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MARCINKIEWICZ-ZYGMUND LAW OF LARGE NUMBERS FOR BLOCKWISE ADAPTED SEQUENCES

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Abstract. In this paper we establish the Marcinkiewicz-Zygmund strong law of large numbers for blockwise adapted sequences. Some related results are considered.

1. Introduction and notations

In [5] and [8] it was shown that some properties of independent sequences of random variables can be applied to the sequences consisting of independent blocks. Particularly, it was proved in [8] that if $(X_i)_{i=1}^{\infty}$, $EX_i = 0$ is a sequence independent in blocks $[2^k, 2^{k+1})$, then it satisfies the Kolmogorov's theorem: the condition $\sum_{i=1}^{\infty}(EX_i^2)i^{-2} < \infty$ implies the strong law large numbers (s.l.l.n.), i.e.,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i = 0 \text{ a.s.}$$

Strong law of large numbers for blockwise independent random variables was studied by V. F. Gaposhkin [4].

Marcinkiewicz-Zygmund type strong law of large numbers was studied by many authors. In 1981, N. Etemadi [3] proved that if $\{X_n, n \geq 1\}$ is a sequence of pairwise i.i.d. random variables with $E|X_1| < \infty$, then $\lim\limits_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}(X_i - EX_1) = 0$ a.s.

Later, in 1985, B. D. Choi and S. H. Sung [2] have shown that if $\{X_n, n \geq 1\}$ are pairwise independent and are dominated in distribution

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by a random variable X with $E|X|^p(\log^+|X|)^2 < \infty$, $1 < p < 2$, then

$$\lim_{n\to\infty} \frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^{n} (X_i - EX_i) = 0 \text{ a.s.}$$

Recently, D. H. Hong and S. Y. Hwang [6], D. H. Hong and A. I. Volodin [7] studied Marcinkiewicz-Zygmund strong law of large numbers for double sequence of random variables.

In this paper we establish the Marcinkiewicz-Zygmund strong law of large numbers for blockwise adapted sequences. Some related results are considered.

Let $\{\omega(n), n \geq 1\}$ be a strictly increasing sequence of positive integers with $\omega(1) = 1$. For each $k \geq 1$, we set $\Delta_k = \big[\omega(k), \omega(k+1)\big)$. We recall that the sequence $\{X_i, i \geq 1\}$ of random variables is blockwise independent with respect to blocks Δ_k, if for any fixed k, the sequences $\{X_i\}_{i\in\Delta_k}$ are independent. Let $\{\mathcal{F}_i, i \geq 1\}$ be a sequence of σ-fields such that for any fixed k, the sequences $\{\mathcal{F}_i, i \in \Delta_k\}$ are increasing. The sequence $\{X_i, i \geq 1\}$ of random variables is said to be blockwise adapted to $\{\mathcal{F}_i, i \geq 1\}$, if each X_i is measurable with respect to $\mathcal{F}_i$. The sequence $\{X_i, \mathcal{F}_i, i \geq 1\}$ is said to be a block martingale difference with respect to blocks Δ_k, if for any fixed k, the sequences $\{X_i, \mathcal{F}_i\}_{i\in\Delta_k}$ are martingale differences. Denote

$$\begin{aligned}
N_m &= \min\{n | \omega(n) \geq 2^m\},\\
s_m &= N_{m+1} - N_m + 1,\\
\varphi(i) &= \max_{k\leq m} s_k \text{ if } i \in [2^m, 2^{m+1}),\\
\Delta^{(m)} &= [2^m, 2^{m+1}), m \geq 0,\\
\Delta_k^{(m)} &= \Delta_k \cap \Delta^{(m)}, m \geq 0, k \geq 1,\\
p_m &= \min\{k : \Delta_k^{(m)} \neq \emptyset\},\\
q_m &= \max\{k : \Delta_k^{(m)} \neq \emptyset\}.
\end{aligned}$$

Since $\omega(N_m - 1) < 2^m, \omega(N_m) \geq 2^m, \omega(N_{m+1}) \geq 2^{m+1}$ for each $m \geq 1$, the number of nonempty blocks $[\Delta_k^{(m)}]$ is not large than $s_m = N_{m+1} - N_m + 1$. Assume $\Delta_k^{(m)} \neq \emptyset$, let $r_k^{(m)} = \min\{r : r \in \Delta_k^{(m)}\}$.

