x}), \eta(x, y) \rangle + H(x, \bar{x}) \notin -\text{int } C(\bar{x})$$

for any $x \in K$ and any $y \in (G_{\bar{x}})_{\beta(\bar{x})}$, where $\{C(x) : x \in K\}$ is a family of closed convex cones in Y .

Replacing a fuzzy mapping $G : K \rightarrow \mathfrak{S}(K)$ with a multivalued mapping $G : K \rightarrow 2^K$ defined by $G(x) = K$ for $x \in K$ and putting $H \equiv 0$ in **(F-VVLI)**, we obtain the following vector variational-like inequalities for multivalued mappings:

(VVLI) Find $\bar{x} \in K$ such that

$$\max\langle S(\bar{x}), \eta(x, y) \rangle \notin -\text{int } C(\bar{x}), \quad x, y \in K,$$

which is a generalized form of the following vector variational-like inequalities for multivalued mappings introduced and studied by Chang, Thompson and Yuan [4]:

Find $\bar{x} \in K$ satisfying the following inequality:

$$\max\langle S(\bar{x}), \eta(x, \bar{x}) \rangle \notin -\text{int } C(\bar{x}) \quad x \in K.$$

2. Preliminaries

Definition 2.1. ([6]) Let X, Y be topological spaces and $T : X \rightarrow 2^Y$ be a multivalued mapping. Let $T^- : Y \rightarrow 2^X$ be a multivalued mapping defined by

$$x \in T^-(y) \quad \text{if and only if} \quad y \in T(x).$$

- (1) T is said to be upper semicontinuous (in short, u.s.c.) at $x \in X$ if, for every open set V in Y containing $T(x)$, there is an open set U containing x such that $T(u) \subseteq V$ for all $u \in U$;
- (2) T is said to be u.s.c. on X if T is u.s.c. at every point of X .
- (3) T is said to be lower semicontinuous (in short, l.s.c.) at $x \in X$ if, for every open set V in Y with $T(x) \cap V \neq \emptyset$, there is an open set U containing x such that $T(u) \cap V \neq \emptyset$ for all $u \in U$;
- (4) T is said to be l.s.c. on X if T is l.s.c. at every point of X .
- (5) T is said to be continuous at x if T is both u.s.c. and l.s.c. at x .
- (6) T is said to be compact if $T(X)$ is contained in some compact subset of Y ,
- (7) T is said to be closed if the graph of T , $G_r T = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed in $X \times Y$.
- (8) T is said to have open lower sections if for any $y \in Y$, $T^-(y)$ is open in X .

Lemma 2.1. T is l.s.c. at $x \in X$ if, and only if, for any $y \in T(x)$ and for any net $\{x_\alpha\}$ in X converging to x , there is a net $\{y_\alpha\}$ such that $y_\alpha \in T(x_\alpha)$ for each α , and y_α converging to y .

Definition 2.2. ([1, 2, 3, 5]) Let X, Y be sets and $F : X \rightarrow \mathfrak{S}(Y)$ be a fuzzy mapping. We denote a fuzzy set $F(x)$ by F_x in Y for $x \in X$.

- (1) F is said to be convex on a set X if Y is a convex subset of a topological vector space and for any $x \in X$, $y, z \in Y$ and $\lambda \in [0, 1]$,

$$F_x(\lambda y + (1 - \lambda)z) \geq \min\{F_x(y), F_x(z)\}.$$

- (2) F is said to be closed fuzzy set-valued if for each $y \in Y$, $F_x(y)$ is u.s.c. on $X \times Y$ as a real ordinary function.

- (3) F is said to be topologically open fuzzy set-valued if, for each $x_0 \in X$ and for each open subset V of Y with $F_{x_0}(y) \geq \gamma$ for some $y \in V$ ($\gamma \in (0, 1]$), there is a neighborhood U of x_0 in X such that if $x \in U$, then $F_x(y) \geq \gamma$ for some $y \in V$.
- (4) F is said to be weakly open fuzzy set-valued if for each $y \in Y$, $F_x(y)$ is l.s.c. on $X \times Y$ as a real ordinary function.

