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TIGHT ASYMMETRIC ORTHOGONAL ARRAYS OF STRENGTH 2 USING FINITE PROJECTIVE GEOMETRY

M. L. AGGARWAL[1], LIH-YUAN DENG[2] AND MUKTA D. MAZUMDER[3]

ABSTRACT

Wu *et al.* (1992) constructed some general classes of tight asymmetric orthogonal arrays of strength 2 using the method of grouping. Rains *et al.* (2002) obtained asymmetric orthogonal arrays of strength 2 using the concept of mixed spread in finite projective geometry. In this paper, we obtain some new tight asymmetric orthogonal arrays of strength 2 using the concept of mixed partition in finite projective geometry.

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1. INTRODUCTION

Rao (1973) introduced asymmetric orthogonal arrays which have found numerous applications for quality improvements in the context of the industrial experiments as pointed out by Taguchi (1987). An asymmetric orthogonal array $\mathrm{OA}(N, k, m_1^{k_1} \times m_2^{k_2} \times \cdots \times m_n^{k_n}, t)$ is an array of size $N \times k$ where $k = k_1 + k_2 + \cdots + k_n$ is the total number of factors in which k_1 columns have m_1 symbols ranging from $\{0, 1, \ldots, m_1 - 1\}$, the next k_2 columns have m_2 symbols ranging from $\{0, 1, \ldots, m_2 - 1\}$ and so on with the property that in any $N \times t$ subarray every possible t tuple occurs an equal number of times as a row. An $\mathrm{OA}(N, k, m_1^{k_1} \times m_2^{k_2} \times \cdots \times m_n^{k_n}, 2)$ attaining Rao's bound $N \geq 1 + k_1(m_1 - 1) + k_2(m_2 - 1) + \cdots + k_n(m_n - 1)$ is called tight. The special case $m_1 = m_2 = \cdots = m_n = m$, (say) corresponds to a symmetric orthogonal array, denoted by an $\mathrm{OA}(N, k, m, t)$.

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[1]Corresponding author. Department of Mathematical Sciences, The University of Memphis, Memphis, TN 38152, USA (e-mail : maggarwl@memphis.edu)
[2]Department of Mathematical Sciences, The University of Memphis, Memphis, TN 38152, USA
[3]Department of Statistics, Ram Lal Anand College, New Delhi-110021, India

Rao (1947, 1949) and Hamming (1950) have constructed tight symmetric orthogonal arrays of strength 2 in the context of fractional factorial experiments and linear codes. One may refer to Hedayat *et al.* (1999) and Dey and Mukerjee (1999) for more details. Various methods of construction of asymmetric orthogonal arrays of strength 2 have been given by Wang and Wu (1991), Hedayat, Pu and Stufken (1992), Dey and Mukerjee (1998), DeCock and Stufken (2000) and Zhang, Pang and Wang (2001). Mukerjee *et al.* (2001) constructed minimum aberration designs for mixed factorials in terms of complementary sets using the concept of finite projective geometry. Wu *et al.* (1992) construced some general classes of tight asymmetric orthogonal arrays using the method of grouping. Suen *et al.* (2003) constructed some tight asymmetric orthogonal arrays of strength 2 through finite projective geometry. Rains *et al.* (2002) have also constructed some tight asymmetric orthogonal arrays of strength 2 using the concept of mixed spread in finite projective geometry and have called them as geometric orthogonal arrays. Wu *et al.* (1992) proposed several methods of partitioning to obtain tight asymmetric orthogonal arrays of strength 2. Here we are trying to explore the theories in finite projective geometry for finding several partitions to obtain tight asymmetric orthogonal arrays. Mainly, three types of partitioning we have discussed in this paper, where one of the methods of partitioning is a special case of a partitioning given in Wu *et al.* (1992).

In this paper, we construct some new tight asymmetric orthogonal arrays of strength 2 using mixed partition. Recently partitions of finite projective spaces have received considerable attention. Baker *et al.* (1999a, b), Bonisoli *et al.* (2000) and Bierbrauer *et al.* (2001) studied the mixed partition of finite projective geometry. In Section 2, we give some preliminaries of finite projective geometry and in Section 3, we construct some tight asymmetric orthogonal arrays of strength 2.

