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SPHERICAL FUNCTIONS ON PROJECTIVE CLASS ALGEBRAS

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ABSTRACT. Let $F^\alpha G$ be a twisted group algebra with basis $\{u_g|g \in G\}$ and $\mathcal{P} = \{\mathcal{C}_g|g \in G\}$ be a partition of G. A projective class algebra associated with $\mathcal{P}$ is a subalgebra of $F^\alpha G$ generated by all class sums $\sum_{x\in\mathcal{C}_g} u_x$. A main object of the paper is to find interrelationships of projective class algebras in $F^\alpha G$ and in $F^\alpha H$ for $H < G$. And the α-spherical function will play an important role for the purpose. We find functional properties of α-spherical functions and investigate roles of α-spherical functions as characters of projective class algebras.

1. Introduction

Let G be a finite group and F^* be the multiplicative group of a field F with trivial G-action. For a 2-cocycle α in $Z^2(G, F^*)$, let $F^\alpha G$ be a twisted group algebra with F-basis $\{u_g|g \in G\}$, $u_1 = 1 = 1_{F^\alpha G}$ such that $u_g u_x = \alpha(g,x)u_{gx}$ for all $g, x \in G$.

Let $\mathcal{P} = \{\mathcal{C}_g|g \in G\}$ be a partition of G consisting of classes $\mathcal{C}_g$ of G containing g and let $c_g^+ = \sum_{x\in\mathcal{C}_g} u_x$ be the class sum of $\mathcal{C}_g$. A subalgebra A of $F^\alpha G$ generated by all class sums c_g^+ (g in distinct class $\mathcal{C}_g$) is called a *projective class algebra* in $F^\alpha G$ associated with $\mathcal{P}$. Moreover if $\mathcal{P}$ satisfies conditions that $\mathcal{C}_1 = \{1\}$ and $\mathcal{C}_g^{-1} = \mathcal{C}_{g^{-1}}$ for all $g \in G$ and if A has a unit element 1 then A is called a *projective Schur algebra* over G in $F^\alpha G$. If $\alpha = 1$ then A is a *Schur algebra* over G in group algebra FG, and we may refer to [1], [3], [9], and [21] for this topic.

In 1933, I. Schur introduced a special class of subalgebras of finite group algebra ([16]). The theory of these algebras, which were named

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Schur rings, was developed by Wielandt [20] in order to study permutation groups. During last 20 years, there have been important developments of Schur ring theory to algebraic combinatorics [2], indeed the Schur ring over cyclic groups was closely related to the isomorphism problem in cyclic graphs theory (see from [11] to [14]).

Let H be a subgroup of G. Let A be a projective Schur algebra over G in $F^\alpha G$ associated with $\mathcal{P}$, and A' be a projective Schur algebra over H in $F^\alpha H$ associated with partition $\mathcal{P}'$. For each $\mathcal{C}' \in \mathcal{P}'$, if $\mathcal{C}' = \cup\mathcal{C}$ for some $\mathcal{C} \in \mathcal{P}$ then A' is called a *projective Schur subalgebra* of A. The centralizer algebra $C_{F^\alpha G}(F^\alpha H)$ and $F^\alpha G$ itself are projective Schur algebras in $F^\alpha G$. And the center algebra $Z(F^\alpha H)$ is a projective Schur subalgebra of both $C_{F^\alpha G}(F^\alpha H)$ and $F^\alpha G$. The algebra $C_{FG}(FH)$ as an example of Schur algebra, and the representations and characters of the algebra have been studied in [18] and [20].

The purpose of the work is to study connections of projective Schur algebras in $F^\alpha G$ and in $F^\alpha H$. For this aim, a (projective) α-spherical function of G associated with H will play a central role. When $\alpha = 1$, the spherical function was discussed in [5] and [19] as a character of the Schur algebra $C_{FG}(FH)$ in FG. Although projective Schur rings share many common properties with Schur rings, projective Schur rings are more complicate since they need 2-cocycle. In section 3, we develop α-spherical functions in accordance with group characters, and show that they are H-class functions. In section 4, we study functional properties of α-characters and α-spherical functions. We then investigate how α-spherical functions on G work over the center algebra as well as over the centralizer algebra in $F^\alpha G$ in section 5. We determine characters and modules of some projective Schur algebras.

Throughout the paper, let G be a finite group and α be a 2-cocycle in $Z^2(G, F^*)$ having trivial action over a field F of characteristic 0. Let $F^\alpha G$ be the twisted group algebra with basis $\{u_g | g \in G\}$, $u_1 = 1_{F^\alpha G}$, satisfying $u_g u_x = \alpha(g, x) u_{gx}$ for $g, x \in G$. For $H < G$, we use the same symbol for $\alpha \in Z^2(G, F^*)$ and its restriction to $Z^2(H, F^*)$, hence $F^\alpha H$ can be regarded as a subalgebra of $F^\alpha G$ consisting of linear combinations of $\{u_h | h \in H\}$. For an algebra A and a subalgebra B of A, we denote by $Z(A)$ the center of A and by $C_A(B)$ the centralizer of B in A.

2. Preliminaries

Let $H < G$. We say $g, x \in G$ are *H-conjugate* if $g = x^h = hxh^{-1}$ for some $h \in H$. The H-conjugacy is an equivalence relation and G is a union of H-(conjugacy) classes. $g \in G$ is said to be *α-H-regular* if $\alpha(g,h) = \alpha(h,g)$ for all $h \in H$ with $gh = hg$. If g is α-H-regular then so is any H-conjugate of g. An H-conjugacy class of G is *α-H-regular class* if it contains at least one α-H-regular element (see [10, v.2, p.182]). In particular if $G = H$ then H-conjugate and α-H-regular are nothing but *conjugate* and *α-regular* respectively. A 2-cocycle $\alpha \in Z^2(G, F^*)$ is called *normal* if $\alpha(x,g) = \alpha(g^x, x)$ for any α-regular $g \in G$ and any $x \in G$.

LEMMA 1. [10, vol. 3, (1.6.2)] *Let θ be an α-character of G and $g, x \in G$.*

(i) $\theta(g) = \alpha(x,g)\alpha^{-1}(g^x,x)\theta(g^x)$. *If g is not α-regular then $\theta(g) = 0$.*
(ii) *If α is normal then θ is a class function, i.e., $\theta(g) = \theta(g^x)$. The converse holds when F is a splitting field for $F^\alpha G$.*

While an ordinary character is always a class function, projective α-character is a class function when α is normal. However since any cocycle is cohomologous to a normal cocycle over arbitrary field F ([10, vol. 3, (1.6.1)]), α-character can be regarded as a class function. Furthermore, α-characters are entirely determined by restrictions to the set of all α-regular elements G_0 because they vanish outside G_0.

