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ESTIMATION OF THE SECOND ORDER PARAMETER CHARACTERIZING THE TAIL BEHAVIOR OF PROBABILITY DISTRIBUTIONS: ASYMPTOTIC NORMALITY†

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ABSTRACT

Yun (2005) introduced an estimator of the second order parameter characterizing the tail behavior of probability distributions and proved its consistency. In this paper we prove its asymptotic normality under a third order condition.

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1. INTRODUCTION

Let $X_1, \ldots, X_n$ be an independent and identically distributed (iid) sample from an unknown distribution function (*d.f.*) F. Suppose F belongs to the domain of attraction of an extreme value *d.f.* G_β for some $\beta \in \mathbb{R}$ $[F \in \mathcal{D}(G_\beta)]$, where

$$G_\beta(x) := \exp\{-(1+\beta x)^{-1/\beta}\},\ 1+\beta x > 0.$$

Throughout the paper the case $\beta = 0$ is interpreted as the limit when $\beta \to 0$, so that $G_0(x) = \exp(-e^{-x})$, $x \in \mathbb{R}$.

The parameter β is called the extreme value index (or tail index) of F, which represents how heavy the right tail of F is. F is said to have a short, medium or heavy tail, respectively, if $\beta < 0$, $\beta = 0$ or $\beta > 0$. There is a rich literature on

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the estimation of β using the sample $X_1, \dots, X_n$. For example, see Hill (1975), Pickands (1975), Dekkers and de Haan (1989), Dekkers *et al.* (1989), Drees (1995) and Yun (2002). To prove asymptotic normality of the estimators of β introduced in these papers, one needs to consider a second order condition which contains another parameter $\rho \leq 0$ called the second order parameter.

Those estimators of β are typically defined using m, say, upper order statistics from $X_1, \dots, X_n$. The optimal value of m minimizing the asymptotic mean squared error of the estimator of β depends on ρ, and therefore one needs to estimate ρ using the sample $X_1, \dots, X_n$. For the estimation of ρ, Gomes *et al.* (2002) dealt with the case of $\beta > 0$, *i.e.* heavy tail.

Yun (2005) considered the general case of $\beta \in \mathbb{R}$ and introduced an estimator of ρ under $\beta \in \mathbb{R}$. In this paper we prove asymptotic normality of the Yun estimator for ρ. For this we need to assume a third order condition.

2. Main Results

Let the function U be defined by $U(x) := F^{-1}(1 - 1/x)$, $x > 1$, where F^{-1} denotes the quantile function of F. Then a necessary and sufficient condition for $F \in \mathcal{D}(G_\beta)$ for some $\beta \in \mathbb{R}$ is the existence of a function $a(t) > 0$ such that, for $x > 0$,

$$\lim_{t \to \infty} \frac{U(tx) - U(t)}{a(t)} = \frac{x^\beta - 1}{\beta} \tag{2.1}$$

(*cf.* de Haan, 1984). In this case the function $a(t)$ is regularly varying at infinity with index β [$a(t) \in RV_\beta$]. If (2.1) holds and $x_F := \sup\{x : F(x) < 1\} > 0$, then it also holds that, for $x > 0$,

$$\lim_{t \to \infty} \frac{\log U(tx) - \log U(t)}{a(t)/U(t)} = \frac{x^{\beta_-} - 1}{\beta_-} =: D(x; \beta_-), \tag{2.2}$$

where $\beta_- := \min\{\beta, 0\}$.

For asymptotic normality of estimators of β appeared in the literature, one needs to consider a second order condition that, for $x > 0$,

$$\frac{\log U(tx) - \log U(t)}{a(t)/U(t)} = D(x; \beta_-) + A(t)H(x; \beta_-, \rho) + o(A(t)) \quad \text{as } t \to \infty, \tag{2.3}$$

where $A(t)$ is a function of constant sign for large values of t, satisfying $A(t) = o(1)$ as $t \to \infty$, and

$$H(x; \beta_-, \rho) := \frac{1}{\rho} \left(\frac{x^{\beta_- + \rho} - 1}{\beta_- + \rho} - \frac{x^{\beta_-} - 1}{\beta_-} \right)$$

for some $\rho \le 0$ (*cf.* de Haan and Stadtmüller, 1996). In this case we must have $|A(t)| \in RV_\rho$. Throughout the case $\rho = 0$ is interpreted as the limit when $\rho \to 0$. Some examples of F with the corresponding β, $A(t)$ and ρ are given in Table 2.1.

TABLE 2.1 *Examples of F with the corresponding β, $A(t)$ and ρ*

$F(x)$	β	$A(t)$	ρ
	>1	$-(1/2)t^{-1}$	-1
	1	$-(3/2)t^{-1}$	-1
	$(0,1)$	$-\beta t^{-\beta}$	$-\beta$
$G_\beta(x)$	0	$-(\log t)^{-1}$	0
	$(-1,0)$	βt^{β}	β
	-1	$-2t^{-1}$	-1
	<-1	$((\beta-1)/2)t^{-1}$	-1
Cauchy	1	$-(4/3)\pi^2 t^{-2}$	-2
Normal	0	$-(\log t)^{-1}$	0
Uniform(0,1)	-1	$-t^{-1}$	-1
	>0	$-\beta t^{-\beta}$	$-\beta$
$1-(1+\beta x)^{-1/\beta}$, $x>0$, $1+\beta x>0$	0	$-(\log t)^{-1}$	0
	<0	βt^{β}	β