2. Finite Projective Geometry

A finite projective geometry of $(r-1)$ dimension $\mathrm{PG}(r-1, m)$ over $\mathrm{GF}(m)$, Galois field of order m, m is a prime power, consists of the ordered set $(x_0, x_1, \ldots, x_{r-1})$ of points where x_i, $i = 0, 1, \ldots, r-1$, are elements of $\mathrm{GF}(m)$ and all of them are not simultaneously zero. For any $\lambda \in \mathrm{GF}(m)$ $(\lambda \neq 0)$, the point $(\lambda x_0, \ldots, \lambda x_{r-1})$ represents the same point as that of $(x_0, \ldots, x_{r-1})$. All those points which satisfy a set of $(r-t-1)$ linearly independent homogeneous equations with coefficients from $\mathrm{GF}(m)$ (all of them are not simultaneously zero within the same equation) is said to represent a t-flat in PG$(r-1,m)$.

In particular a 0-flat, a 1-flat, ..., a $(r-2)$-flat respectively in PG$(r-1,m)$ are known as a point, a line ..., a hyperplane of PG$(r-1,m)$. The number of points lying on a $(t-1)$-flat in PG$(r-1,m)$ is $(m^t-1)/(m-1)$ and the number of independent points lying on a $(t-1)$-flats is t. A s-spread F of PG$(r-1,m)$ is a set of s-spaces which partitions PG$(r-1,m)$; that is, every point of PG$(r-1,m)$ lies in exactly one s-space of F. Hence any two s-spaces of F are disjoint. One may refer to Hirschfeld (1998) for more details. A non-uniform partition of a finite projective geometry is known as a mixed partition of a finite projective geometry. These partitions differ from the spreads, essentially; in the sense that underlying projective geometry is splitted into different spaces; more specifically the members of the partitions may not have the same geometric structure. The points of PG$(r-1,m)$ can always be divided into n disjoint flats, *i.e.* a (u_1-1)-flat, ..., a (u_n-1)-flat if $u_1+u_2+\cdots+u_n=r$, $u_i\geq 1$.

3. CONSTRUCTION OF ORTHOGONAL ARRAY

Wu *et al.* (1992) constructed tight asymmetric orthogonal arrays of the type OA$\left(m^r, k+\sum_{i=1}^t n_i, m^k\times(m^{r_1})^{n_1}\times\cdots\times(m^{r_t})^{n_t},2\right)$, $\sum_{j=1}^t r_j\leq r$, using the method of grouping where

$$\begin{cases} n_j\leq m^{r-\sum_{i=1}^{j} r_i}, & \text{if } r-\sum_{i=1}^{j} r_i\geq r_j \quad \text{and} \\ n_j\leq 1, & \text{if } r-\sum_{i=1}^{j} r_i< r_j\,. \end{cases}$$

Here we construct tight asymmetric orthogonal arrays of the type OA$(m^r, n+c,(m^{u_1})\times\cdots\times(m^{u_n})\times m^c,2)$ using the concept of disjoint flats in finite projective geometry where

$$u_1+u_2+\cdots+u_n=r\,.$$

In other words, the first r independent columns $x_1,\ldots,x_r$ in the array can be partitioned into n mutually exclusive sets each containing $u_1,\ldots,u_n$ columns respectively, to generate mutually exclusive sets of points containing a (u_1-1)-flat, ..., a (u_n-1)-flat in PG$(r-1,m)$. It is a special case of Theorem 4 in Wu *et al.* (1992).

THEOREM 3.1. *Consider PG(r-1,m) over GF(m). Let $n<r$ be any integer such that $u_1+u_2+\cdots+u_n=r$ for all $u_i>1$, $i=1,2,\ldots,n$, then there exists*

(u_1-1)-flat, (u_2-1)-flat, ..., (u_n-1)-flat, which are disjoint in PG(r-1,m). If $u_i > 1$, $i = 1,2,\ldots,n$, $r \geq 4$, then by using disjoint (u_1-1)-flat, ...,(u_n-1)-flat, one can construct a tight asymmetric $OA(m^r, n+c, (m^{u_1}) \times (m^{u_2}) \times \cdots \times (m^{u_n}) \times m^c, 2)$ where $c = \{(m^r-1) - \sum_{i=1}^{n}(m^{u_i}-1)\}/(m-1)$.