The sequence $\{X_n, n \geq 1\}$ is said to be stochastically dominated by a random variable X if there exists a constant $C > 0$ such that $P\{|X_n| > t\} \leq CP\{|X| > t\}$ for all nonnegative real numbers t and for all $n > 1$.

Finally, the symbol C denotes throughout a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance.

2. Lemmas

In the sequel we will need the following lemmas.

LEMMA 2.1. (Doob's Inequality) *If* $\{X_i, \mathcal{F}_i\}_{i=1}^N$ *is a martingale difference,* $E|X_i|^p < \infty$ $(1 < p < \infty)$, *then*

$$E\left|\max_{k\leq N}\sum_{i=1}^{k} X_i\right|^p \leq (\frac{p}{p-1})^p E\left|\sum_{i=1}^{N} X_i\right|^p .$$

The next lemma is due to von Bahr and Esseen [1].

LEMMA 2.2. (von Bahr and Esseen [1]) *Let* $\{X_i,\}_{i=1}^N$ *be random variables such that* $E\{X_{m+1}|S_m\} = 0$ *for* $0 \leq m \leq N-1$, *where* $S_0 = 0$ *and* $S_m = \sum_{i=1}^m X_i$ *for* $1 \leq m \leq N$, *then*

$$E|S_N|^p \leq C\sum_{i=1}^{N} E|X_i|^p \text{ for all } 1 \leq p \leq 2,$$

where C *is a constant independent of* N.

By lemmas 2.1 and 2.2, we get the following lemma.

LEMMA 2.3. *If* $\{X_i, \mathcal{F}_i\}_{i=1}^N$ *is a martingale difference,* $E|X_i|^p \leq \infty$ $(1 \leq p \leq 2)$, *then*

$$E\left|\max_{k\leq N}\sum_{i=1}^{k} X_i\right|^p \leq C\sum_{i=1}^{N} E|X_i|^p,$$

where C *is a constant independent of* N.

Proof. In the case $p = 1$, we have

$$E\left|\max_{k\leq N}\sum_{i=1}^{k} X_i\right| \leq E(\sum_{i=1}^{N}|X_i|) = \sum_{i=1}^{N} E|X_i|.$$

In the case $1 < p \leq 2$,

$$E\left|\max_{k\leq N}\sum_{i=1}^{k} X_i\right|^p \leq (\frac{p}{p-1})^p E\left|\sum_{i=1}^{N} X_i\right|^p \quad \text{(By Lemma 2.1)}$$

$$\leq C\sum_{i=1}^{N} E|X_i|^p \text{ (By Lemma 2.2).}$$

The proof of the lemma is completed. □

LEMMA 2.4. *If $q > 1$ and $\{x_n, n \geq 0\}$ is a sequence of constants such that $\lim_{n\to\infty} x_n = 0$, then*

$$\lim_{n\to\infty} q^{-n} \sum_{k=0}^{n} q^{k+1} x_k = 0.$$

Proof. Let $s = q + \sum_{i=0}^{\infty} q^{-i}$. For any $\epsilon > 0$, there exists k_0 such that $|x_k| < \frac{\epsilon}{2s}$ for all $k \geq k_0$. Since $\lim_{n\to\infty} q^{-n} = 0$, so, there exists $n_0 \geq k_0$ such that $\left|q^{-n} \sum_{k=0}^{k_0} q^{k+1} x_k\right| < \frac{\epsilon}{2}$. It follows that, for all $n \geq n_0$,

$$\begin{aligned}\left|q^{-n}\sum_{k=0}^{n} q^{k+1}x_k\right| &\leq \left|q^{-n}\sum_{k=0}^{k_0} q^{k+1}x_k\right| + \left|q^{-n}\sum_{k_0+1}^{n} q^{k+1}x_k\right| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2s}\left(q + 1 + \frac{1}{q} + \cdots\right) \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\end{aligned}$$

which completes the proof. □

3. Main result

With the notations and lemmas accounted for, main results may now be established. Theorem 3.1 establishes the strong law of large numbers for block martingale differences.