Under (2.3) which is clearly a stronger condition than (2.2), Yun (2005) introduced an estimator of the second order parameter ρ, based on the sample $X_1, \ldots, X_n$. Specifically, let $X_1^{(n)} \ge X_2^{(n)} \ge \cdots \ge X_n^{(n)}$ denote the descending order statistics of $X_1, \ldots, X_n$. For $k = 1, 2, \ldots$, we define

$$M_{n,m}^{(k)} := \frac{1}{m}\sum_{j=1}^{m}(\log X_j^{(n)} - \log X_{m+1}^{(n)})^k \quad \text{and}$$

$$N_{n,m}^{(k)} := \frac{\left(M_{n,m}^{(1)}\right)^k}{M_{n,m}^{(k)}},$$

provided that $X_{m+1}^{(n)} > 0$, where $1 \le m < n$. For $k = 2, 3, \ldots$, we also define the function $\phi_k : (-\infty, 1/k) \to (0,1)$ by

$$\phi_k(x) := \frac{\prod_{j=2}^{k}(1-jx)}{k!(1-x)^{k-1}}, \quad x < \frac{1}{k},$$

which is strictly decreasing, and let $\phi_k^{-1} : (0,1) \to (-\infty, 1/k)$ denote its inverse function. Let the functions c_i, $i = 1, 2, 3, 4$, be defined by

$$c_1(x) := 2(1-4x)(3-17x+23x^2),$$

$$c_2(x) := -(1-3x)(6-33x+46x^2),$$
$$c_3(x) := 2(3-17x+23x^2),$$
$$c_4(x) := -(1-3x)(3-8x).$$

Yun (2005) defined an estimator $\hat{\rho}_{n,m}$ of ρ by

$$\hat{\rho}_{n,m} := \frac{c_1(\widehat{\beta_-}) + c_2(\widehat{\beta_-})R_{n,m}}{c_3(\widehat{\beta_-}) + c_4(\widehat{\beta_-})R_{n,m}},$$

where $1 \leq m < n$ and

$$\widehat{\beta_-} := \phi_2^{-1}(N_{n,m}^{(2)}),$$
$$R_{n,m} := \frac{\phi_2^{-1}(N_{n,m}^{(2)}) - \phi_3^{-1}(N_{n,m}^{(3)})}{\phi_3^{-1}(N_{n,m}^{(3)}) - \phi_4^{-1}(N_{n,m}^{(4)})},$$

and proved its consistency.

In this paper we prove asymptotic normality of $\hat{\rho}_{n,m}$. For this we here assume a third order condition, a stronger condition than (2.3), that, for $x > 0$,

$$\frac{\log U(tx) - \log U(t)}{a(t)/U(t)} = D(x;\beta_-) + A(t)H(x;\beta_-,\rho) + A(t)A_1(t)H_1(x;\beta_-,\rho,\rho_1) + o(A(t)A_1(t)) \quad \text{as } t \to \infty, \tag{2.4}$$

where $A_1(t)$ is a function of constant sign for large values of t, satisfying $A_1(t) = o(1)$ as $t \to \infty$, and

$$\begin{aligned} & H_1(x;\beta_-,\rho,\rho_1) \\ & := \frac{1}{\rho_1}\left\{\frac{1}{\rho+\rho_1}\left(\frac{x^{\beta_-+\rho+\rho_1}-1}{\beta_-+\rho+\rho_1} - \frac{x^{\beta_-}-1}{\beta_-}\right) - \frac{1}{\rho}\left(\frac{x^{\beta_-+\rho}-1}{\beta_-+\rho} - \frac{x^{\beta_-}-1}{\beta_-}\right)\right\} \\ & = \frac{1}{\rho_1}\Big\{H(x;\beta_-,\rho+\rho_1) - H(x;\beta_-,\rho)\Big\} \end{aligned}$$

for some $\rho_1 \leq 0$. In this case we must have $|A_1(t)| \in RV_{\rho_1}$. As in the case of $\beta = 0$ and $\rho = 0$, the case $\rho_1 = 0$ is also interpreted as the limit when $\rho_1 \to 0$.

For $k = 0, 1, 2, \ldots$ and for $\alpha \leq 0$ and $\beta \in \mathbb{R}$, define

$$s(k;\alpha,\beta_-) := \int_1^\infty x^{\alpha-2} D^k(x;\beta_-)\,dx = \frac{k!}{\prod_{j=0}^k (1-\alpha-j\beta_-)}.$$

Notice that $s(k;\alpha,\beta_-) = E(Z^\alpha D^k(Z;\beta_-))$ if Z is a random variable with *d.f.* $1 - 1/x$, $x > 1$. The following lemma, an extension of Lemma 5.1 of Draisma *et al.* (1999), is needed to prove asymptotic normality of $\hat{\rho}_{n,m}$. For the proof, the reader is referred to Lemma 2.1 of Yun (2005). By $\xrightarrow{d}$ and $\xrightarrow{p}$ we denote convergence in distribution and convergence in probability, respectively.