Let λ_i be any function on G. The *α-inner product* $\langle\ ,\ \rangle_{\alpha,G}$ is defined by

$$(1)\qquad \langle \lambda_1, \lambda_2 \rangle_{\alpha,G} = \frac{1}{|G|} \sum_{x\in G} \alpha^{-1}(x, x^{-1})\lambda_1(x)\lambda_2(x^{-1})$$

([10, vol. 3, (1.11.8), p.69]). Hence for any $g \in G_0$ and for Kronecker delta δ_{ij}, the orthogonality relation of χ_i and $\chi_j \in \mathrm{Irr}_\alpha(G)$ follows that

$$(2)\qquad \frac{1}{|G|} \sum_{x\in G} \alpha^{-1}(x, x^{-1})\alpha(x,g)\chi_i(xg)\chi_j(x^{-1}) = \delta_{ij}\, \frac{\chi_i(g)}{\chi_i(1)}.$$

If $\chi \in \mathrm{Irr}_\alpha(G)$ then restriction $\chi|_H$ to H is a sum of irreducible α-characters of H, say $\chi|_H = \sum c_{\chi\phi_i}\phi_i$ for $\phi_i \in \mathrm{Irr}_\alpha(H)$ and $c_{\chi\phi_i} \geq 0$. The $c_{\chi\phi_i}$ is the *multiplicity* of ϕ_i in $\chi|_H$, and is equal to $\langle \chi|_H, \phi_i\rangle_{\alpha,G}$. In particular if $\phi \in \mathrm{Irr}_\alpha(H)$ is contained in $\chi|_H$ (denote $\phi \subset \chi|_H$), we write $\chi|_H = c_{\chi\phi}\phi + \sum_{\phi\neq\phi_i\in\mathrm{Irr}_\alpha(H)} c_{\chi\phi_i}\phi_i$.

Let $\chi \in \mathrm{Irr}_\alpha(G)$ and $\phi \in \mathrm{Irr}_\alpha(H)$ with $\phi \subset \chi|_H$. A map $Y_{\chi\phi} : G \to F^*$ defined by

$$(3)\qquad Y_{\chi\phi}(g) = \frac{1}{|H|} \sum_{h\in H} \alpha^{-1}(h, h^{-1})\alpha(h,g)\chi(hg)\phi(h^{-1})$$

is called a *(projective) α-spherical function* attached to χ and ϕ. If $\alpha = 1$, the (ordinary) spherical function $Y_{\chi\phi}$ has analogous properties of group characters ([5], [19]). The classical theory of spherical functions is a well developed part of harmonic analysis that studies the functions on a real reductive Lie group. Group theoretical spherical functions were discussed in [6] and [7].

3. Projective spherical functions

We begin with easy calculations of 2-cocycles for next use.

LEMMA 2. (i) $\alpha(xz, z^{-1}y) = \alpha^{-1}(x,z)\alpha^{-1}(z^{-1},y)\alpha(z,z^{-1})\alpha(x,y)$ *for* $x, y, z \in G$.

(ii) *In particular,* $\alpha(zx, yz^{-1}) = \alpha^{-1}(z,x)\alpha^{-1}(y,z^{-1})\alpha(z,z^{-1})\alpha(x,y)$, *if* x, y *are* α*-regular in group* G_0 *and* α *is normal. Moreover* $\alpha(x^z, y^z) = \alpha(x,y)$.

Proof. Since $u_{x^{-1}} = \alpha(x, x^{-1})u_x^{-1}$, it is obvious that

$$\alpha(xz, z^{-1}y) = u_{xz}u_{z^{-1}y}u^{-1}_{xzz^{-1}y} = \alpha^{-1}(x,z)\alpha^{-1}(z^{-1},y)\alpha(z,z^{-1})\alpha(x,y).$$

Let α be normal and x, y be α-regular elements in group G_0. Then $\alpha(z,x) = \alpha(x^z, z)$ and $u_z u_x u_z^{-1} = u_{x^z}$ for any $z \in G$. Thus

$$\begin{aligned}\alpha(zx, yz^{-1}) &= u_{zx}u_{yz^{-1}}u^{-1}_{zxyz^{-1}} \\ &= \alpha^{-1}(z,x)\alpha^{-1}(y,z^{-1})u_z u_x u_y u_{z^{-1}}(u_z u_{xy} u_z^{-1})^{-1} \\ &= \alpha^{-1}(z,x)\alpha^{-1}(y,z^{-1})\alpha(z,z^{-1})\alpha(x,y).\end{aligned}$$

Moreover,

$$\alpha(x^z, y^z) = u_{x^z}u_{y^z}u^{-1}_{(xy)^z} = u_z u_x u_y u_z^{-1}(u_z u_{xy} u_z^{-1})^{-1} = \alpha(x,y).$$

□

Let us define a *convolution* $*_\alpha$ (with respect to α) of functions λ_i on G by

$$\begin{aligned}\lambda_1 *_\alpha \lambda_2(g) &= \frac{1}{|G|}\sum_{xy=g} \alpha^{-1}(x,x^{-1})\alpha(x^{-1},g)\lambda_1(y)\lambda_2(x) \\ &= \frac{1}{|G|}\sum_{x\in G} \alpha^{-1}(x,x^{-1})\alpha(x^{-1},g)\lambda_1(x^{-1}g)\lambda_2(x) \quad \text{for } g \in G.\end{aligned}$$