THEOREM 3.1. *Let $\{X_i, \mathcal{F}_i\}_{i=1}^{\infty}$ be a block martingale difference with respect to blocks Δ_k, $(1 \leq p \leq 2)$. If*

$$\sum_{i=1}^{\infty} \frac{E|X_i|^2}{i^{\frac{2}{p}}} < \infty,$$

then

$$\frac{\sum_{i=1}^{n} X_i}{n^{\frac{1}{p}} \varphi^{\frac{1}{2}}(n)} \to 0 \quad a.s. \text{ as } n \to \infty.$$

Proof. Let

$$\gamma_k^{(m)} = \max_{n\in\Delta_k^{(m)}} \left| \sum_{i=r_k^{(m)}}^{n} X_i \right|, \quad m \geq 0, k \geq 1;$$

$$\gamma_m = 2^{\frac{-m-1}{p}} \varphi^{-\frac{1}{2}}(2^m) \sum_{p_m \le k \le q_m} \gamma_k^{(m)}, \quad m \ge 0.$$

Using Lemma 2.3 for martingale differences $\{X_i, \mathcal{F}_i, i \in \Delta_k^{(m)}\}$, we have

$$E|\gamma_k^{(m)}|^2 \le C \sum_{i \in \Delta_k^{(m)}} EX_i^2, \text{ for all } m \ge 0, k \ge 1.$$

It implies

$$\begin{aligned} E|\gamma_m|^2 &\le 2^{\frac{-2m-2}{p}} \varphi^{-1}(2^m) s_m \sum_{k=p_m}^{q_m} E|\gamma_k^{(m)}|^2 \\ &\le 2^{\frac{-2m-2}{p}} \sum_{k=p_m}^{q_m} E|\gamma_k^{(m)}|^2 \\ &\le C 2^{\frac{-2m-2}{p}} \sum_{i=2^m}^{2^{m+1}-1} EX_i^2 \\ &\le C \sum_{i=2^m}^{2^{m+1}-1} \frac{X_i^2}{i^{\frac{2}{p}}}. \end{aligned}$$

Thus

$$\sum_{m=0}^{\infty} E|\gamma_m|^2 \le C \sum_{i=1}^{\infty} \frac{X_i^2}{i^{\frac{2}{p}}} < \infty.$$

By the Markov inequality and the Borel-Cantelli lemma, we get

$$\lim_{m \to \infty} \gamma_m = 0 \quad a.s. \tag{3.1}$$

On the other hand

$$0 \le 2^{-\frac{m}{p}} \varphi^{-\frac{1}{2}}(2^m) \sum_{k=0}^{m} \sum_{i=p_k}^{q_k} \gamma_i^{(k)} \le 2^{-\frac{m}{p}} \sum_{k=0}^{m} 2^{\frac{k+1}{p}} \gamma_k. \tag{3.2}$$

By (3.1), (3.2) and Lemma 2.4, we get

$$\lim_{m \to \infty} 2^{-\frac{m}{p}} \varphi^{-\frac{1}{2}}(2^m) \sum_{k=0}^{m} \sum_{i=p_k}^{q_k} \gamma_i^{(k)} = 0 \text{ a.s.} \tag{3.3}$$

Assume $n \in \Delta_k^{(m)}$, we have

$$0 \leq \left| n^{-\frac{1}{p}} \varphi^{-\frac{1}{2}}(n) \sum_{i=1}^{n} X_i \right|$$
$$\leq 2^{-\frac{m}{p}} \varphi^{-\frac{1}{2}}(2^m) \sum_{k=0}^{m} \sum_{i=p_k}^{q_k} \gamma_i^{(k)}. \tag{3.4}$$

By (3.3) and (3.4), we get

$$n^{-\frac{1}{p}} \varphi^{-\frac{1}{2}}(n) \sum_{i=1}^{n} X_i \to 0 \text{ a.s. (as } n \to \infty).$$

The proof is completed. □

In the next theorem, we set up the Marcinkiewicz-Zygmund law of large numbers for blockwise adapted sequences which are stochastically dominated by a random variable X.