LEMMA 2.1. *Let $Y_1, \ldots, Y_n$ be iid random variables with d.f. $1-1/x$, $x>1$, and let $Y_1^{(n)} \geq Y_2^{(n)} \geq \cdots \geq Y_n^{(n)}$ be their descending order statistics. Let $m = m(n)$ be any sequence of integers such that $1 \leq m < n$ and $m \to \infty$ as $n \to \infty$.*

(i) *For $k = 1, 2, \ldots$ and for $\beta \in \mathbb{R}$, define*

$$Q_{n,m}^{(k)}(\beta_-) := \frac{1}{m}\sum_{j=1}^{m} D^k\left(\frac{Y_j^{(n)}}{Y_{m+1}^{(n)}}; \beta_-\right) - s(k; 0, \beta_-).$$

Then, for $k = 1, 2, \ldots$ and for $\beta \in \mathbb{R}$,

$$\sqrt{m}(Q_{n,m}^{(1)}(\beta_-), \ldots, Q_{n,m}^{(k)}(\beta_-)) \xrightarrow{d} (Q_1(\beta_-), \ldots, Q_k(\beta_-)) \text{ as } n \to \infty,$$

where $(Q_1(\beta_-), \ldots, Q_k(\beta_-))$ has a k-variate normal distribution with mean vector $(0, \ldots, 0)$ and covariance matrix $(v_{ij}(\beta_-))$ given by

$$v_{ij}(\beta_-) := s(i+j; 0, \beta_-) - s(i; 0, \beta_-)s(j; 0, \beta_-), \quad i, j = 1, \ldots, k.$$

(ii) *For $k = 1, 2, \ldots$ and for $\beta \in \mathbb{R}$, $\rho \leq 0$ and $\rho_1 \leq 0$, as $n \to \infty$,*

$$\frac{1}{m}\sum_{j=1}^{m} H\left(\frac{Y_j^{(n)}}{Y_{m+1}^{(n)}}; \beta_-, \rho\right) D^{k-1}\left(\frac{Y_j^{(n)}}{Y_{m+1}^{(n)}}; \beta_-\right) \xrightarrow{p} b_1(k; \beta_-, \rho),$$

$$\frac{1}{m}\sum_{j=1}^{m} H_1\left(\frac{Y_j^{(n)}}{Y_{m+1}^{(n)}}; \beta_-, \rho, \rho_1\right) D^{k-1}\left(\frac{Y_j^{(n)}}{Y_{m+1}^{(n)}}; \beta_-\right) \xrightarrow{p} b_2(k; \beta_-, \rho, \rho_1),$$

$$\frac{1}{m}\sum_{j=1}^{m} H^2\left(\frac{Y_j^{(n)}}{Y_{m+1}^{(n)}}; \beta_-, \rho\right) D^{k-2}\left(\frac{Y_j^{(n)}}{Y_{m+1}^{(n)}}; \beta_-\right) \xrightarrow{p} b_3(k; \beta_-, \rho), \ k \geq 2,$$

where

$$\begin{aligned} b_1(k; \beta_-, \rho) &:= \int_1^\infty x^{-2} H(x; \beta_-, \rho) D^{k-1}(x; \beta_-)\, dx \\ &= \frac{1}{\rho}\left\{\frac{s(k-1; \beta_- + \rho, \beta_-) - s(k-1; 0, \beta_-)}{\beta_- + \rho} - s(k; 0, \beta_-)\right\}, \end{aligned}$$

$$
\begin{aligned}
b_2(k;\beta_-,\rho,\rho_1) &:= \int_1^\infty x^{-2}H_1(x;\beta_-,\rho,\rho_1)D^{k-1}(x;\beta_-)\,dx \\
&= \frac{1}{\rho_1}\Big\{b_1(k;\beta_-,\rho+\rho_1)-b_1(k;\beta_-,\rho)\Big\},\\
b_3(k;\beta_-,\rho) &:= \int_1^\infty x^{-2}H^2(x;\beta_-,\rho)D^{k-2}(x;\beta_-)\,dx \\
&= \frac{1}{\rho^2}\Bigg[\frac{1}{(\beta_-+\rho)^2}\Big\{s(k-2;2\beta_-+2\rho,\beta_-)\\
&\quad -2s(k-2;\beta_-+\rho,\beta_-)+s(k-2;0,\beta_-)\Big\}\\
&\quad -2\rho b_1(k;\beta_-,\rho)-s(k;0,\beta_-)\Bigg].
\end{aligned}
$$

In the following theorem we establish asymptotic normality of $\hat{\rho}_{n,m}$. A sequence of positive integers $m=m(n)$ is called an intermediate sequence if $m\to\infty$ and $m/n\to 0$ as $n\to\infty$.