LEMMA 3. *The convolution* $*_\alpha$ *satisfies the associative law.*

Proof. For any functions λ_i $(i = 1, \dots, 3)$ of G and $g \in G$, we have

$$\begin{aligned}
&\lambda_1 *_\alpha (\lambda_2 *_\alpha \lambda_3)(g) \\
&= \frac{1}{|G|}\sum_{z\in G} \alpha^{-1}(z, z^{-1})\alpha(z^{-1}, g)\lambda_1(z^{-1}g)(\lambda_2 *_\alpha \alpha_3)(z) \\
&= \frac{1}{|G|^2}\sum_{z,y\in G} \alpha^{-1}(z, z^{-1})\alpha(z^{-1}, g)\cdot\alpha^{-1}(y, y^{-1})\alpha(y^{-1}, z) \\
&\qquad \cdot \lambda_1(z^{-1}g)\lambda_2(y^{-1}z)\lambda_3(y) \\
&= \frac{1}{|G|^2}\sum_{x,y\in G} \alpha^{-1}(yx, (yx)^{-1})\alpha((yx)^{-1}, g)\alpha^{-1}(y, y^{-1})\alpha(y^{-1}, yx) \\
&\qquad \cdot\ \lambda_1(x^{-1}y^{-1}g)\lambda_2(x)\lambda_3(y) \qquad \text{(by substituting } \ y^{-1}z = x).
\end{aligned}$$

Easy computations in Lemma 2 (i) give rise to the next relations that

$$\alpha((yx)^{-1}, g) = \alpha(x^{-1}, y^{-1}g)\alpha(y^{-1}, g)\alpha^{-1}(x^{-1}, y^{-1}),$$
$$\alpha(y^{-1}, yx) = \alpha(y^{-1}, y)\alpha^{-1}(y, x),$$

and

$$\alpha^{-1}(yx, (yx)^{-1}) = \alpha^{-1}(y, y^{-1})\alpha^{-1}(x, x^{-1})\alpha(y, x)\alpha(x^{-1}, y^{-1}).$$

Therefore, the associative law follows immediately from the calculation that

$$\begin{aligned}
&\lambda_1 *_\alpha (\lambda_2 *_\alpha \lambda_3)(g) \\
&= \frac{1}{|G|^2}\sum_{x,y\in G} \alpha^{-1}(y, y^{-1})\alpha^{-1}(x, x^{-1})\alpha(x^{-1}, y^{-1}g)\alpha(y^{-1}, g) \\
&\qquad \cdot \lambda_1(x^{-1}y^{-1}g)\lambda_2(x)\lambda_3(y) \\
&= \frac{1}{|G|}\sum_{y\in G} \alpha^{-1}(y, y^{-1})\alpha(y^{-1}, g)\frac{1}{|G|}\sum_{x\in G} \alpha^{-1}(x, x^{-1})\alpha(x^{-1}, y^{-1}g) \\
&\qquad \cdot \lambda_1(x^{-1}y^{-1}g)\lambda_2(x)\lambda_3(y) \\
&= \frac{1}{|G|}\sum_{y\in G} \alpha^{-1}(y, y^{-1})\alpha(y^{-1}, g)(\lambda_1 *_\alpha \lambda_2)(y^{-1}g)\lambda_3(y) \\
&= (\lambda_1 *_\alpha \lambda_2) *_\alpha \lambda_3(g).
\end{aligned}$$

□

If no confusion can occur, notation $*$ will be used for $*_\alpha$. We show that the α-spherical function $Y_{\chi\phi}$ is a convolution product of $\chi \in \mathrm{Irr}_\alpha(G)$ and $\phi \in \mathrm{Irr}_\alpha(H)$.

THEOREM 4. (i) *If $H = 1$ or $H = G$ then all $Y_{\chi\phi}$ are α-characters of G.*

(ii) *For any functions λ_i on G, $\lambda_1 * \lambda_2(1) = \langle \lambda_1, \lambda_2 \rangle_{\alpha,G}$. And*

$$Y_{\chi\chi} = \chi * \chi = \frac{1}{\chi(1)}\chi, \quad \text{and } Y_{\chi\phi}(1) = \langle \chi|_H, \phi \rangle_{\alpha,H} = c_{\chi\phi}.$$

(iii) *$Y_{\chi\phi} = \chi * \tilde{\phi}$, where $\tilde{\phi}(x) = \frac{|G|}{|H|}\phi(x)$ if $x \in H$ and 0 otherwise.*

Proof. (i) If $G = H$, then $Y_{\chi\phi}(g) = \frac{1}{|G|}\sum_{x \in G} \alpha^{-1}(x, x^{-1})\alpha(x, g)$ $\chi(xg)\chi(x^{-1})$ which is equal to $\frac{\chi(g)}{\chi(1)}$ by the orthogonality (2), i.e., $Y_{\chi\phi} = \frac{1}{\chi(1)}\chi$. In particular if $H = 1$ then $Y_{\chi\phi}(g) = \chi(g)\phi(1) = \chi(g)$ for all $g \in G$, since $\phi \in \mathrm{Irr}_\alpha(1)$.

(ii) $\lambda_1 * \lambda_2(1) = \frac{1}{|G|}\sum_{x \in G} \alpha^{-1}(x, x^{-1})\lambda_1(x^{-1})\lambda_2(x) = \langle \lambda_1, \lambda_2 \rangle_{\alpha,G}$ from (1). And for any $g \in G$, we have

$$\begin{aligned} \chi * \chi(g) &= \frac{1}{|G|}\sum_{x \in G} \alpha^{-1}(x, x^{-1})\alpha(x^{-1}, g)\chi(x^{-1}g)\chi(x) \\ &= \frac{1}{\chi(1)}\chi(g) \\ &= Y_{\chi\chi}(g), \end{aligned}$$

by (i). Moreover due to (1), we obtain

$$Y_{\chi\phi}(1) = \frac{1}{|H|}\sum_{h \in H} \alpha^{-1}(h, h^{-1})\chi(h)\phi(h^{-1}) = \langle \chi|_H, \phi \rangle_{\alpha,H} = c_{\chi\phi}.$$

(iii) Since $\tilde{\phi}(z) = 0$ for $z \in G - H$, we have

$$\begin{aligned} \chi * \tilde{\phi}(g) &= \frac{1}{|G|}\frac{|G|}{|H|}\sum_{h \in G \cap H} \alpha^{-1}(h, h^{-1})\alpha(h^{-1}, g)\chi(h^{-1}g)\phi(h) \\ &= \frac{1}{|H|}\sum_{h \in H} \alpha^{-1}(h, h^{-1})\alpha(h^{-1}, g)\chi(h^{-1}g)\phi(h) \\ &= Y_{\chi\phi}(g). \end{aligned}$$

□

When α and β are cohomologous cocycles in $Z^2(G, F^*)$, there is a complete parallelism between α- and β-characters. In fact, if $\alpha = \beta(\delta t)$ for some $t : G \to F^*$ with $t(1) = 1$ (δt : coboundary), and if χ is an α-character of G, then there is a unique β-character χ' such that $\chi = t\chi'$ ([10, vol. 3, (1.2.5)]).