Lemma 2.2. ([1]) Let K be a nonempty closed convex subset of a real Hausdorff topological vector space X , D a nonempty closed convex subset of a real Hausdorff topological vector space Y and $\beta : K \rightarrow (0, 1]$ a l.s.c. function. Let $F : K \rightarrow \mathfrak{S}(D)$ be a fuzzy mapping with nonempty cut set $(F_x)_{\beta(x)}$ for any $x \in K$. Let $\bar{F} : K \rightarrow 2^D$ be a multivalued mapping defined by $\bar{F}(x) = (F_x)_{\beta(x)}$. If F is a convex fuzzy mapping with closed fuzzy set-values, then \bar{F} is a closed mapping with nonempty convex-values.

Lemma 2.3. ([2]) Let X and Y be topological spaces, and $F : X \rightarrow \mathfrak{S}(Y)$ be a fuzzy mapping such that for any $x \in X$, the cut set $(F_x)_\gamma$ is nonempty for $\gamma \in (0, 1]$. Let $\bar{F} : X \rightarrow 2^Y$ be a multivalued mapping defined by $\bar{F}(x) = (F_x)_\gamma$. If F is convex and topologically open fuzzy set-valued, then \bar{F} is a l.s.c. mapping with nonempty convex-values.

Lemma 2.4. ([3]) Let K be a nonempty closed convex subset of a real Hausdorff topological vector space X , D a nonempty closed convex subset of a Hausdorff topological vector space Y and $\beta : X \rightarrow (0, 1]$ an u.s.c. function. Let $F : K \rightarrow \mathfrak{S}(D)$ be a fuzzy mapping such that for any $x \in K$, the strong cut set $[F_x]_{\beta(x)}$ is nonempty. Let $\bar{F} : K \rightarrow 2^D$ be a multivalued mapping defined by $\bar{F}(x) = [F_x]_{\beta(x)}$.

- (1) If F is convex, then \bar{F} has nonempty convex-values.
- (2) If F is weakly open fuzzy set-valued, then \bar{F} has open lower sections.

Definition 2.3. Let X, Z be two vector spaces, $K \subset X$ be a nonempty convex set and $P \subset Z$ a pointed, closed convex cone with apex at the origin and nonempty interior $int P$. A multivalued mapping $H : K \times K \rightarrow 2^Z$ is said to be P -convex with respect to the first variable if, for $x_1, x_2, y \in K, u_1 \in H(x_1, y), u_2 \in H(x_2, y)$ and $\lambda \in [0, 1]$, there exists $u \in H(\lambda x_1 + (1 - \lambda)x_2, y)$ such that

$$\lambda u_1 + (1 - \lambda)u_2 \in u + P.$$

Definition 2.4. Let X, Z be vector spaces. A mapping $\eta : X \times X \rightarrow Z$ is said to be linear if

$$\eta(\lambda(x_1, y_1) + (x_2, y_2)) = \lambda\eta(x_1, y_1) + \eta(x_2, y_2)$$

for all $(x_1, x_2), (y_1, y_2) \in X \times X$ and $\lambda \in R$.

Definition 2.5. ([18]) A point z_0 in a nonempty subset C of Z is called a vector maximal point of C if the set $\{z \in C : z_0 \leq_P z, z \neq z_0\} = \emptyset$, which is equivalent to

$$C \cap (z_0 + P) = \{z_0\}.$$

Lemma 2.5. ([14]) Let C be a nonempty compact subset of an ordered Banach space Z . Then $\max C \neq \emptyset$, where $\max C$ denotes the set of all vector maximal points of C .

The following particular form of the generalized Ky Fan's section theorem which is due to Park [15] will be used in dealing with (F-WVQVLI) for the noncompact set case.

Theorem 2.1. ([15]) Let K be a nonempty convex subset of X and $A \subset K \times K$ satisfy the following conditions:

- (i) $(x, x) \in A, x \in K$;
- (ii) $A_x = \{y \in K : (x, y) \in A\}, x \in K$, is closed;
- (iii) $A^y = \{x \in K : (x, y) \in A\}, y \in K$, is convex or empty;
- (iv) there exists a nonempty compact subset B of K such that for each finite subset N of K there exists a nonempty compact convex subset L_N of K containing N such that

$$L_N \cap \{y \in K : (x, y) \in A \text{ for any } x \in L_N\} \subset B.$$

Then there exists a $y_0 \in B$ such that $K \times \{y_0\} \subset A$.

In particular, if $K = B$, that is, K is a compact convex subset of X , then the condition (iv) is obviously true and thus we obtain Ky Fan's section theorem [7], which will be used in considering (F-VQVLI) under compact assumption.