THEOREM 2.2. *Suppose $F\in\mathcal{D}(G_\beta)$ for some $\beta\in\mathbb{R}$ and $x_F>0$ and that (2.4) holds for some $\rho<0$ and $\rho_1\le 0$. Let $m=m(n)$ be an intermediate sequence such that $\sqrt{m}|A(n/m)|\to\infty$, $\sqrt{m}A(n/m)A_1(n/m)\to\lambda_1$ and $\sqrt{m}A^2(n/m)\to\lambda_2$ as $n\to\infty$ for some $\lambda_1\in(-\infty,\infty)$ and $\lambda_2\in[0,\infty)$. Then we have*

$$
\sqrt{m}A(n/m)(\hat{\rho}_{n,m}-\rho)\xrightarrow{d} N(e_0(\lambda_1,\lambda_2,\beta_-,\rho,\rho_1),\sigma^2(\beta_-,\rho)) \text{ as } n\to\infty,
$$

where

$$
\begin{aligned}
\sigma^2(\beta_-,\rho) := \left(\frac{\prod_{j=1}^4(1-j\beta_--\rho)^2}{\rho^2\prod_{j=2}^8(1-j\beta_-)}\right)&\Big\{(1-2\rho+2\rho^2)-\beta_-(13-4\rho+4\rho^2)\\
&+19\beta_-^2(7-2\rho+2\rho^2)-3\beta_-^3(241-28\rho+28\rho^2)\\
&+2\beta_-^4(1129-48\rho+48\rho^2)-3720\beta_-^5+2496\beta_-^6\Big\}
\end{aligned}
$$

and the formula for the bias term $e_0(\lambda_1,\lambda_2,\beta_-,\rho,\rho_1)$ is given in the proof.

PROOF. Let $m=m(n)$ be any intermediate sequence. Let $Y_1,\dots,Y_n$ be *iid* random variables with *d.f.* $1-1/x$, $x>1$, and let $Y_1^{(n)}\ge Y_2^{(n)}\ge\cdots\ge Y_n^{(n)}$ be their order statistics. Then $(X_j)_{j=1}^n\overset{d}{=}(U(Y_j))_{j=1}^n$ and so $(X_j^{(n)})_{j=1}^n\overset{d}{=}(U(Y_j^{(n)}))_{j=1}^n$. By (2.4), we have, for $k=1,2,\dots$ and for $x>0$, as $t\to\infty$,

$$\begin{aligned}&\left(\frac{\log U(tx)-\log U(t)}{a(t)/U(t)}\right)^k\\&= D^k(x;\beta_-)+k\Big\{A(t)H(x;\beta_-,\rho)+A(t)A_1(t)H_1(x;\beta_-,\rho,\rho_1)\Big\}D^{k-1}(x;\beta_-)\\&\quad+\frac{k(k-1)}{2}A^2(t)H^2(x;\beta_-,\rho)D^{k-2}(x;\beta_-)+I(k\geq 3)\cdot o(A^2(t))\\&\quad+o(A(t)A_1(t)),\end{aligned}$$

where $I(k \geq 3)$ denotes the indicator function of the set $\{k \geq 3\}$. Replacing t by $Y_{m+1}^{(n)}$ and x by $Y_j^{(n)}/Y_{m+1}^{(n)}$, adding the equalities for $j = 1, \ldots, m$ and dividing by m, we have, for $k = 1, 2, \ldots$, as $n \to \infty$,

$$\begin{aligned}&\frac{1}{m}\sum_{j=1}^m\left(\frac{\log U(Y_j^{(n)})-\log U(Y_{m+1}^{(n)})}{a(Y_{m+1}^{(n)})/U(Y_{m+1}^{(n)})}\right)^k\\&=\frac{1}{m}\sum_{j=1}^m\Bigg[D^k\left(\frac{Y_j^{(n)}}{Y_{m+1}^{(n)}};\beta_-\right)+k\Bigg\{A(Y_{m+1}^{(n)})H\left(\frac{Y_j^{(n)}}{Y_{m+1}^{(n)}};\beta_-,\rho\right)\\&\quad+A(Y_{m+1}^{(n)})A_1(Y_{m+1}^{(n)})H_1\left(\frac{Y_j^{(n)}}{Y_{m+1}^{(n)}};\beta_-,\rho,\rho_1\right)\Bigg\}D^{k-1}\left(\frac{Y_j^{(n)}}{Y_{m+1}^{(n)}};\beta_-\right)\\&\quad+\frac{k(k-1)}{2}A^2(Y_{m+1}^{(n)})H^2\left(\frac{Y_j^{(n)}}{Y_{m+1}^{(n)}};\beta_-,\rho\right)D^{k-2}\left(\frac{Y_j^{(n)}}{Y_{m+1}^{(n)}};\beta_-\right)\Bigg]\\&\quad+I(k\geq 3)\cdot o_p(A^2(Y_{m+1}^{(n)}))+o_p(A(Y_{m+1}^{(n)})A_1(Y_{m+1}^{(n)})),\end{aligned}$$

which can be written by Lemma 2.1 as

$$\begin{aligned}&s(k;0,\beta_-)+Q_{n,m}^{(k)}(\beta_-)\\&+k\left\{b_1(k;\beta_-,\rho)A\left(\frac{n}{m}\right)+b_2(k;\beta_-,\rho,\rho_1)A\left(\frac{n}{m}\right)A_1\left(\frac{n}{m}\right)\right\}\\&+\frac{k(k-1)}{2}b_3(k;\beta_-,\rho)A^2\left(\frac{n}{m}\right)+I(k\geq 3)\cdot o_p\left(A^2\left(\frac{n}{m}\right)\right)\\&+o_p\left(A\left(\frac{n}{m}\right)A_1\left(\frac{n}{m}\right)\right)\end{aligned}$$