THEOREM 5. *Let α, $\beta \in Z^2(G, F^*)$ be cohomologous such that $\alpha = \beta(\delta t)$ for $t : G \to F^*$, $t(1) = 1$. Let χ_i $(i = 1, 2)$ be irreducible α-characters of G and χ_i' be corresponding β-characters of G such $\chi_i = t\chi_i'$. Then*

(i) $\chi_1 *_\alpha \chi_2 = t\,(\chi_1' *_\beta \chi_2')$.

(ii) *For $H < G$, let ϕ_1 be an irreducible α-character of H contained in $\chi_1|_H$. Then there exists unique irreducible β-character ϕ_1' of H contained in $\chi_1'|_H$ where the multiplicity of ϕ_1' in $\chi_1'|_H$ equals that of ϕ_1 in $\chi_1|_H$, i.e., $c_{\chi_1\phi_1} = c_{\chi_1'\phi_1'}$.*

(iii) *For α- and β-spherical functions $Y_{\chi_1\phi_1}$ and $Y_{\chi_1'\phi_1'}$, we have $Y_{\chi_1\phi_1} = t\,Y_{\chi_1'\phi_1'}$.*

Proof. Let $g \in G$. Then (i) follows immediately from the calculation:

$$\begin{aligned}
&\chi_1 *_\alpha \chi_2(g)\\
&= \frac{1}{|G|}\sum_{x\in G}\alpha^{-1}(x,x^{-1})\alpha(x^{-1},g)\chi_1(x^{-1}g)\chi_2(x)\\
&= \frac{1}{|G|}\sum_{x\in G}t^{-1}(x)t^{-1}(x^{-1})t(xx^{-1})t(x^{-1})t(g)t^{-1}(x^{-1}g)t(x^{-1}g)t(x)\\
&\qquad \beta^{-1}(x,x^{-1})\beta(x^{-1},g)\chi_1'(x^{-1}g)\chi_2'(x)\\
&= \frac{t(g)}{|G|}\sum_{x\in G}\beta^{-1}(x,x^{-1})\beta(x^{-1},g)\chi_1'(x^{-1}g)\chi_2'(x)\\
&= t(g)\cdot\chi_1' *_\beta \chi_2'(g).
\end{aligned}$$

Define $\phi_1 = t\phi_1'$. Then ϕ_1' is a β-character of H, and

$$\begin{aligned}
\chi_1'|_H = t^{-1}\chi_1|_H &= t^{-1}(c_{\chi_1\phi_1}\phi_1 + \sum_{j\neq 1} c_{\chi_j\phi_j}\phi_j)\\
&= c_{\chi_1\phi_1}\phi_1' + t^{-1}(\sum_{j\neq 1} c_{\chi_j\phi_j}\phi_j).
\end{aligned}$$

Thus $\phi_1' \subset \chi_1'|_H$ and $c_{\chi_1\phi_1} = c_{\chi_1'\phi_1'}$.

Obviously $Y_{\chi_1\phi_1}(1) = c_{\chi_1\phi_1} = c_{\chi_1'\phi_1'} = Y_{\chi_1'\phi_1'}(1) = (tY_{\chi_1'\phi_1'})(1)$. And it is easy to see that $\tilde{\phi}_1 = t\tilde{\phi}_1'$, for $\tilde{\phi}_1(g) = \frac{|G|}{|H|}\phi_1(g) = \frac{|G|}{|H|}t(g)\phi_1'(g) = (t\,\tilde{\phi}_1')(g)$ for any $g \in G$. Thus due to (i) and Theorem 4 (iii), we obtain

$$Y_{\chi_1\phi_1} = \chi_1 *_\alpha \tilde{\phi}_1 = t\chi_1' *_\alpha t\tilde{\phi}_1' = t(\chi_1' *_\beta \tilde{\phi}_1') = t\,Y_{\chi_1'\phi_1'}.$$

□

THEOREM 6. *Let $\chi \in Irr_\alpha(G)$ and $\phi \in Irr_\alpha(H)$. Then*

(i) $\chi * \tilde{\phi} = \tilde{\phi} * \chi$, $\phi * \phi = \frac{1}{\phi(1)}\phi$, *and* $\tilde{\phi} * \tilde{\phi} = \frac{1}{\phi(1)}\tilde{\phi}$.

(ii) *We assume that* $\phi \subset \chi|_H$. *Then* $Y_{\chi\phi} * \tilde{\phi} = \frac{1}{\phi(1)} Y_{\chi\phi}$ *and* $Y_{\chi\phi} * \chi = \frac{1}{\chi(1)} Y_{\chi\phi}$. *Thus,* $Y_{\chi\phi} * Y_{\chi\phi} = \frac{1}{\chi(1)\phi(1)} Y_{\chi\phi}$.

(iii) *Furthermore,* $\langle Y_{\chi\phi}, \chi\rangle_{\alpha,G} = \frac{c_{\chi\phi}}{\chi(1)}$ *and* $\langle Y_{\chi\phi}, \tilde{\phi}\rangle_{\alpha,G} = \frac{c_{\chi\phi}}{\phi(1)}$.
Thus $\langle Y_{\chi\phi}, Y_{\chi\phi}\rangle_{\alpha,G} = \frac{c_{\chi\phi}}{\chi(1)\phi(1)} = \frac{1}{c_{\chi\phi}}\langle Y_{\chi\phi}, \chi\rangle_{\alpha,G}\langle Y_{\chi\phi}, \tilde{\phi}\rangle_{\alpha,G}$.
Hence $Y_{\chi\phi} = 0$ *if and only if* $c_{\chi\phi} = 0$.