THEOREM 3.2. *Let $\{\mathcal{F}_i, i \geq 1\}$ be a sequence of σ-fields such that for any fixed k, the sequences $\{\mathcal{F}_i, i \in \Delta_k\}$ are increasing and $\{X_i, i \geq 1\}$ is blockwise adapted to $\{\mathcal{F}_i, i \geq 1\}$. If $\{X_i, i \geq 1\}$ is stochastically dominated by a random variable X such that either*

$$E|X| \log^+ |X| < \infty \text{ if } p = 1,$$

or

$$E|X|^p < \infty \text{ if } 1 < p < 2,$$

then

$$\lim_{n \to \infty} \frac{1}{n^{\frac{1}{p}} \varphi^{\frac{1}{2}}(n)} \sum_{i=1}^{n} (X_i - a_i) = 0 \quad a.s.,$$

where $a_i = EX_i$ if $i = r_k^{(m)}$ and $a_i = E(X_i | \mathcal{F}_{i-1})$ if $i \neq r_k^{(m)}$ for $k \geq 1$ and $m \geq 0$.

Proof. Let $X_i' = X_i I\{|X_i| \leq i^{\frac{1}{p}}\}$, $b_i = EX_i'$ if $i = r_k^{(m)}$ and $b_i = E(X_i' | \mathcal{F}_{i-1})$ if $i \neq r_k^{(m)}$ for $k \geq 1$ and $m \geq 0$. We have

$$E(X_i' - b_i)^2 \leq E|X_i'|^2$$
$$= \int_0^{i^{\frac{2}{p}}} P(|X_i|^2 > t) dt$$

$$\le C\int_0^{i^{\frac{2}{p}}} P(|X|^2 > t)dt$$
$$= C\int_0^{i^{\frac{2}{p}}} \big(P(t < |X|^2 < i^{\frac{2}{p}}) + P(i^{\frac{2}{p}} \le |X|^2)\big)dt$$
$$= C(\int_0^{i^{\frac{1}{p}}} x^2 dF(x) + i^{\frac{2}{p}} P(i^{\frac{2}{p}} \le |X|^2)),$$

where $F(x)$ is the distribution function of X.

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{1}{i^{\frac{2}{p}}}\int_0^{i^{\frac{1}{p}}} x^2 dF(x) &\le C\sum_{i=1}^{\infty}\frac{1}{i^{\frac{2}{p}}}\sum_{k=1}^{i}\int_{(k-1)^{\frac{1}{p}}}^{k^{\frac{1}{p}}} x^2 dF(x)\\ &\le C\sum_{k=1}^{\infty}\sum_{i=k}^{\infty}\frac{1}{i^{\frac{2}{p}}}\int_{(k-1)^{\frac{1}{p}}}^{k^{\frac{1}{p}}} x^2 dF(x)\\ &\le C\sum_{k=1}^{\infty} k^{\frac{p-2}{p}}\int_{(k-1)^{\frac{1}{p}}}^{k^{\frac{1}{p}}} x^2 dF(x)\\ &\le C\sum_{k=1}^{\infty}\int_{(k-1)^{\frac{1}{p}}}^{k^{\frac{1}{p}}} x^p dF(x)\\ &\le CE|X|^p < \infty,\end{aligned}$$

and

$$\sum_{i=1}^{\infty} P(i^{\frac{2}{p}} \le |X|^2) = \sum_{i=1}^{\infty} P(i \le |X|^p) \le CE|X|^p < \infty.$$

Hence

$$\sum_{i=1}^{\infty}\frac{E(X_i' - b_i)^2}{i^{\frac{2}{p}}} < \infty.$$

For each $k > 1$ and $m \ge 0$, sequence $\{X_i' - b_i, \mathcal{F}_i, i \in \Delta_k^{(m)}\}$ is a martingale difference. By using the proof of Theorem 3.1, we get

$$n^{-\frac{1}{p}}\varphi^{-\frac{1}{2}}(n)\sum_{i=1}^{n}(X_i' - b_i) \to 0 \text{ a.s. } (\text{as } n \to \infty). \tag{3.5}$$