since $Y_{m+1}^{(n)} \overset{p}{\sim} n/m$ and so since $A(Y_{m+1}^{(n)}) \overset{p}{\sim} A(n/m)$ and $A_1(Y_{m+1}^{(n)}) \overset{p}{\sim} A_1(n/m)$ by applying the uniform convergence theorem to $|A(t)| \in RV_\rho$ and $|A_1(t)| \in RV_{\rho_1}$, respectively. Thus we have, for $k = 1, 2, \ldots$, as $n \to \infty$,

$$\begin{aligned}
M_{n,m}^{(k)} &\overset{d}{=} \left(\frac{a(Y_{m+1}^{(n)})}{U(Y_{m+1}^{(n)})}\right)^k \cdot \frac{1}{m}\sum_{j=1}^m \left(\frac{\log U(Y_j^{(n)}) - \log U(Y_{m+1}^{(n)})}{a(Y_{m+1}^{(n)})/U(Y_{m+1}^{(n)})}\right)^k \\
&= \left(\frac{a(Y_{m+1}^{(n)})}{U(Y_{m+1}^{(n)})}\right)^k \Big[s(k;0,\beta_-) + Q_{n,m}^{(k)}(\beta_-) + k\Big\{ b_1(k;\beta_-,\rho) A\left(\frac{n}{m}\right) \\
&\quad + b_2(k;\beta_-,\rho,\rho_1) A\left(\frac{n}{m}\right) A_1\left(\frac{n}{m}\right)\Big\} + \frac{k(k-1)}{2} b_3(k;\beta_-,\rho) A^2\left(\frac{n}{m}\right) \\
&\quad + I(k \ge 3) \cdot o_p\left(A^2\left(\frac{n}{m}\right)\right) + o_p\left(A\left(\frac{n}{m}\right) A_1\left(\frac{n}{m}\right)\right)\Big]. \qquad (2.5)
\end{aligned}$$

For $k = 2, 3, \ldots$, define

$$B_1(k;\beta_-,\rho) := \frac{k\{b_1(1;\beta_-,\rho)s^{k-1}(1;0,\beta_-) - b_1(k;\beta_-,\rho)\phi_k(\beta_-)\}}{\phi_k'(\beta_-)s(k;0,\beta_-)},$$

$$B_2(k;\beta_-,\rho,\rho_1) := \frac{k\{b_2(1;\beta_-,\rho,\rho_1)s^{k-1}(1;0,\beta_-) - b_2(k;\beta_-,\rho,\rho_1)\phi_k(\beta_-)\}}{\phi_k'(\beta_-)s(k;0,\beta_-)},$$

$$\begin{aligned}
B_3(k;\beta_-,\rho) &:= \frac{1}{\phi_k'(\beta_-)s(k;0,\beta_-)}\left[\frac{k(k-1)}{2}\Big\{ b_1^2(1;\beta_-,\rho)s^{k-2}(1;0,\beta_-) \right. \\
&\quad \left. - b_3(k;\beta_-,\rho)\phi_k(\beta_-)\Big\} - k b_1(k;\beta_-,\rho)\phi_k'(\beta_-)B_1(k;\beta_-,\rho)\right].
\end{aligned}$$

By the Taylor expansion, we have, for $k = 2, 3, \ldots$, as $n \to \infty$,

$$\phi_k^{-1}(N_{n,m}^{(k)}) - \beta_- = \frac{1}{\phi_k'(\beta_-)}(N_{n,m}^{(k)} - \phi_k(\beta_-)) + o_p(N_{n,m}^{(k)} - \phi_k(\beta_-))$$

and thus

$$\begin{aligned}
&\phi_k^{-1}(N_{n,m}^{(k)}) - \beta_- - B_1(k;\beta_-,\rho)A\left(\frac{n}{m}\right) \\
&= \frac{(a(Y_{m+1}^{(n)})/U(Y_{m+1}^{(n)}))^k}{\phi_k'(\beta_-)M_{n,m}^{(k)}}\left[\left(\frac{M_{n,m}^{(1)}}{a(Y_{m+1}^{(n)})/U(Y_{m+1}^{(n)})}\right)^k \right. \\
&\quad \left. - \frac{\{\phi_k(\beta_-) + \phi_k'(\beta_-)B_1(k;\beta_-,\rho)A(n/m)\}M_{n,m}^{(k)}}{(a(Y_{m+1}^{(n)})/U(Y_{m+1}^{(n)}))^k}\right] \\
&\quad + o_p(N_{n,m}^{(k)} - \phi_k(\beta_-)), \qquad (2.6)
\end{aligned}$$