Proof. We first assume that α is a normal 2-cocycle. Then α-character is a class function, and satisfies $\alpha^{-1}(gy^{-1}, yg^{-1})\alpha(yg^{-1}, g) = \alpha^{-1}(y, y^{-1})\alpha(g, y^{-1})$ and $\alpha(g, y^{-1})\chi(gy^{-1}) = \alpha(y^{-1}, g)\chi(y^{-1}g)$ for any $g, y \in G$. So it follows that

$$\begin{aligned}
&\tilde{\phi} *_\alpha \chi(g)\\
&= \frac{1}{|G|}\sum_{xy=g} \alpha^{-1}(x, x^{-1})\alpha(x^{-1}, g)\tilde{\phi}(y)\chi(x)\\
&= \frac{1}{|G|}\sum_{y\in G} \alpha^{-1}(gy^{-1}, yg^{-1})\alpha(yg^{-1}, g)\chi(gy^{-1})\tilde{\phi}(y)\\
&= \frac{1}{|G|}\sum_{y\in G} \alpha^{-1}(y, y^{-1})\alpha(g, y^{-1})\chi(gy^{-1})\tilde{\phi}(y)\\
&= \frac{1}{|G|}\sum_{y\in G} \alpha^{-1}(y, y^{-1})\alpha(y^{-1}, g)\chi(y^{-1}g)\tilde{\phi}(y)\\
&= \chi *_\alpha \tilde{\phi}(g).
\end{aligned}$$

Now for any cocycle α, there always exists a normal 2-cocycle β cohomologous to α. We write $\alpha = \beta(\delta t)$ for $t : G \to F^*$. Then there are $\chi' \in \mathrm{Irr}_\beta(G)$ and $\phi' \in \mathrm{Irr}_\beta(H)$ such that $\chi = t\chi'$, $\phi = t\phi'$ and $\phi' \subset \chi'|_H$. And $\tilde{\phi} = t\tilde{\phi}'$. Thus we have

$$\tilde{\phi} *_\alpha \chi = t\tilde{\phi}' *_\alpha t\chi' = t\ (\tilde{\phi}' *_\beta \chi') = t\ (\chi' *_\beta \tilde{\phi}') = \chi *_\alpha \tilde{\phi}.$$

In what follows, without loss of generality we may assume α is normal. Similar to Theorem 4 (ii), it is obvious that $\phi * \phi = \frac{1}{\phi(1)}\phi$. Now let $g \in G$.

Then

$$\tilde{\phi} * \tilde{\phi}(g) = \frac{1}{|G|} \sum_{x \in G_0} \alpha^{-1}(x, x^{-1}) \alpha(x^{-1}, g) \tilde{\phi}(x^{-1}g) \tilde{\phi}(x)$$
$$= \frac{1}{|H|} \sum_{x \in H_0} \alpha^{-1}(x, x^{-1}) \alpha(x^{-1}, g) \tilde{\phi}(x^{-1}g) \phi(x).$$

If $g \in H_0$ then $x^{-1}g \in H$, so it follows from (2) that

$$\tilde{\phi} * \tilde{\phi}(g) = \frac{|G|}{|H||H|} \sum_{x \in H_0} \alpha^{-1}(x, x^{-1}) \alpha(x^{-1}, g) \phi(x^{-1}g) \phi(x)$$
$$= \frac{|G|}{|H|} \frac{\phi(g)}{\phi(1)} = \frac{\tilde{\phi}(g)}{\phi(1)}.$$

On the other hand, if $g \notin H_0$ then $x^{-1}g \notin H_0$ (otherwise if $x^{-1}g = a \in H_0$ then $g = xa \in H_0$, contradicts). Hence $\tilde{\phi}(g) = \tilde{\phi}(x^{-1}g) = 0$, thus $\tilde{\phi} * \tilde{\phi}(g) = 0 = \frac{1}{\phi(1)} \tilde{\phi}(g) = 0$.

Now the rest parts follow immediately from Theorem 4 (iii) that

$$Y_{\chi\phi} * \chi = \chi * \tilde{\phi} * \chi = \chi * \chi * \tilde{\phi} = \frac{1}{\chi(1)} \chi * \tilde{\phi} = \frac{1}{\chi(1)} Y_{\chi\phi},$$
$$Y_{\chi\phi} * \tilde{\phi} = \chi * \tilde{\phi} * \tilde{\phi} = \frac{1}{\phi(1)} Y_{\chi\phi}$$

and $Y_{\chi\phi} * Y_{\chi\phi} = \chi * \tilde{\phi} * \chi * \tilde{\phi} = \frac{1}{\chi(1)\phi(1)} Y_{\chi\phi}$. Thus by Theorem 4 (ii) we have

$$\langle Y_{\chi\phi}, \chi \rangle_G = Y_{\chi\phi} * \chi(1) = \frac{Y_{\chi\phi}(1)}{\chi(1)} = \frac{c_{\chi\phi}}{\chi(1)} ;$$
$$\langle Y_{\chi\phi}, Y_{\chi\phi} \rangle_G = \frac{Y_{\chi\phi}(1)}{\chi(1)\phi(1)} = \frac{c_{\chi\phi}}{\chi(1)\phi(1)}.$$

□

For $H < G$, assume $\chi_1 \in \mathrm{Irr}_\alpha(G)$ and $\phi_1 \in \mathrm{Irr}_\alpha(H)$. If ϕ_1 is contained in $\chi_1|_H$ with $\chi_1 \neq \chi$ and $\phi_1 \neq \phi$, then it is clear that $Y_{\chi\phi} * \chi_1 = Y_{\chi\phi} * \tilde{\phi}_1 = 0$. Thus $\langle Y_{\chi\phi}, \chi_1 \rangle_G = \langle Y_{\chi\phi}, \tilde{\phi}_1 \rangle_G = 0$, and $\langle Y_{\chi\phi}, Y_{\chi_1\phi_1} \rangle_G = 0$. Hence $Y_{\chi\phi}$ are orthogonal, so are linearly independent.

A map $\theta : G \to F^*$ is called an *H-class function* if θ is constant on H-conjugacy classes of G, i.e., $\theta(g) = \theta(g^h)$ for $h \in H$, $g \in G$. We will show that the α-spherical function $Y_{\chi\phi}$ attached to χ and ϕ is an H-class function.