Next,

$$\begin{aligned}\sum_{i=1}^{\infty} P(X_i \neq X_i^{'}) &= \sum_{i=1}^{\infty} P(|X_i| > i^{\frac{1}{p}}) \\ &\leq C\sum_{i=1}^{\infty} P(|X| > i^{\frac{1}{p}}) \\ &\leq C\sum_{i=1}^{\infty} P(|X|^p > i) \leq CE|X|^p < \infty. \end{aligned} \tag{3.6}$$

Finally, we prove that

$$\lim_{n\to\infty} \frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^{n} (a_i - b_i) = 0 \text{ a.s.} \tag{3.7}$$

In the case p=1,

$$\begin{aligned}\sum_{n=1}^{\infty} n^{-1} E\big[|X_n| I(|X_n| > n)\big] &= \sum_{n=1}^{\infty} n^{-1} \int_n^{\infty} P\big(|X_n| > x\big)dx \\ &\leq C\sum_{n=1}^{\infty} n^{-1} \int_n^{\infty} P(|X| > x)dx \\ &= C\sum_{n=1}^{\infty} n^{-1} \sum_{i=n}^{\infty} \int_{i<x\leq i+1} P(|X| > x)dx \\ &\leq C\sum_{i=1}^{\infty} P(|X| > i) \sum_{n=1}^{i} n^{-1} \\ &\leq C\sum_{i=1}^{\infty} (1 + \log i) P(|X| > i) < \infty. \end{aligned}$$

This implies that $\sum_{n=1}^{\infty} n^{-1}(a_n - b_n) < \infty$ a.s. By using Kronecker's lemma, we get (3.7).

In the case $1 < p < 2$, since

$$\sum_{n=1}^{\infty} n^{-\frac{1}{p}} E\big[|X_n| I(|X_n| > n^{\frac{1}{p}})\big]$$

$$\le C\sum_{n=1}^{\infty} n^{-\frac{1}{p}} \int_{n^{\frac{1}{p}}}^{\infty} xdF(x)$$

$$= C\sum_{n=1}^{\infty} n^{-\frac{1}{p}} \sum_{i=n}^{\infty} \int_{i^{\frac{1}{p}}}^{(i+1)^{\frac{1}{p}}} xdF(x)$$

$$\le C\sum_{i=1}^{\infty}\sum_{n=1}^{i} n^{-\frac{1}{p}} \int_{i^{\frac{1}{p}}}^{(i+1)^{\frac{1}{p}}} xdF(x)$$

$$\le C\sum_{i=1}^{\infty} i^{\frac{p-1}{p}} \int_{i^{\frac{1}{p}}}^{(i+1)^{\frac{1}{p}}} xdF(x)$$

$$\le C\sum_{i=1}^{\infty} \int_{i^{\frac{1}{p}}}^{(i+1)^{\frac{1}{p}}} x^p dF(x) < \infty.$$

It follows that

$$\sum_{n=1}^{\infty} n^{-\frac{1}{p}}(a_n - b_n) < \infty \text{ a.s..}$$

By Kronecker's lemma, we get (3.7).

Combining (3.5), (3.6) and (3.7) we obtain

$$\lim_{n\to\infty} \frac{1}{n^{\frac{1}{p}}\varphi^{\frac{1}{2}}(n)} \sum_{i=1}^{n}(X_i - a_i) = 0 \text{ a.s.}$$

This completes the proof of theorem. □

The following corollaries extend the classical Marcinkiewicz-Zygmund strong law of large numbers.

COROLLARY 3.3. *Let $\{X_i, i \ge 1\}$ be a sequence of blockwise independent random variables with respect to blocks Δ_k. If $\{X_i, i \ge 1\}$ is stochastically dominated by a random variable X, $E|X|^p < \infty$ $(1 \le p < 2)$, then*

$$\lim_{n\to\infty} \frac{1}{n^{\frac{1}{p}}\varphi^{\frac{1}{2}}(n)} \sum_{i=1}^{n}(X_i - EX_i) = 0 \quad a.s.$$

Proof. Let $\mathcal{F}_i = \sigma(X_{r_k^{(m)}}, \dots, X_i)$ (the σ-field generated by $X_{r_k^{(m)}}, \dots, X_i$) if $i \in \Delta_k^{(m)}$. Then $\{X_i, i \ge 1\}$ is blockwise adapted to $\{\mathcal{F}_i, i \ge 1\}$. From the independence of sequence $\{X_i, i \in \Delta_k^{(m)}\}$ we get for all k and m

$$E(X_i|\mathcal{F}_{i-1}) = EX_i \text{ if } i \ne r_k^{(m)}.$$

By the proof of Theorem 3.2, we only need prove for the case $p = 1$.