where we have from (2.5), as $n \to \infty$,

$$\begin{aligned}
&\left(\frac{M_{n,m}^{(1)}}{a(Y_{m+1}^{(n)})/U(Y_{m+1}^{(n)})}\right)^k - \frac{\{\phi_k(\beta_-)+\phi_k'(\beta_-)B_1(k;\beta_-,\rho)A(n/m)\}M_{n,m}^{(k)}}{(a(Y_{m+1}^{(n)})/U(Y_{m+1}^{(n)}))^k} \\
&\overset{d}{=} \Big\{s(1;0,\beta_-)+Q_{n,m}^{(1)}(\beta_-)+b_1(1;\beta_-,\rho)A\left(\frac{n}{m}\right) \\
&\quad +b_2(1;\beta_-,\rho,\rho_1)A\left(\frac{n}{m}\right)A_1\left(\frac{n}{m}\right)+o_p\left(A\left(\frac{n}{m}\right)A_1\left(\frac{n}{m}\right)\right)\Big\}^k \\
&\quad -\Big\{\phi_k(\beta_-)+\phi_k'(\beta_-)B_1(k;\beta_-,\rho)A\left(\frac{n}{m}\right)\Big\}\Big[s(k;0,\beta_-)+Q_{n,m}^{(k)}(\beta_-) \\
&\quad +k\Big\{b_1(k;\beta_-,\rho)A\left(\frac{n}{m}\right)+b_2(k;\beta_-,\rho,\rho_1)A\left(\frac{n}{m}\right)A_1\left(\frac{n}{m}\right)\Big\} \\
&\quad +\frac{k(k-1)}{2}b_3(k;\beta_-,\rho)A^2\left(\frac{n}{m}\right)+I(k\ge 3)\cdot o_p\left(A^2\left(\frac{n}{m}\right)\right) \\
&\quad +o_p\left(A\left(\frac{n}{m}\right)A_1\left(\frac{n}{m}\right)\right)\Big] \\
&= \Big\{ks^{k-1}(1;0,\beta_-)Q_{n,m}^{(1)}(\beta_-)-\phi_k(\beta_-)Q_{n,m}^{(k)}(\beta_-)\Big\} \\
&\quad +k\Big\{b_2(1;\beta_-,\rho,\rho_1)s^{k-1}(1;0,\beta_-)-b_2(k;\beta_-,\rho,\rho_1)\phi_k(\beta_-)\Big\}A\left(\frac{n}{m}\right)A_1\left(\frac{n}{m}\right) \\
&\quad +\Bigg[\frac{k(k-1)}{2}\Big\{b_1^2(1;\beta_-,\rho)s^{k-2}(1;0,\beta_-)-b_3(k;\beta_-,\rho)\phi_k(\beta_-)\Big\} \\
&\quad -kb_1(k;\beta_-,\rho)\phi_k'(\beta_-)B_1(k;\beta_-,\rho)\Bigg]A^2\left(\frac{n}{m}\right)+o_p(Q_{n,m}^{(1)}(\beta_-)) \\
&\quad +o_p(Q_{n,m}^{(k)}(\beta_-))+I(k\ge 3)\cdot o_p\left(A^2\left(\frac{n}{m}\right)\right)+o_p\left(A\left(\frac{n}{m}\right)A_1\left(\frac{n}{m}\right)\right) \qquad (2.7)
\end{aligned}$$

since $s^k(1;0,\beta_-) = \phi_k(\beta_-)s(k;0,\beta_-)$. Now suppose that $\sqrt{m}|A(n/m)| \to \infty$, $\sqrt{m}A(n/m)A_1(n/m) \to \lambda_1$ and $\sqrt{m}A^2(n/m) \to \lambda_2$ as $n \to \infty$ for some $\lambda_1 \in (-\infty,\infty)$ and $\lambda_2 \in [0,\infty)$. Then, combining (2.5)–(2.7) with Lemma 2.1, we have, for $k = 2, 3, \ldots$, as $n \to \infty$,

$$\begin{aligned}
&\sqrt{m}A\left(\frac{n}{m}\right)\left(\frac{\phi_k^{-1}(N_{n,m}^{(k)})-\beta_-}{A(n/m)}-B_1(k;\beta_-,\rho)\right) \\
&= \sqrt{m}\left(\phi_k^{-1}(N_{n,m}^{(k)})-\beta_- - B_1(k;\beta_-,\rho)A\left(\frac{n}{m}\right)\right) \\
&\overset{d}{\longrightarrow} \frac{ks^{k-1}(1;0,\beta_-)Q_1(\beta_-)-\phi_k(\beta_-)Q_k(\beta_-)}{\phi_k'(\beta_-)s(k;0,\beta_-)}
\end{aligned}$$

$$+\lambda_1 B_2(k;\beta_-,\rho,\rho_1)+\lambda_2 B_3(k;\beta_-,\rho) =: L_k. \tag{2.8}$$

Applying the Taylor expansion

$$\begin{aligned}
\frac{x_1-x_2}{x_2-x_3} &= \frac{B_1(2;\beta_-,\rho)-B_1(3;\beta_-,\rho)}{B_1(3;\beta_-,\rho)-B_1(4;\beta_-,\rho)}+\frac{x_1-B_1(2;\beta_-,\rho)}{B_1(3;\beta_-,\rho)-B_1(4;\beta_-,\rho)}\\
&\quad +\frac{(B_1(4;\beta_-,\rho)-B_1(2;\beta_-,\rho))}{(B_1(3;\beta_-,\rho)-B_1(4;\beta_-,\rho))^2}(x_2-B_1(3;\beta_-,\rho))\\
&\quad +\frac{(B_1(2;\beta_-,\rho)-B_1(3;\beta_-,\rho))}{(B_1(3;\beta_-,\rho)-B_1(4;\beta_-,\rho))^2}(x_3-B_1(4;\beta_-,\rho))\\
&\quad +o(x_1-B_1(2;\beta_-,\rho))+o(x_2-B_1(3;\beta_-,\rho))+o(x_3-B_1(4;\beta_-,\rho))
\end{aligned}$$