THEOREM 7. *If α is normal then $Y_{\chi\phi}$ is an H-class function. Moreover, $Y_{\chi\phi}$ is a class function on G_0 if and only if $\chi|_H = c_{\chi\phi}\phi$.*

Proof. Let $g \in G$ and $k \in H$, and we will show that $Y_{\chi\phi}(g^k) = Y_{\chi\phi}(g)$. Since α is normal, the projective α-characters χ and ϕ are class functions so $\chi(g) = \chi(g^z)$ and $\phi(k) = \phi(k^h)$ for any $z \in G$ and $h \in H$. Hence it follows that

$$\begin{aligned} Y_{\chi\phi}(g^k) &= \frac{1}{|H|} \sum_{h \in H_0} \alpha^{-1}(h, h^{-1})\alpha(h, g^k)\chi(hg^k)\phi(h^{-1}) \\ &= \frac{1}{|H|} \sum_{h \in H_0} \alpha^{-1}(h, h^{-1})\alpha(h, g^k)\chi(hg^k)\phi((h^{-1})^{k^{-1}}). \end{aligned}$$

Set $(h^{-1})^{k^{-1}} = s^{-1} \in H$. Then $hg^k = (sg)^k$, and due to Lemma 2 we have

$$\begin{aligned} Y_{\chi\phi}(g^k) &= \frac{1}{|H|} \sum_{s \in H_0} \alpha^{-1}(s^k, (s^{-1})^k)\alpha(s^k, g^k)\chi((sg)^k)\phi(s^{-1}) \\ &= \frac{1}{|H|} \sum_{s \in H_0} \alpha^{-1}(s, s^{-1})\alpha(s, g)\chi(sg)\phi(s^{-1}) \; = \; Y_{\chi\phi}(g). \end{aligned}$$

Now we assume that $\chi|_H = c_{\chi\phi}\phi$. Then $\chi(1) = c_{\chi\phi}\phi(1)$ and

$$\langle Y_{\chi\phi}, \chi\rangle^2_{\alpha,G} = \left(\frac{c_{\chi\phi}}{\chi(1)}\right)^2 = \frac{c_{\chi\phi}}{\phi(1)\chi(1)} = \langle Y_{\chi\phi}, Y_{\chi\phi}\rangle_{\alpha,G} \cdot \langle \chi, \chi\rangle_{\alpha,G},$$

by Theorem 5 (iii). The Cauchy Schwarz theorem on inner product implies that $Y_{\chi\phi}$ is a scalar multiple of χ. Since χ is a class function on G, so is $Y_{\chi\phi}$. Conversely, suppose that $Y_{\chi\phi}$ is a class function on G_0. Since all irreducible α-characters χ_i vanishes outside G_0 and every class function on G_0 is spanned by all $\chi_i|_{G_0}$ ([10, v.3, (1.6.3)]), we may write, for $g \in G_0$,

$$Y_{\chi\phi}(g) = \sum_{\chi_i \in \mathrm{Irr}_\alpha(G)} b_i\chi_i(g) \quad \text{with } b_i \in F^* \text{ and } \chi = \chi_1.$$

Then $b_1 = \langle Y_{\chi\phi}, \chi\rangle_{\alpha,G_0}$ and $b_i = \langle Y_{\chi\phi}, \chi_i\rangle_{\alpha,G_0} = 0$ for $i \neq 1$. So $Y_{\chi\phi} = b_1\chi$, and

$$\begin{aligned} \left(\frac{c_{\chi\phi}}{\chi(1)}\right)^2 &= \langle Y_{\chi\phi}, \chi\rangle^2_{\alpha,G} = b_1^2\langle \chi, \chi\rangle_{\alpha,G} = \langle b_1\chi, b_1\chi\rangle_{\alpha,G} \\ &= \langle Y_{\chi\phi}, Y_{\chi\phi}\rangle_{\alpha,G} \\ &= \frac{c_{\chi\phi}}{\phi(1)\chi(1)}. \end{aligned}$$

Thus $\chi(1) = c_{\chi\phi}\phi(1)$. But since $\chi|_H = c_{\chi\phi}\phi + \sum_{\phi \neq \phi_i \in \mathrm{Irr}_\alpha(H)} c_{\chi\phi_i}\phi_i$, it follows that all $c_{\chi\phi_i} = 0$ and $\chi|_H = c_{\chi\phi}\phi$. □

4. Functional properties of α-spherical functions

This section is devoted to study functional properties of projective α-characters and α-spherical functions on G. We may refer to [5] and [19] in case of $\alpha = 1$.

For each $j = 1, \dots, s$, let χ_j be an irreducible α-character of G, $\mathcal{C}_j$ be the distinct α-regular class of G with class sum $c_j^+ = \sum_{x \in \mathcal{C}_j} u_x$, and let g_j be a representative of $\mathcal{C}_j$. We assume that α is normal, so the α-characters χ_j are class functions.

Define a map ω_χ with respect to an irreducible character $\chi \in \mathrm{Irr}_\alpha(G)$ by

$$\omega_\chi(c_j^+) = |\mathcal{C}_j| \frac{\chi(g_j)}{\chi(1)} \quad \text{for each } j$$

and assume ω_χ extends F-linearly that $\omega_\chi(\sum_{j=1}^s b_j c_j^+) = \sum_{j=1}^s b_j \omega_\chi(c_j^+)$ for $b_j \in F^*$. Since χ is a class function, $\chi(g_j) = \chi(x)$ for any $x \in \mathcal{C}_j$, so ω_χ is well defined. We may refer to [8, p.35] when $\alpha = 1$.

LEMMA 8. [10, vol. 2, (3.3.1)] *Let T be an α-representation of G. Then T^* such that $T^*(\sum_{g \in G} b_g u_g) = \sum_{g \in G} b_g T(g)$ is an F-algebra homomorphism on $F^\alpha G$.*

THEOREM 9. *Let T be an irreducible α-representation of G and let χ be an α-character afforded by T. Then ω_χ is a mapping from the center $Z(F^\alpha G)$ to F, and $T^*(c_j^+) = \omega_\chi(c_j^+)I$, where I is the identity matrix.*

Proof. Clearly all class sums c_j^+ belong to $Z(F^\alpha G)$ for they constitute an F-basis of $Z(F^\alpha G)$ [10, vol. 2,(2.6.3)]. Set $c_j^+ = \sum_{g \in \mathcal{C}_j} u_g$, with an α-regular element $g \in G$. Due to Lemma 8, $T^*(c_j^+) = \sum_{g \in \mathcal{C}_j} T(g)$ and the trace $\mathrm{tr}(T^*(c_j^+)) = |\mathcal{C}_j| \chi(g)$.