In the case $p = 1$, also using the proof of Theorem 3.2, we get

$$n^{-1}\varphi^{-\frac{1}{2}}(n)\sum_{i=1}^{n}(X_i - EX_i^{'}) \to 0 \text{ (as } n \to \infty), \tag{3.8}$$

where $X_i^{'} = X_i I(|X_i| \le i)$. On the other hand

$$\begin{aligned} E[|X_i|I(|X_i| > i)] &= \int_i^{\infty} P(|X_i| > x)dx \\ &\le C\int_i^{\infty} P(|X| > x)dx \to 0 \text{ as } i \to \infty. \end{aligned}$$

Thus

(3.9)

$$|n^{-1}\sum_{i=1}^{n}(EX_i - EX_i^{'})| \le n^{-1}\sum_{i=1}^{n} E[|X_i|I(|X_i| > i)] \to 0 \text{ as } n \to \infty.$$

Combining (3.8) and (3.9) we obtain

$$\lim_{n\to\infty} \frac{1}{n\varphi^{\frac{1}{2}}(n)}\sum_{i=1}^{n}(X_i - EX_i) = 0 \text{ a.s.}$$

□

COROLLARY 3.4. *If $\omega(k) = 2^k$ (or $\omega(k) = [q^k], q > 1$) and $\{X_i, i \ge 1\}$ is Δ_k-independent, $P\{|X_i| \ge t\} \le CP\{|X| \ge t\}$ for all nonnegative real numbers t, $E|X|^p < \infty$, $(1 \le p < 2)$, then*

$$\lim_{n\to\infty} \frac{1}{n^{\frac{1}{p}}}\sum_{i=1}^{n}(X_i - EX_i) = 0 \text{ a.s.}$$

Proof. Really, in that case $\varphi(i) = O(1)$, so, from Corollary 3.3, we obtain

$$\lim_{n\to\infty} \frac{1}{n^{\frac{1}{p}}}\sum_{i=1}^{n}(X_i - EX_i) = 0 \text{ a.s.}$$

□

References

[1] B. von Bahr and C. G. Esseen, *Inequalities for the r-th absolute moment of a sum of random variables,* $1 \le r \le 2$, Ann. Math. Statist. **36** (1965), 299–303.

[2] B. D. Choi and S. H. Sung, *On convergence of $(S_n - ES_n)/n^{1/r}$, $1 < r < 2$, for pairwise independent random variables*, Bull. Korean Math. Soc. **22** (1985), no. 2, 79–82.

[3] N. Etemadi, *An elementary proof of the strong law of large numbers*, Z. Wahrsch. Verw. Gebiete **55** (1981), no. 1, 119–122.

[4] V. F. Gaposhkin, *On the strong law of large numbers for blockwise-independent and blockwise-orthogonal random variables*, Theory Probab. Appl. **39** (1994), no. 4, 677–684.

[5] ______, *Series of block-orthogonal and block-independent systems*, Izv. Vyssh. Uchebn. Zaved. Mat. (1990), no. 5, 12–18.

[6] D. H. Hong and S. Y. Hwang, *Marcinkiewicz-type Strong law of large numbers for double arrays of pairwise independent random variables*, Int. J. Math. Math. Sci. **22** (1999), no. 1, 171–177.

[7] D. H. Hong and A. I. Volodin, *Marcinkiewicz-type law of large numbers for double array*, J. Korean Math. Soc. **36** (1999), no. 6, 1133–1143.

[8] F. Móricz, *Strong limit theorems for blockwise m-independent and blockwise quasi-orthogonal sequences of random variables*, Proc. Amer. Math. Soc. **101** (1987), no. 4, 709–715.

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