as $(x_1,x_2,x_3)\to(B_1(2;\beta_-,\rho),B_1(3;\beta_-,\rho),B_1(4;\beta_-,\rho))$, we have from (2.8), as $n\to\infty$,

$$\begin{aligned}
&\sqrt{m}A\left(\frac{n}{m}\right)(R_{n,m}-r(\beta_-,\rho))\\
&= \sqrt{m}A\left(\frac{n}{m}\right)\left\{\frac{(\phi_2^{-1}(N_{n,m}^{(2)})-\beta_-)/A(n/m)-(\phi_3^{-1}(N_{n,m}^{(3)})-\beta_-)/A(n/m)}{(\phi_3^{-1}(N_{n,m}^{(3)})-\beta_-)/A(n/m)-(\phi_4^{-1}(N_{n,m}^{(4)})-\beta_-)/A(n/m)}\right.\\
&\quad\left. -\frac{B_1(2;\beta_-,\rho)-B_1(3;\beta_-,\rho)}{B_1(3;\beta_-,\rho)-B_1(4;\beta_-,\rho)}\right\}\\
&= \frac{1}{B_1(3;\beta_-,\rho)-B_1(4;\beta_-,\rho)}\sqrt{m}A\left(\frac{n}{m}\right)\left\{\frac{\phi_2^{-1}(N_{n,m}^{(2)})-\beta_-}{A(n/m)}-B_1(2;\beta_-,\rho)\right\}\\
&\quad +\frac{(B_1(4;\beta_-,\rho)-B_1(2;\beta_-,\rho))}{(B_1(3;\beta_-,\rho)-B_1(4;\beta_-,\rho))^2}\sqrt{m}A\left(\frac{n}{m}\right)\left\{\frac{\phi_3^{-1}(N_{n,m}^{(3)})-\beta_-}{A(n/m)}-B_1(3;\beta_-,\rho)\right\}\\
&\quad +\frac{(B_1(2;\beta_-,\rho)-B_1(3;\beta_-,\rho))}{(B_1(3;\beta_-,\rho)-B_1(4;\beta_-,\rho))^2}\sqrt{m}A\left(\frac{n}{m}\right)\left\{\frac{\phi_4^{-1}(N_{n,m}^{(4)})-\beta_-}{A(n/m)}-B_1(4;\beta_-,\rho)\right\}\\
&\quad +o_p(1)\\
&\xrightarrow{d} \frac{L_2}{B_1(3;\beta_-,\rho)-B_1(4;\beta_-,\rho)}+\frac{(B_1(4;\beta_-,\rho)-B_1(2;\beta_-,\rho))L_3}{(B_1(3;\beta_-,\rho)-B_1(4;\beta_-,\rho))^2}\\
&\quad +\frac{(B_1(2;\beta_-,\rho)-B_1(3;\beta_-,\rho))L_4}{(B_1(3;\beta_-,\rho)-B_1(4;\beta_-,\rho))^2}\\
&= \frac{L_2-(r(\beta_-,\rho)+1)L_3+r(\beta_-,\rho)L_4}{B_1(3;\beta_-,\rho)-B_1(4;\beta_-,\rho)},
\end{aligned} \tag{2.9}$$

where

$$r(\beta_-,\rho) := \frac{B_1(2;\beta_-,\rho) - B_1(3;\beta_-,\rho)}{B_1(3;\beta_-,\rho) - B_1(4;\beta_-,\rho)} = \frac{-c_1(\beta_-) + \rho c_3(\beta_-)}{c_2(\beta_-) - \rho c_4(\beta_-)}.$$

Again, applying the Taylor expansion

$$\frac{c_1(x_1) + c_2(x_1)x_2}{c_3(x_1) + c_4(x_1)x_2} = \frac{c_1(\beta_-) + c_2(\beta_-)r(\beta_-,\rho)}{c_3(\beta_-) + c_4(\beta_-)r(\beta_-,\rho)} + d_1(\beta_-,\rho)(x_1 - \beta_-)$$
$$+ d_2(\beta_-,\rho)(x_2 - r(\beta_-,\rho)) + o(x_1 - \beta_-) + o(x_2 - r(\beta_-,\rho))$$

as $(x_1, x_2) \to (\beta_-, r(\beta_-,\rho))$, where

$$\begin{aligned} d_1(\beta_-,\rho) &:= \frac{1}{(1-2\beta_-)(1-3\beta_-)(3-7\beta_-)(3-17\beta_- + 23\beta_-^2)} \\ &\quad \times\{-3(7+5\rho) + 6\beta_-(37+32\rho-\rho^2) - \beta_-^2(850+855\rho-17\rho^2) \\ &\quad +92\beta_-^3(15+17\rho) - 46\beta_-^4(17+21\rho)\}, \\ d_2(\beta_-,\rho) &:= \frac{-(1-3\beta_-)(6-33\beta_- + 46\beta_-^2 - 3\rho + 8\beta_-\rho)^2}{2(1-2\beta_-)(3-7\beta_-)(3-17\beta_- + 23\beta_-^2)}, \end{aligned}$$