PROOF. Consider (u_1-1)-flat, (u_2-1)-flat, ..., (u_n-1)-flat for $u_i > 1$, $i = 1,2,\ldots,n$, such that $u_1+u_2+\cdots+u_n = r$ in PG$(r-1,m)$. Let $P_1, P_2, \ldots, P_n$ be the set of points containing (u_1-1)-flat, ..., (u_n-1)-flat in PG$(r-1,m)$ respectively. The cardinality of P_i is $(m^{u_i}-1)/(m-1)$ for $i = 1,2,\ldots,n$. Let P be a set of points in PG$(r-1,m)$ excluding the points of (u_1-1)-flat,(u_2-1)-flat, ..., (u_n-1)-flat. The cardinality of P is $\{(m^r-1)-\sum_{i=1}^{n}(m^{u_i}-1)\}/(m-1) = c$ (say). □

Let $G_1, G_2, \ldots, G_n$ and R be the matrices whose columns are the points of the sets $P_1, P_2, \ldots, P_n$ and P respectively. Let $G = [G_1|G_2|\cdots|G_n|R]$ be of order $r(m^r-1)/(m-1)$. $G_1, G_2, \ldots, G_n$ and R are disjoint matrices since $P_1, P_2, \ldots, P_n$ and P are all mutually exclusive. The rank of matrix G has full row rank r because the columns of G are the points of PG$(r-1,m)$. Consider the matrix G_i, $i = 1,2,\ldots,n$, which is of order $r(m^{u_i}-1)/(m-1)$. The rank of the matrix G_i is u_i, $i = 1,2,\ldots,n$, since columns of G_i are the points lying on a (u_i-1) flat, $i = 1,2,\ldots,n$, in PG$(r-1,m)$, where only u_i, $i = 1,2,\ldots,n$, points are independent, and also $u_1+u_2+\cdots+u_n = r$. Thus generating the row space of G_i, we have m^r rows but only m^{u_i}, $i = 1,2,\ldots,n$, rows will be distinct. We identify the distinct rows with m^{u_i}, $i = 1,2,\ldots,n$, symbols. It is obvious that each symbol is repeated m^{r-u_i} times in each row of the matrix G_i, $i = 1,2,\ldots,n$. Next we have to show that we will get an asymmetric orthogonal array by generating the matrix G.

From the earlier paragraph we observe that each of the symbol occurs equally often in each of the row when the matrix G is generated.

Now, we consider the row space of matrix G. Then each symbol appears equal number of times with respect to column and also the number of times each of the ordered pair (say column i and column i' in the row space of G, $i \neq i'$) appears as same as

$$\frac{\text{number of rows in the array}}{\text{symbol in column } i \times \text{ symbol in column } i'}.$$

Hence, by using the result of the Theorem 11.25 given in Hedayat *et al.* (1999) we get an $\mathrm{OA}(m^r, n+c, (m^{u_1}) \times (m^{u_2}) \times \cdots \times (m^{u_n}) \times m^c, 2)$. That is,

$$1 + \sum_{i=1}^{n} m^{u_i} - n + c(m-1) = m^r, \ \text{where } c = \frac{m^r - 1}{m-1} - \left(\sum_{i=1}^{n} \frac{m^{u_i} - 1}{m-1} \right).$$

This asymmetric orthogonal array is tight since it attains Rao's bound. We explain Theorem 3.1 with the help of examples.

Example 1. Consider $\mathrm{PG}(4,2)$ over $\mathrm{GF}(2)$. Here $r = 5$, $m = 2$. Using Theorem 3.1, $u_1 + u_2 = 5 \Rightarrow u_1 = 2, u_2 = 3$ and $n = 2$. P_1 is the set of points of 1-flat in $\mathrm{PG}(4,2)$, that is P_1 ={00100, 01000, 01100}. P_2 is the set of the points of 2-flat in $\mathrm{PG}(4,2)$, that is P_2={00001, 00010, 00011, 10000, 10001, 10010, 10011}. Thus P={00101, 00110, 00111, 01001, 01010, 01011, 01101, 01110, 01111, 10100, 10101, 10110, 10111, 11000, 11001, 11010, 11011, 11100, 11101, 11110, 11111}.

Now G_1 and G_2 and R are the matrices of order 5×3, 5×7 and 5×21, respectively. Also $G = [G_1 | G_2 | R]$. Now, generating the matrix G and identifying

$$\left[\begin{array}{ccc|ccccccc|ccccccccccccccccccccc}
0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1\\
1&0&1&0&0&0&0&0&0&0&1&1&1&0&0&0&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1\\
0&0&0&0&1&1&0&0&1&1&0&1&1&0&1&1&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1\\
0&0&0&1&0&1&0&1&0&1&1&0&1&1&0&1&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1
\end{array}\right]$$

the rows of G_1 and G_2 with symbols as $0, 1, 2, 3$ and $0, 1, 2, \ldots, 7$, respectively, we have the following matrix:

$$\left[\begin{array}{cc|ccccccccccccccccccccc}
0&1&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
1&0&0&0&0&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1\\
2&0&1&1&1&0&0&0&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1\\
0&2&0&1&1&0&1&1&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1\\
0&3&1&0&1&1&0&1&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\
1&1&0&0&0&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
2&1&1&1&1&0&0&0&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0\\
0&4&0&1&1&0&1&1&0&1&1&1&1&0&0&1&1&0&0&1&1&0&0\\
0&5&1&0&1&1&0&1&1&0&1&1&0&1&0&1&0&1&0&1&0&1&0\\
3&0&1&1&1&1&1&1&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0
\end{array}\right.$$