Let x be any element in G. Then $\alpha(x, g) = \alpha(g^x, x)$, so

$$\alpha^{-1}(x,g)T(x)T(g) = T(xg) = T(xgx^{-1}x) = \alpha^{-1}(g^x,x)T(g^x)T(x)$$

and $T(x)T(g)T(x)^{-1} = T(g^x)$. Hence we have

$$\begin{aligned} T(x)T^*(c_j^+)T(x)^{-1} &= \sum_{g \in \mathcal{C}_j} T(x)T(g)T(x)^{-1} = \sum_{g \in \mathcal{C}_j} T(g^x) \\ &= \sum_{y \in \mathcal{C}_j} T(y) = T^*(c_j^+), \end{aligned}$$

which shows that $T^*(c_j^+)$ commutes with $T(x)$ for all $x \in G$. Therefore $T^*(c_j^+)$ is a scalar matrix (see [4, (1.7)]), that is, $T^*(c_j^+) = \varepsilon I$ with

$\chi(1) \times \chi(1)$-identity matrix I and $\varepsilon \in F^*$. Taking the trace of both sides, it follows that

$$\varepsilon\chi(1) = \mathrm{tr}T^*(c_j^+) = |\mathcal{C}_j|\chi(g)$$

and $\varepsilon = |\mathcal{C}_j|\frac{\chi(g)}{\chi(1)} = \omega_\chi(c_j^+)$, thus $T^*(c_j^+) = \varepsilon I = \omega_\chi(c_j^+)I$. □

Corollary 10. *Let g be an α-regular element in the center $Z(G)$. Then $T(g) = T^*(u_g) = \omega_\chi(u_g)I$. Conversely if $T(g) = \varepsilon I$ for some $\varepsilon \in F^*$ then $\varepsilon = \omega_\chi(u_g)$.*

Proof. Since $\alpha(g,x) = \alpha(x,g)$ and $T(g)T(x) = T(x)T(g)$ for any $x \in G$, we have $T(g) = \varepsilon I$ for some $\varepsilon \in F^*$. Taking trace, it follows $\chi(g) = \varepsilon\chi(1)$ and $\varepsilon = \frac{\chi(g)}{\chi(1)}$. Since $|\mathcal{C}_g| = 1$, $c_g^+ = u_g$ and $\varepsilon = \frac{\chi(g)}{\chi(1)} = \omega_\chi(u_g)$, thus $T(g) = \omega_\chi(u_g)I$. On the other hand if $T(g) = \varepsilon I$ for $\varepsilon \in F^*$ then $\chi(g) = \varepsilon\chi(1)$ and $\varepsilon = \frac{\chi(g)}{\chi(1)} = \omega_\chi(u_g)$. □

We now have formulae for multiplications of ω_χ's and of α-characters χ in Theorem 11, and of α-spherical functions in Theorem 12.

Theorem 11. *Let χ be an irreducible α-character of G. Then*

(i) $\omega_\chi(c_i^+)\omega_\chi(c_j^+) = \alpha(g_i,g_j)\sum_{m=1}^{s} a_{ijm}\ \omega_\chi(c_m^+)$, *where a_{ijm} is the class algebra constant, i.e., the number of pairs (x,y) for $x \in \mathcal{C}_i$, $y \in \mathcal{C}_j$ with $xy \in \mathcal{C}_m$.*

(ii) $\chi(g)\chi(x) = \frac{\chi(1)}{|G|}\alpha(g,x)\cdot\sum_{z\in G}\chi(gx^z) = \frac{\chi(1)}{|G|}\alpha(x,g)\cdot\sum_{z\in G}\chi(xg^z)$.

Proof. Keep the same notation $\mathcal{C}_j$ for α-regular class with representative g_j. Since $\omega_\chi(c_j^+)I = T^*(c_j^+) = \sum_{\mathcal{C}_j} T(g_j)$ by Theorem 9, (i) follows immediately from

$$\begin{aligned}\omega_\chi(c_i^+)\omega_\chi(c_j^+)I &= \sum_{\mathcal{C}_i}\sum_{\mathcal{C}_j}\alpha(g_i,g_j)T(g_ig_j)\\ &= \sum_{m=1}^{s}\alpha(g_i,g_j)a_{ijm}\sum_{g_m\in\mathcal{C}_m}T(g_m)\\ &= \alpha(g_i,g_j)\sum_{m=1}^{s}a_{ijm}T^*(c_m^+)\\ &= \alpha(g_i,g_j)\sum_{m=1}^{s}a_{ijm}\omega_\chi(c_m^+)I.\end{aligned}$$

Since χ is a class function, we have, for any g, x and $y \in G$,

$$\sum_{z\in G}\chi(gx^z) = \sum_{z\in G}\chi((gx^z)^y) = \sum_{z\in G}\chi(g^y x^{yz}) = \sum_{t\in G}\chi(g^y x^t) = \sum_{z\in G}\chi(g^y x^z).$$

Let $g = g_i \in \mathcal{C}_i$ and $x = g_j \in \mathcal{C}_j$ for some $1 \leq i, j \leq s$. Then

$$\begin{aligned}\sum_{z\in G}\chi(gx^z) &= \frac{1}{|G|}\sum_{y\in G}\sum_{z\in G}\chi(g_i^y \cdot g_j^z)\\ &= \frac{1}{|G|}\frac{|G|}{|\mathcal{C}_i|}\frac{|G|}{|\mathcal{C}_j|}\sum_{a_i\in\mathcal{C}_i}\sum_{a_j\in\mathcal{C}_j}\chi(a_i a_j)\\ &= \frac{|G|}{|\mathcal{C}_i||\mathcal{C}_j|}\sum_{m=1}^{s}|\mathcal{C}_m|a_{ijm}\chi(g_m)\\ &= \frac{|G|}{|\mathcal{C}_i||\mathcal{C}_j|}\chi(1)\sum_{m=1}^{s}a_{ijm}\omega_\chi(c_m^+).\end{aligned}$$

Therefore (ii) follows from (i) that

$$\begin{aligned}\frac{\chi(1)}{|G|}\alpha(g,x)\sum_{z\in G}\chi(gx^z) &= \frac{\chi(1)\chi(1)}{|\mathcal{C}_i||\mathcal{C}_j|}\alpha(g_i,g_j)\sum_{m=1}^{s}a_{ijm}\omega_\chi(c_m^+)\\ &= \frac{\chi(1)}{|\mathcal{C}_i|}\frac{\chi(1)}{|\mathcal{C}_j|}\omega_\chi(c_i^+)\omega_\chi(c_j^+)\\ &= \chi(g)\chi(x).\end{aligned}$$