3. Main results

Now, we consider the existence of solutions to the Stampacchia type of (F-WVQVLI) for non-compact set case studied in [9]. The following Proposition 3.1 is an existence theorem to a kind of vector variational-like inequalities for multi-valued mappings. With Lemma 3.1, it is an important tool for the following main results Theorem 3.1 and Theorem 3.2.

Lemma 3.1. ([17]) Let X be a paracompact Hausdorff topological space and Y be a topological vector space. Let $F : X \rightarrow 2^Y$ be a multivalued mapping with nonempty convex-values. If F has open lower sections, then there exists a continuous function $f : X \rightarrow Y$ such that $f(x) \in F(x)$ for any $x \in X$.

Proposition 3.1. Let K be a nonempty convex subset of X and D be a nonempty subset of Y . Let $f : K \rightarrow D$ be a continuous function and $G : K \rightarrow 2^K$ be a l.s.c. multivalued mapping with convex-values. Let $M : K \times D \rightarrow 2^{L(X,Z)}$ be a multivalued mapping, and a multivalued mapping $W : K \rightarrow 2^Z$ defined by $W(x) = Z \setminus \{-int C(x)\}$, $x \in K$, be closed. Let $\eta : X \times X \rightarrow X$ be linear, $y \mapsto \eta(\cdot, y)$ be continuous and $H : K \times K \rightarrow 2^Z$ be P -convex with respect to the first variable and l.s.c. with respect to the second, where $P = \bigcap_{x \in K} C(x)$. Suppose further that

- (i) $\max\langle M(y_\alpha, s_\alpha), \eta(x, z_\alpha) \rangle$ converges to $\max\langle M(y, s), \eta(x, z) \rangle$ provided that $y_\alpha \rightarrow y$, $s_\alpha \rightarrow s$ and $z_\alpha \rightarrow z$;
- (i) $\langle M(x, \cdot), \eta(x, \cdot) \rangle = 0$ and $H(x, x) = \{0\}$ for all $x \in K$,
- (ii) there is a nonempty compact subset B of K such that for each nonempty finite subset N of K , there is a nonempty compact convex subset L_N of K containing N such that, for $y \in L_N \setminus B$, there exist $x \in L_N$, $z \in G(y)$ and $u \in H(x, y)$ such that

$$\max\langle M(y, f(y)), \eta(x, z) \rangle + u \in -int C(y).$$

Then there exists $\bar{x} \in K$ such that

$$\max\langle M(\bar{x}, f(\bar{x})), \eta(x, z) \rangle + u \notin -int C(\bar{x})$$

for any $x \in K$, $z \in G(\bar{x})$ and $u \in H(x, \bar{x})$.

Proof. Let $A = \{(x, y) \in K \times K : \max\langle M(y, f(y)), \eta(x, z) \rangle + u \notin -int C(y) \text{ for any } z \in G(y) \text{ and } u \in H(x, y)\}$. It is easily shown that $(x, x) \in A$ for $x \in K$ from the condition (i). And $A_x = \{y \in K : (x, y) \in A\}$, $x \in K$, is closed. In

fact, for any net $\{y_\alpha\}$ in A_x converging to y , we have $\max\langle M(y_\alpha, f(y_\alpha)), \eta(x, z_\alpha) \rangle + u_\alpha \notin -int C(y_\alpha)$ for any $z_\alpha \in G(y_\alpha)$ and $u_\alpha \in H(x, y_\alpha)$. From Lemma 2.1 and the condition (i), $\max\langle M(y, f(y)), \eta(x, z) \rangle + u \notin -int C(y)$ for any $z \in G(y)$ and $u \in H(x, y)$, so that we have $y \in A_x$, which shows that A_x is closed for $x \in K$. And $A^y = \{x \in K : (x, y) \notin A\}$, $y \in K$, is convex. Indeed, let $x_1, x_2 \in A^y$ and $\lambda \in [0, 1]$. Then, from the fact that $(x_1, y) \notin A$ for any $s \in \overline{F}(y)$, there exist $z_1 \in \overline{G}(y)$ and $u_1 \in H(x_1, y)$ such that

$$\max\langle M(y, s), \eta(x_1, z_1) \rangle + u_1 \in -int C(y)$$

and from the fact that $(x_2, y) \notin A$ for any $s \in \overline{F}(y)$, there exist $z_2 \in \overline{G}(y)$ and $u_2 \in H(x_2, y)$ such that