we finally have from (2.8) and (2.9), as $n \to \infty$,

$$\begin{aligned} &\sqrt{m}A\left(\frac{n}{m}\right)(\hat{\rho}_{n,m} - \rho) \\ &= \sqrt{m}A\left(\frac{n}{m}\right)\left\{\frac{c_1(\widehat{\beta_-}) + c_2(\widehat{\beta_-})R_{n,m}}{c_3(\widehat{\beta_-}) + c_4(\widehat{\beta_-})R_{n,m}} - \frac{c_1(\beta_-) + c_2(\beta_-)r(\beta_-,\rho)}{c_3(\beta_-) + c_4(\beta_-)r(\beta_-,\rho)}\right\} \\ &= d_1(\beta_-,\rho)\sqrt{m}A\left(\frac{n}{m}\right)(\widehat{\beta_-} - \beta_-) + d_2(\beta_-,\rho)\sqrt{m}A\left(\frac{n}{m}\right)(R_{n,m} - r(\beta_-,\rho)) \\ &\quad + o_p(1) \\ &\xrightarrow{d} \lambda_2 B_1(2;\beta_-,\rho)d_1(\beta_-,\rho) + d_2(\beta_-,\rho)\frac{\{L_2 - (r(\beta_-,\rho)+1)L_3 + r(\beta_-,\rho)L_4\}}{B_1(3;\beta_-,\rho) - B_1(4;\beta_-,\rho)} \\ &= e_0(\lambda_1,\lambda_2,\beta_-,\rho,\rho_1) + \sum_{i=1}^{4} e_i(\beta_-,\rho)Q_i(\beta_-) \\ &\sim N\Bigg(e_0(\lambda_1,\lambda_2,\beta_-,\rho,\rho_1), \sum_{i=1}^{4}\sum_{j=1}^{4} e_i(\beta_-,\rho)e_j(\beta_-,\rho)v_{ij}(\beta_-)\Bigg), \end{aligned}$$

where

$$e_0(\lambda_1,\lambda_2,\beta_-,\rho,\rho_1) := \frac{\lambda_1 d_2(\beta_-,\rho)}{B_1(3;\beta_-,\rho) - B_1(4;\beta_-,\rho)} \sum_{k=2}^{4} B_2(k;\beta_-,\rho,\rho_1)$$

$$+\lambda_2\left\{B_1(2;\beta_-,\rho)d_1(\beta_-,\rho)\right.$$

$$\left.+\frac{d_2(\beta_-,\rho)}{B_1(3;\beta_-,\rho)-B_1(4;\beta_-,\rho)}\sum_{k=2}^{4}B_3(k;\beta_-,\rho)\right\},$$

$$e_1(\beta_-,\rho) := \frac{s(1;0,\beta_-)d_2(\beta_-,\rho)}{B_1(3;\beta_-,\rho)-B_1(4;\beta_-,\rho)}\left\{\frac{2}{\phi_2'(\beta_-)s(2;0,\beta_-)}\right.$$

$$\left.-\frac{3(r(\beta_-,\rho)+1)s(1;0,\beta_-)}{\phi_3'(\beta_-)s(3;0,\beta_-)}+\frac{4r(\beta_-,\rho)s^2(1;0,\beta_-)}{\phi_4'(\beta_-)s(4;0,\beta_-)}\right\},$$

$$e_2(\beta_-,\rho) := -\frac{d_2(\beta_-,\rho)\phi_2(\beta_-)}{(B_1(3;\beta_-,\rho)-B_1(4;\beta_-,\rho))\phi_2'(\beta_-)s(2;0,\beta_-)},$$

$$e_3(\beta_-,\rho) := \frac{(r(\beta_-,\rho)+1)d_2(\beta_-,\rho)\phi_3(\beta_-)}{(B_1(3;\beta_-,\rho)-B_1(4;\beta_-,\rho))\phi_3'(\beta_-)s(3;0,\beta_-)},$$

$$e_4(\beta_-,\rho) := -\frac{r(\beta_-,\rho)d_2(\beta_-,\rho)\phi_4(\beta_-)}{(B_1(3;\beta_-,\rho)-B_1(4;\beta_-,\rho))\phi_4'(\beta_-)s(4;0,\beta_-)}.$$

With an extra effort one can show that

$$e_1(\beta_-,\rho) = -\frac{(4-9\beta_- -\rho)\prod_{j=1}^{4}(1-j\beta_- -\rho)}{\rho},$$

$$e_2(\beta_-,\rho) = \frac{(6-33\beta_- +46\beta_-^2-3\rho+8\beta_-\rho)\prod_{j=1}^{4}(1-j\beta_- -\rho)}{2\rho},$$

$$e_3(\beta_-,\rho) = -\frac{(1-3\beta_-)(4-27\beta_- +46\beta_-^2-3\rho+10\beta_-\rho)\prod_{j=1}^{4}(1-j\beta_- -\rho)}{6\rho},$$

$$e_4(\beta_-,\rho) = \frac{(1-3\beta_-)(1-4\beta_-)^2(1-4\beta_- -\rho)^2\prod_{j=1}^{3}(1-j\beta_- -\rho)}{24\rho},$$

and thus that

$$\sum_{i=1}^{4}\sum_{j=1}^{4}e_i(\beta_-,\rho)e_j(\beta_-,\rho)v_{ij}(\beta_-) = \sigma^2(\beta_-,\rho),$$

which completes the proof. □

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