(*continued*)

(continued)

1	2	0	1	1	1	0	0	1	0	0	0	0	1	1	1	1	0	0	1	1	0	0
1	3	1	0	1	0	1	0	0	1	0	0	1	0	1	1	0	1	0	1	0	1	0
2	2	1	0	0	0	1	1	1	0	0	1	1	0	0	0	0	1	1	1	1	0	0
2	3	0	1	0	1	0	1	0	1	0	1	0	1	0	0	1	0	1	1	0	1	0
0	6	1	1	0	1	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0
3	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
1	4	0	1	1	1	0	0	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1
1	5	1	0	1	0	1	0	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1
2	4	1	0	0	0	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1
2	5	0	1	0	1	0	1	0	1	0	0	1	0	1	1	0	1	0	0	1	0	1
0	7	1	1	0	1	1	0	1	1	0	1	0	0	1	1	0	0	1	1	0	0	1
3	2	1	0	0	1	0	0	0	1	1	1	1	0	0	1	1	0	0	0	0	1	1
3	3	0	1	0	0	1	0	1	0	1	1	0	1	0	1	0	1	0	0	1	0	1
1	6	1	1	0	0	0	1	0	0	1	0	1	1	0	1	0	0	1	1	0	0	1
2	6	0	0	1	1	1	0	0	0	1	1	0	0	1	0	1	1	0	1	0	0	1
3	4	1	0	0	1	0	0	0	1	1	0	0	1	1	0	0	1	1	1	1	0	0
3	5	0	1	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	0	1	0
2	7	0	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	1	0	1	1	0
3	6	0	0	1	0	0	1	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0
1	7	1	1	0	0	0	1	0	0	1	1	0	0	1	0	1	1	0	0	1	1	0
3	7	0	0	1	0	0	1	1	0	0	0	1	1	0	0	1	1	0	1	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

which is an $OA(2^5, 23, (2^2) \times (2^3) \times 2^{21}, 2)$. This orthogonal array attains the Rao's bound.

Example 2. Consider $PG(3,3)$ over $GF(3)$. Here $r = 4$, $m = 3$. Using Theorem 3.1, $u_1 + u_2 = 4 \Rightarrow u_1 = u_2 = 2$ and $n = 2$. P_1 and P_2 are the set of two distinct 1-flat in $PG(3,3)$, that is P_1={0010,1100,1110,1120}, P_2={0001,0110,0111,0112}. Thus P= {0011, 0012,0100, 0101, 0102, 0112, 0120, 0121, 1000, 1001, 1002, 1010, 1011, 1012, 1020, 1021,1022, 1101, 1102, 1111, 1112, 1121, 1122, 1200, 1201, 1202, 1210, 1211, 1212, 1220, 1221, 1222}

Now, G_1 and G_2 and R are the matrices of order 4×4, 4×4 and 4×32 respectively. Also $G = [G_1|G_2|R]$ has order 4×40. Now generating the matrix G and identifying the rows of G_1 and G_2 with symbols as $0, 1, 2, 3, \ldots, 8$, we get tight $OA(3^4, 34, (3^2)^2 \times 3^{32}, 2)$. We give some more tight asymmetric orthogonal arrays of strength 2 in Table 3.1.

We can get some new tight asymmetric orthogonal arrays of strength 2 using Theorem 3.1 and contractive replacement method (CRM) given in Hedayat *et al.*