$$\max\langle M(y, s), \eta(x_2, z_2) \rangle + u_2 \in -int C(y).$$

On the other hand, from the convexity of G , \overline{G} is convex-valued due to Lemma 2.4(1). Hence, for any $s \in \overline{F}(y)$, there exist $u \in H(\lambda x_1 + (1 - \lambda)x_2, y)$ and $z := \lambda z_1 + (1 - \lambda)z_2 \in \overline{G}(y)$ for $\lambda \in [0, 1]$ such that

$$\begin{aligned} & \max\langle M(y, s), \eta(\lambda x_1 + (1 - \lambda)x_2, z) \rangle + u \\ &= \max\langle M(y, s), \eta(\lambda x_1 + (1 - \lambda)x_2, \lambda z_1 + (1 - \lambda)z_2) \rangle \\ & \quad + u \\ &= \max\langle M(y, s), (\lambda \eta(x_1, z_1) + (1 - \lambda)\eta(x_2, z_2)) \rangle + u \\ & \leq \lambda \max\langle M(y, s), \eta(x_1, z_1) \rangle + (1 - \lambda) \max\langle M(y, s), \eta(x_2, z_2) \rangle \\ & \quad + u \\ & \in \lambda \max\langle M(y, s), \eta(x_1, z_1) \rangle + (1 - \lambda) \max\langle M(y, s), \eta(x_2, z_2) \rangle \\ & \quad + \lambda u_1 + (1 - \lambda)u_2 - P \\ &= \lambda(\max\langle M(y, s), \eta(x_1, z_1) \rangle + u_1) \\ & \quad + (1 - \lambda)(\max\langle M(y, s), \eta(x_2, z_2) \rangle + u_2) - P \\ & \subseteq -int C(y) - int C(y) - C(y) \\ &= -int C(y). \end{aligned}$$

Thus $\lambda x_1 + (1 - \lambda)x_2 \in A^y$, which shows that A^y is convex. Further, note that the condition (ii) implies that, for $y \in L_N \setminus B$, there exists $x \in L_N$ such that $y \notin A_x$. Hence the condition (iv) of Theorem 2.1 is satisfied. Thus there exists $\bar{x} \in K$ such that

$$\max\langle M(\bar{x}, f(\bar{x})), \eta(x, z) \rangle + u \notin -int C(\bar{x})$$

for any $x \in K$, $z \in G(\bar{x})$ and $u \in H(x, \bar{x})$. This completes the proof.

Now, we show the existence of solution for the problem (F-WVQVLI).

Theorem 3.1. Let K be a nonempty paracompact convex subset of X and D be a nonempty closed convex subset of Y . Let $F : K \rightarrow \mathfrak{S}(D)$ be a convex fuzzy mapping with weakly open fuzzy set-values and nonempty strong cut set $[F_x]_{\beta(x)}$ for an u.s.c. function $\beta : X \rightarrow (0, 1]$, $G : K \rightarrow \mathfrak{S}(K)$ be a convex fuzzy mapping with topologically open fuzzy set-values and nonempty cut set $(G_x)_\gamma$ for $\gamma \in (0, 1]$, a multivalued mapping $W : K \rightarrow 2^Z$ defined by $W(x) = Z \setminus \{-int C(x)\}$, $x \in K$, be closed, and $M : K \times D \rightarrow 2^{L(X,Z)}$ be a multivalued mapping. Let $\eta : X \times X \rightarrow X$ be linear, $y \mapsto \eta(\cdot, y)$ continuous and $H : K \times K \rightarrow 2^Z$ be P -convex with respect to the first variable and l.s.c. with respect to the second, where $P = \bigcap_{x \in K} C(x)$.

Suppose further that

- (i) $\max\langle M(y_\alpha, s_\alpha), \eta(x, z_\alpha) \rangle \rightarrow \max\langle M(y, s), \eta(x, z) \rangle$ provided that $y_\alpha \rightarrow y, s_\alpha \rightarrow s$ and $z_\alpha \rightarrow z$,
- (ii) $\langle M(x, \cdot), \eta(x, \cdot) \rangle = 0$ and $H(x, x) = \{0\}$ for all $x \in K$,
- (iii) there is a nonempty compact subset B of K such that for any nonempty finite subset N of K , there is a nonempty compact convex subset L_N of K containing N such that for any $y \in L_N \setminus B$, there exist $x \in L_N, z \in (G_y)_\gamma$ and $u \in H(x, y)$ such that

$$\max\langle M(y, s), \eta(x, z) \rangle + u \in -int C(y)$$

for any $s \in (F_y)_{\beta(y)}$.

Then the problem (F-WVQVLI) is solvable, i.e., there exist $\bar{x} \in K$ and $\bar{s} \in (F_{\bar{x}})_{\beta(\bar{x})}$ such that

$$\max\langle M(\bar{x}, \bar{s}), \eta(x, z) \rangle + u \notin -int C(\bar{x})$$

for any $x \in K, z \in (G_{\bar{x}})_\gamma$ and $u \in H(x, \bar{x})$.

Proof. Define two multivalued mappings $\bar{F} : K \rightarrow 2^D$ and $\bar{G} : K \rightarrow 2^K$ by $\bar{F}(x) = [F_x]_{\beta(x)}$ and $\bar{G}(x) = (G_x)_\gamma$, for $x \in K$, respectively. It follows from Lemma 2.3 that \bar{G} is l.s.c. and has nonempty convex-values and, from Lemma 2.4, that \bar{F} has nonempty convex-values such that $\bar{F}^-(y)$ is open in X for $y \in D$. Thus, by Lemma 3.1, there exists a continuous function $f : K \rightarrow D$ such that $f(x) \in \bar{F}(x)$ for $x \in K$. So, by Proposition 3.1, there exists $\bar{x} \in K$ such that

$$\max\langle M(\bar{x}, f(\bar{x})), \eta(x, z) \rangle + u \notin -int C(\bar{x})$$

for any $x \in K, z \in (G_{\bar{x}})_\gamma$ and $u \in H(x, \bar{x})$. Letting $\bar{s} = f(\bar{x})$, we obtain the desired conclusion of Theorem 3.1.

From Theorem 3.1, we obtain the following theorem for Stampacchia type of the generalized weak vector quasivariational-like inequalities (WVQVLI) for multivalued mappings.

Theorem 3.2. Let K be a nonempty paracompact convex subset of X and D be a nonempty closed convex subset of Y . Let $F : K \rightarrow 2^D$ be a multivalued mapping with nonempty convex-values and open lower sections, $G : K \rightarrow 2^K$ be a multivalued l.s.c. mapping with nonempty convex-values, a multivalued mapping $W : K \rightarrow 2^Z$ defined by $W(x) = Z \setminus \{-int C(x)\}$, $x \in K$, closed, and $M : K \times D \rightarrow 2^{L(X,Z)}$ be a multivalued mapping. Let $\eta : X \times X \rightarrow X$ be linear, $y \mapsto \eta(\cdot, y)$ be continuous and $H : K \times K \rightarrow 2^Z$ be P -convex with respect to the first variable and l.s.c. with respect to the second, where $P = \bigcap_{x \in K} C(x)$.

Suppose further that

- (i) $\max\langle M(y_\alpha, s_\alpha), \eta(x, z_\alpha) \rangle \rightarrow \max\langle M(y, s), \eta(x, z) \rangle$ provided that $y_\alpha \rightarrow y, s_\alpha \rightarrow s$ and $z_\alpha \rightarrow z$,
- (ii) $\langle M(x, \cdot), \eta(x, \cdot) \rangle = 0$ and $H(x, x) = \{0\}$ for all $x \in K$,
- (iii) there is a nonempty compact subset B of K such that, for any nonempty finite subset N of K , there is a nonempty compact convex subset L_N of K containing N such that, for any $y \in L_N \setminus B$, there exist $x \in L_N, z \in G(y)$ and $u \in H(x, y)$ such that

$$\max\langle M(y, s), \eta(x, z) \rangle + u \in -int C(y), \quad s \in F(y).$$

Then the problem (WVQVLI) is solvable.

For the compact set case, by using Ky Fan's section theorem [7], we obtain the following existence of solutions for the vector variational inequalities (F-VQVLI), (F-VVI), (VQVLI) as special cases of (WVQVLI).

Theorem 3.3. ([8]) Let K be a nonempty compact convex subset of X and D be a nonempty closed convex subset of Y . Let $F : K \rightarrow \mathfrak{S}(D)$ be a convex fuzzy mapping with closed fuzzy set-values, $G : K \rightarrow \mathfrak{S}(K)$ be a convex fuzzy mapping with topologically open fuzzy set-values, $M : K \times D \rightarrow 2^{L(X,Z)}$ be a multivalued mapping and a multivalued mapping $W : K \rightarrow 2^Z$ defined by $W(x) = Z \setminus \{-int C(x)\}$, $x \in K$, be closed. Let $\eta : X \times X \rightarrow X$ be linear, $y \mapsto \eta(\cdot, y)$ be continuous, and $H : K \times K \rightarrow 2^Z$ be P -convex with respect to the first variable and l.s.c. with respect to the second, where $P := \bigcap_{x \in K} C(x)$.