TABLE 3.1 *Tight asymmetric orthogonal arrays of strength* 2

S. No.	*PG*$(r-1,m)$	$u_1, u_2, \ldots, u_n$	*Orthogonal Array*
1.	PG$(3,2)$	$u_1 = u_2 = 2$	OA$(2^4, 11, (2^2)^2 \times 2^9, 2)$
2.	PG$(4,2)$	$u_1 = 2, u_2 = 3$	OA$(2^5, 23, (2^2) \times (2^3) \times 2^{21}, 2)$
3.	PG$(5,2)$	$u_1 = u_2 = u_3 = 2$	OA$(2^6, 57, (2^2)^3 \times 2^{54}, 2)$
4.	PG$(5,2)$	$u_1 = 2, u_2 = 4$	OA$(2^6, 47, (2^2) \times (2^4) \times 2^{45}, 2)$
5.	PG$(5,2)$	$u_1 = u_2 = 3$	OA$(2^6, 51, (2^3)^2 \times 2^{49}, 2)$
6.	PG$(6,2)$	$u_1 = u_2 = 2, u_3 = 3$	OA$(2^7, 117, (2^2)^2 \times (2^3) \times 2^{114}, 2)$
7.	PG$(6,2)$	$u_1 = 2, u_2 = 5$	OA$(2^7, 95, (2^2) \times (2^5) \times 2^{93}, 2)$
8.	PG$(6,2)$	$u_1 = 3, u_2 = 4$	OA$(2^7, 107, (2^3) \times (2^4) \times 2^{105}, 2)$
9.	PG$(7,2)$	$u_1 = u_2 = u_3 = u_4 = 2$	OA$(2^8, 247, (2^2)^4 \times 2^{243}, 2)$
10.	PG$(7,2)$	$u_1 = u_2 = 2, u_3 = 4$	OA$(2^8, 237, (2^2)^2 \times (2^4) \times 2^{234}, 2)$
11.	PG$(7,2)$	$u_1 = 2, u_2 = u_3 = 3$	OA$(2^8, 241, (2^2) \times (2^3)^2 \times 2^{238}, 2)$
12.	PG$(7,2)$	$u_1 = 2, u_2 = 6$	OA$(2^8, 191, (2^2) \times (2^6) \times 2^{189}, 2)$
13.	PG$(7,2)$	$u_1 = 3, u_2 = 5$	OA$(2^8, 219, (2^3) \times (2^5) \times 2^{217}, 2)$
14.	PG$(7,2)$	$u_1 = u_2 = 4$	OA$(2^8, 227, (2^4)^2 \times 2^{225}, 2)$
15.	PG$(3,3)$	$u_1 = u_2 = 2$	OA$(3^4, 34, (3^2)^2 \times 3^{32}, 2)$
16.	PG$(4,3)$	$u_1 = 2, u_2 = 3$	OA$(3^5, 106, (3^2) \times (3^3) \times 3^{104}, 2)$
17.	PG$(3,4)$	$u_1 = u_2 = 2$	OA$(4^4, 77, (4^2)^2 \times 4^{75}, 2)$
18.	PG$(3,5)$	$u_1 = u_2 = 2$	OA$(5^4, 146, (5^2)^2 \times 5^{144}, 2)$

(1999). Here we will state the contractive replacement method. Let A be an orthogonal array OA$(N, n, m_1 \times m_2 \times \cdots \times m_n, 2)$, where m_i's are not necessarily all distinct, such that for a subset of p factors, say the first p, the runs of the array obtained from A by removing $n-p$ factors consist of N/N_1 copies of each of runs of an OA$(N_1, p, m_1 \times m_2 \times \cdots \times m_p, 2)$ say B, which is tight. After labeling the runs of B by $0, 1, \ldots, N_1 - 1$, we replace each level combination of the first p factors in A by corresponding label of B. The resultant one is an OA$(N, n-p+1, N_1 \times m_{p+1} \times \cdots \times m_n, 2)$.

Now we construct tight asymmetric orthogonal arrays of strength 2 using the above replacement method and Theorem 3.1. Here CRM can be regarded as a method of finding partitions. Because the orthogonal arrays A are of specific type originated from Theorem 3.1 and any tight p columns in A, *i.e.* tight orthogonal arrays B is corresponding to the points of a $(t-1)$-flat where $t = 2, 3, \ldots$ in PG$(r-1, m)$.

Consider OA$(2^r, L, 2, 2)$, $L = (m^r - 1)/(m-1)$. For maximal number of mutually exclusive lines in OA$(2^r, L, 2, 2)$, the maximal number of disjoint 1-flats (lines) in PG$(r-1, 2)$ depends on r. If r is even, $3|(2^r - 1)$, (where the number of points in any 1-flat is 3), then there are $(2^r - 1)/3$ mutually exclusive 1-flats in $\text{PG}(r-1,2)$. Let $\text{PG}(5,2)$. Here $r = 6$.

(*a*) Let $u_1 = 2$, $u_2 = 4$, then $\text{OA}(2^6, 47, (2^2) \times (2^4) \times 2^{45}, 2)$. Let $A = \text{OA}(2^6, 47, (2^2) \times (2^4) \times 2^{45}, 2)$ and $B = \text{OA}(4, 3, 2, 2)$. Using CRM we get $\text{OA}(2^6, 17, 4^{16} \times 16, 2)$ which is included in Rains *et al.* (2002).