Suppose further that

- (i) there exist a l.s.c. function $\beta : X \rightarrow (0, 1]$ and a constant $\gamma \in (0, 1]$ such that for any $x \in K$, the cut sets $(F_x)_{\beta(x)}$ and $(G_x)_\gamma$ are nonempty;

- (i) $\bigcup_{x \in K} (F_x)_{\beta(x)}$ is contained in some compact subset of D ;
- (ii) $\max\langle M(y_\alpha, s_\alpha), \eta(x, z_\alpha) \rangle$ converges to $\max\langle M(y, s), \eta(x, z) \rangle$ provided that $y_\alpha \rightarrow y, s_\alpha \rightarrow s$ and $z_\alpha \rightarrow z$;
- (iii) $\langle M(x, \cdot), \eta(x, \cdot) \rangle = 0$ and $H(x, x) = \{0\}$ for all $x \in K$.

Then the problem (F-VQVLI) is solvable from Theorem 2.1.

Corollary 3.4. ([13]) Let K be a nonempty compact convex subset of X . Let $F : K \rightarrow \mathfrak{S}(L(X, Y))$ be a fuzzy mapping with closed fuzzy set-values, $G : K \rightarrow \mathfrak{S}(K)$ be a convex fuzzy mapping with topologically open fuzzy set-values and a multivalued mapping $W : K \rightarrow 2^Y$ defined by $W(x) = Y \setminus [-int C(x)]$, $x \in K$, be closed, where $\{C(x) : x \in K\}$ is a family of solid convex cones in Y . Let $P = \bigcap_{x \in K} C(x)$ and $h : K \rightarrow Y$ be a continuous P -convex function. Suppose further that

- (i) there exist a l.s.c. function $\beta : X \rightarrow (0, 1]$ and a constant $\gamma \in (0, 1]$ such that for any $x \in K$ the cut sets $(F_x)_{\beta(x)}$ and $(G_x)_\gamma$ are nonempty;
- (ii) $\bigcup_{x \in K} (F_x)_{\beta(x)}$ is contained in some compact subset of $L(X, Y)$;
- (iii) for any $x \in K$, there exists $s \in (F_x)_{\beta(x)}$ such that $\langle s, x - z \rangle \notin -int C(x)$ for any $z \in (G_x)_\gamma$.

Then the following variational inequality:

(F-VVI) Find $\bar{x} \in K$ such that, for any $x \in K$, there exists $\bar{s} \in (F_{\bar{x}})_{\beta(\bar{x})}$ such that

$$\langle \bar{s}, x - z \rangle + h(x) - h(\bar{x}) \notin -int C(\bar{x}), \quad z \in (G_{\bar{x}})_\gamma,$$

is solvable.

The following theorem for the existence of solutions for (VQVLI) is a special case of Theorem 3.3.

Corollary 3.5. ([8]) Let K be a nonempty compact convex subset of X and D be a nonempty subset of Y . Let $F : K \rightarrow 2^D$ be closed, $G : K \rightarrow 2^K$ be l.s.c. and nonempty convex-valued, $M : K \times D \rightarrow 2^{L(X, Z)}$ be nonempty compact-valued and a multivalued mapping $W : K \rightarrow 2^Z$ defined by $W(x) = Z \setminus \{-int C(x)\}$, $x \in K$, be closed. Let $\eta : X \times X \rightarrow X$ be linear and $H : K \times K \rightarrow 2^Z$ be P -convex with respect to the first variable and l.s.c. with respect to the second, where $P := \bigcap_{x \in K} C(x)$.

Suppose further that

- (i) $\langle M(x, \cdot), \eta(x, \cdot) \rangle = 0$ and $H(x, x) = \{0\}$ for all $x \in K$;
- (ii) F is compact;
- (iii) $\max\langle M(y_\alpha, s_\alpha), \eta(x, z_\alpha) \rangle$ converges to $\max\langle M(y, s), \eta(x, z) \rangle$ provided that $y_\alpha \rightarrow y, s_\alpha \rightarrow s$ and $z_\alpha \rightarrow z$.

Then the problem (VQVLI) is solvable.

Remark 3.1. Corollary 3.5 is a generalization of many outcomes in [1-5, 8, 10, 11, 14] and theirin.

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