(*b*) Let $u_1 = u_2 = 3$, then $\text{OA}(2^6, 51, (2^3)^2 \times 2^{49}, 2)$. Let $A = \text{OA}(2^6, 51, (2^3)^2 \times 2^{49}, 2)$, $B = \text{OA}(4, 3, 2, 2)$. In array A, there are two 2-flats and 49 points. Also there are 9 disjoint 2-flats in $\text{PG}(5,2)$. Thus 49 points of the array correspond to 7 disjoint 2-flats, *i.e.* 15 disjoint lines and 4 points. Hence using CRM we get $\text{OA}(2^6, 21, 8^2 \times 4^{15} \times 2^4, 2)$ which is included in Rains *et al.* (2002).

For maximal number of mutually exclusive planes in $\text{OA}(2^r, L, 2, 2)$, the maximal number of disjoint 2-flats (planes) in $\text{PG}(r-1,2)$ depends on r. Here, if $3 \mid r$, then there are $(2^r - 1)/7$, (where the number of points in any 2-flat in 7) mutually exclusive 2-flats in $\text{PG}(r-1,2)$.

(*c*) Let $u_1 = u_2 = u_3 = 2$, then $\text{OA}(2^6, 57, (2^2)^3 \times 2^{54}, 2)$. Let $A = \text{OA}(2^6, 57, (2^2)^3 \times 2^{54}, 2)$ and $B = \text{OA}(8, 7, 2, 2)$. In array A there are 3 1-flats and 54 points. In $\text{PG}(5,2)$, there are 9 disjoint 2-flats. The 3 1-flats can be obtained from 3 disjoint 2-flats. Thus 9 2-flats can be divided into 6 2-flats, 3 1-flats and 12 points. The 54 points in the array correspond to the points of 6 disjoint 2-flats and 12 points. Hence using CRM we get $\text{OA}(2^6, 21, 8^6 \times 4^3 \times 2^{12}, 2)$ which is not included in Rains *et al.* (2002) and hence a new orthogonal array.

(*d*) Let $u_1 = 2$, $u_2 = 4$, then $\text{OA}(2^6, 47, (2^2) \times (2^4) \times 2^{45}, 2)$. Let $A = \text{OA}(2^6, 47, (2^2) \times (2^4) \times 2^{45}, 2)$ and $B = \text{OA}(8, 7, 2, 2)$. There are 1 3-flat, 1 1-flat and 45 points in array A. In $\text{PG}(5,2)$, there are 21 disjoint lines. Also one 3-flat can be obtained from the unions of 5 disjoint lines. Thus the 45 points of $\text{PG}(5,2)$ are the points of 15 disjoint lines. From these 15 disjoint lines, we have 6 disjoint 2-flats and 1 line (disjoint from the 6 2-flats). Hence using CRM we get $\text{OA}(2^6, 11, 4 \times 16 \times 8^6 \times 2^3, 2)$ which is not included in Rains *et al.* (2002) and hence a new orthogonal array.

For maximal number of mutually exclusive 3-flats in $\text{OA}(2^r, L, 2, 2)$, the maximal number of disjoint 3-flats (solids) in $\text{PG}(r-1,2)$ depends on r. If $15 \mid (2^r - 1)$, (where the number of points of any 3-flat in PG$(r-1,2)$ is 15) then there are $(2^r-1)/(15)$ mutually exclusive 3-flats in PG$(r-1,2)$. Let $B = \text{OA}(16,15,2,2)$. Now, we will give some more examples.

(*a*) Let $A = \text{OA}(2^8, 247, (2^2)^4 \times 2^{243}, 2)$. In PG$(7,2)$, there are 85 disjoint lines. Also a 3-flat can be obtained from the unions of 5 disjoint lines. Array A have 4 1-flats and 243 points. Now these 243 points are the points of 81 disjoint lines. From these 81 disjoint lines, we have 16 3-flats (since $81 = 16 \times 5 + 1$) and 1 line. All these 3-flats and line are mutually disjoint. Hence we have $\text{OA}(2^8, 23, 4^4 \times 16^{16} \times 2^3, 2)$.

(*b*) Let $A = \text{OA}(2^8, 237, (2^2)^2 \times (2^4) \times 2^{234}, 2)$. In PG$(7,2)$, there are 17 disjoint 3-flats. There are 1 3-flats and 2 lines (disjoint) in array A. Also there are 234 points in the array. Here the 234 points are the points of 78 disjoint lines. Thus, these 234 points are the points of 15 3-flats and 9 points. Hence we have $\text{OA}(2^8, 27, 4^2 \times 16^{16} \times 2^9, 2)$.

(*c*) Let $A = \text{OA}(2^8, 241, (2^2) \times (2^3)^2 \times 2^{238}, 2)$. In the array, there are 1 line, 2 2-flats and 238 points. From 9 disjoint lines, *i.e.* a 3-flat and 4 lines we can have 2 disjoint 2-flats and 13 points. The 238 points are the points of 15 3-flats and 13 points since there are 17 3-flats (disjoint) in PG$(7,2)$. These 17 flats can be divided into 15 3-flats and 10 lines. Again using 9 lines out of 10 -lines we have 2 disjoint 2-flats and 13 points. Hence we have $\text{OA}(2^8, 31, 4 \times 8^2 \times 16^{15} \times 2^{13}, 2)$.

(*d*) Let $A = \text{OA}(2^8, 191, (2^2) \times (2^6) \times 2^{189}, 2)$. There are 1 line, 1 5-flat and 189 points in the array. 5-flat is the points of unions of 21 disjoint lines. Thus 189 points are points of the unions of 63 disjoint lines $(85 - 22 = 63)$. These 63 disjoint lines can be divided into 12 group of 5 disjoint lines and 3 disjoint lines. Unions of the points of 5 disjoint lines are the points of a 3-flat. Thus 189 points are the points of 12 disjoint 3-flats and 9 points (3 lines). Hence we have $\text{OA}(2^8, 23, 4 \times 64 \times 16^{12} \times 2^9, 2)$.

(*e*) Let $A = \text{OA}(2^8, 219, (2^3) \times (2^5) \times 2^{217}, 2)$. The array A have 1 2-flat, 1 4-flat and 217 points. Using 5 disjoint lines we have a 2-flat and 8 points. Again from 21 disjoint lines we have 1 4-flat and 32 points. There are 85 disjoint lines in PG$(7,2)$. Thus 59 (*i.e.* $85 - (21 + 5)$) disjoint lines can be divided into 11 sets of 5 disjoint lines and 4 disjoint lines. 11 sets of 5 disjoint lines correspond to 11 3-flats. The 217 points of the array correspond to the points of 11 3-flats, 52 points (*i.e.*, 8 points from the 5 disjoint lines, 32 from the 21 disjoint lines and 12 points from the 4 disjoint lines). Hence we have $\mathrm{OA}(2^8, 65, 8 \times 32 \times 16^{11} \times 2^{52}, 2)$.

We can also construct some new tight asymmetric orthogonal arrays using a result given in Rains *et al.* (2002). We rewrite Lemma 12 given in Rains *et al.* (2002) in terms of finite projective geometry, suited for our purpose.

LEMMA 3.1. *Let F_1, F_2, F_3 be three $(k-1)$-flats in $PG(2k-1, m)$ such that they are mutually disjoint, then their union can be replaced by $2^k - 1$ disjoint 1-flats (line) in $PG(2k-1, m)$.*

We will make use of this Lemma to construct some more tight asymmetric orthogonal arrays of strength 2 as explained below:

(*a*) Consider $\mathrm{OA}(2^6, 21, 8^6 \times 4^3 \times 2^{12}, 2)$. Here $2k = 6$, $k = 3$. Thus F_1, F_2, F_3 are 3 2-flats in $\mathrm{PG}(5, 2)$ such that they are mutually disjoint. Then their union can be replaced by $2^3 - 1 = 7$ 1-flats in $\mathrm{PG}(5, 2)$. In the above array, there are 6 columns with symbols 8. Hence, there are 6 disjoint 2-flats in $\mathrm{PG}(5, 2)$. Using the Lemma 3.1, out of these 6 disjoint 2-flats, consider the first three disjoint 2-flats and replace their union by 7 1-flats. Finally the 6 columns with symbols 8 can be replaced by 14 columns with symbol 4. Hence we have $\mathrm{OA}(2^6, 29, 4^7 \times 4^7 \times 4^3 \times 2^{12}, 2) = \mathrm{OA}(2^6, 29, 2^{12} \times 4^{17}, 2)$ which is not included in Rains *et al.* (2002).

(*b*) Consider $\mathrm{OA}(2^6, 11, 4 \times 16 \times 8^6 \times 2^3, 2)$. Here $2k = 6$, $k = 3$. Arguing the same way as in (a) we have $\mathrm{OA}(2^6, 19, 4 \times 16 \times 4^7 \times 4^7 \times 2^3, 2) = \mathrm{OA}(2^6, 19, 2^3 \times 4^{15} \times 16, 2)$ which is not included in Rains *et al.* (2002) and hence a new array. Consider $\mathrm{PG}(7, 2)$ over $\mathrm{GF}(2)$. Here $k = 4$, $m = 2$. Thus if there are three disjoint 3-flats in $\mathrm{PG}(7, 2)$, then their union can be replaced by $2^4 - 1 = 15$ disjoint 1-flats in $\mathrm{PG}(7, 2)$.

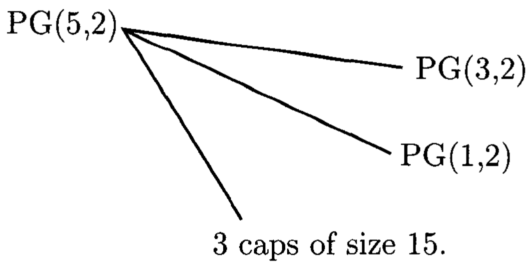
In Table 3.2, we have tabulated some more new tight asymmetric orthogonal arrays of strength 2 in the following table based on Lemma 3.1.

It may be remarked that some of the well known mixed partition of finite projective spaces can be used to construct tight asymmetric orthogonal arrays of strength 2.

TABLE 3.2 *New tight asymmetric orthogonal arrays*

S. No.	*Orthogonal Array*	*Orthogonal Array using Lemma* 3.1
1.	$OA(2^8, 23, 4^4 \times 16^{16} \times 2^3, 2)$	$OA(2^8, 83, 16 \times 4^{79} \times 2^3, 2)$
2.	$OA(2^8, 27, 4^2 \times 16^{16} \times 2^9, 2)$	$OA(2^8, 87, 16 \times 4^{77} \times 2^9, 2)$
3.	$OA(2^8, 31, 4 \times 8^2 \times 16^{15} \times 2^{13}, 2)$	$OA(2^8, 91, 8^2 \times 4^{76} \times 2^{13}, 2)$
4.	$OA(2^8, 23, 4 \times 64 \times 16^{12} \times 2^9, 2)$	$OA(2^8, 71, 64 \times 4^{61} \times 2^9, 2)$
5.	$OA(2^8, 65, 8 \times 32 \times 16^{11} \times 2^{52}, 2)$	$OA(2^8, 101, 32 \times 16^2 \times 8 \times 4^{45} \times 2^{52}, 2)$

1. Bierbrauer *et al.* (2001) have stated that the points of $PG(3r-1, m)$ can be partitioned into the points of a subspace $PG(2r-1, m)$, the points of a subspace $PG(r-1, m)$ and $m^r - 1$ caps of size $(m^{2r}-1)/(m-1)$ each. We have $OA(m^{3r}, a+2, (m^{2r}) \times (m^r) \times m^a, 2)$, where a=$(m^r-1)\{(m^{2r}-1)/(m-1)\}$. Let $r = 2$, $m = 2$. Then



Hence we have $OA(2^6, 47, (2^4) \times (2^2) \times 2^{45}, 2)$, which is same as Serial No.4 in Table 3.1.

2. Baker *et al.* (1999a) have given that $PG(5, m)$ can be partitioned into 2-planes and $m^3 - 1$ caps of size $m^2 + m + 1$. We have $OA(m^6, a+2, (m^3)^2 \times m^a, 2)$ where $a = (m^3-1)(m^2+m+1)$. Let $m = 2$. Then we have $OA(2^6, 51, (2^3)^2 \times 2^{49}, 2)$ which is same as Serial No.5 in Table 3.1.

3. Kestenband (1981) constructed partitions using $PG(2r-1, m^2)$ which consist of 2 $(r-1)$-dimensional subspaces and $(m^{2r}-1)$ caps of size $(m^{2r}-1)/(m^2-1)$. Thus we have $OA((m^2)^{2r}, a+2, (m^{2r})^2 \times (m^2)^a, 2)$, where $a = (m^{2r}-1)\{(m^{2r}-10)/(m^2-1)\}$. Let $m = 2$, $r = 2$. Then we have $OA(4^4, 77, (4^2)^2 \times 4^{75}, 2)$ which is same as Serial No. 17 in Table 3.1.

4. Baker *et al.* (1999b) have given that $PG(3, m)$ can be divided into 2-lines and $(m-1)$ hyperbolic quadrics. Thus we have $OA(m^4, a+2, (m^2)^2 \times m^a, 2)$

where, $a = (m-1)(m+1)^2$. Since, there are $(m+1)^2$ points in any hyperbolic quadric $xy = zw$, where x, y, z, w are in PG$(3, m)$. Let $m = 3$. Then we have OA$(3^4, 34, (3^2)^2 \times 3^{32}, 2)$ which is same as Serial No.15 in Table 3.1.

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