□

Now for an arithmetic property of α-spherical functions, let $g, x \in G$. Then

$$\begin{aligned}&Y_{\chi\phi}(g)Y_{\chi\phi}(x)\\ &= \frac{1}{|H|^2}\sum_{v,k\in H}\alpha^{-1}(v,v^{-1})\alpha^{-1}(k,k^{-1})\alpha(v,g)\alpha(k,x)\\ &\quad\cdot\ \chi(vg)\chi(kx)\phi(k^{-1})\phi(v^{-1})\\ &= \frac{\phi(1)}{|H|^3}\sum_{v,k,h\in H}\alpha^{-1}(v,v^{-1})\alpha^{-1}(k,k^{-1})\alpha(v,g)\alpha(k,x)\alpha(k^{-1},v^{-1})\\ &\quad\cdot\ \chi(vg)\chi(kx)\phi(k^{-1}v^{-h})\end{aligned}$$

due to Theorem 11 (ii). Because $\chi(g) = 0$ when g is not α-regular, we may assume $g, x \in G$ and $v, k, h \in H$ are all α-regular. If α is a normal 2-cocycle then ϕ is a class function so $\phi(k^{-1}v^{-h}) = \phi(v^{-h}k^{-1})$, thus we obtain

$$\begin{aligned}
&Y_{\chi\phi}(g)Y_{\chi\phi}(x)\\
&= \frac{\phi(1)}{|H|^3} \sum_{v,k,h\in H} \alpha^{-1}(v, v^{-1})\alpha^{-1}(k, k^{-1})\alpha(v, g)\alpha(k, x)\alpha(k^{-1}, v^{-1})\\
&\quad \cdot\ \chi(vg)\chi(kx)\phi(v^{-h}k^{-1})\\
&= \frac{\phi(1)}{|H|^3} \sum_{a,k,h\in H} \alpha^{-1}\big((k^{-1}a)^{h^{-1}}, (a^{-1}k)^{h^{-1}}\big)\alpha^{-1}(k, k^{-1})\alpha\big((k^{-1}a)^{h^{-1}}, g\big)\\
&\quad \cdot\ \alpha(k, x)\alpha\big(k^{-1}, (a^{-1}k)^{h^{-1}}\big)\chi\big((k^{-1}a)^{h^{-1}}g\big)\chi(kx)\phi(a^{-1})
\end{aligned}$$

by substituting $v^{-h}k^{-1} = a^{-1}$ (so $v = (k^{-1}a)^{h^{-1}}$). But by Theorem 11 (ii), since

$$\begin{aligned}
\chi\big((k^{-1}a)^{h^{-1}}g\big)\chi(kx) &= \chi(k^{-1}ag^h)\chi(xk)\\
&= \frac{\chi(1)}{|G|}\alpha(k^{-1}ag^h, xk)\sum_{y\in G}\chi\big(k^{-1}ag^h(xk)^y\big)\\
&= \frac{\chi(1)}{|G|}\alpha(k^{-1}ag^h, xk)\sum_{y\in G}\chi(ag^hyxy^{-k}),
\end{aligned}$$

we have a multiplication formula of the spherical function $Y_{\chi\phi}$ that

$$\begin{aligned}
&Y_{\chi\phi}(g)Y_{\chi\phi}(x)\\
&= \frac{\phi(1)}{|H|^3}\frac{\chi(1)}{|G|} \sum_{a,k,h\in H}\sum_{y\in G} \alpha^{-1}\big((k^{-1}a)^{h^{-1}}, (a^{-1}k)^{h^{-1}}\big)\alpha^{-1}(k, k^{-1})\\
&\quad \cdot \alpha\big((k^{-1}a)^{h^{-1}}, g\big)\alpha(k, x)\alpha\big(k^{-1}, (a^{-1}k)^{h^{-1}}\big)\\
&\quad \cdot \alpha(k^{-1}ag^h, xk)\chi(ag^hyxy^{-k})\phi(a^{-1}).
\end{aligned} \tag{4}$$

In particular if $\alpha = 1$ then the (ordinary) spherical function $Y_{\chi\phi}$ satisfies

$$\begin{aligned}
Y_{\chi\phi}(g)Y_{\chi\phi}(x) &= \frac{\phi(1)}{|H|^3}\frac{\chi(1)}{|G|} \sum_{a,k,h\in H}\sum_{y\in G}\chi(ag^hyxy^{-k})\phi(a^{-1})\\
&= \frac{\phi(1)}{|H|^2}\frac{\chi(1)}{|G|} \sum_{k,h\in H}\sum_{y\in G} Y_{\chi\phi}(g^hyxy^{-k}),
\end{aligned}$$

this is the result obtained by Gallagher in [5]. Moreover if G is abelian then

$$Y_{\chi\phi}(g)Y_{\chi\phi}(x) = \frac{\phi(1)}{|H|^2}\frac{\chi(1)}{|G|}\sum_{k,h\in H}\sum_{y\in G} Y_{\chi\phi}(gx) = Y_{\chi\phi}(gx).$$

Prior to this, it was proved that $Y_{\chi\phi}(g)Y_{\chi\phi}(x) = \frac{1}{|H|}\sum_{h\in H} Y_{\chi\phi}(gx^h)$ if $c_{\chi\phi} = 1$ in [19, Corollary 1] by using more representation-theoretic interpretations.

Now in order to have a simple formula for product of projective spherical functions, we will assume that G is an abelian group in Theorem 12. As mentioned before, the structure of Schur algebra over abelian group has wide application in combinatorics, graph and design theory.

THEOREM 12. *If G is abelian then the multiplication formula of $Y_{\chi\phi}$ is*

$$Y_{\chi\phi}(g)Y_{\chi\phi}(x) = \chi(1)\phi(1)\alpha(g,x)Y_{\chi\phi}(gx) \quad \textit{for any } g, x \in G.$$

Proof. We first notice that $\alpha \in Z^2(G, F^*)$ is normal because G is abelian. In fact, if u is any α-regular element and v is any element in G, then $v \in C_G(u)$ and $\alpha(u, v) = \alpha(v, u) = \alpha(v, u^v)$. Thus from (4), we have

$$\begin{aligned}
&Y_{\chi\phi}(g)Y_{\chi\phi}(x)\\
&= \frac{\phi(1)}{|H|^3}\frac{\chi(1)}{|G|}\sum_{a,k,h\in H}\sum_{y\in G}\alpha^{-1}(k^{-1}a, a^{-1}k)\alpha^{-1}(k, k^{-1})\alpha(k^{-1}a, g)\alpha(k, x)\\
&\quad\cdot\ \alpha(k^{-1}, a^{-1}k)\alpha(k^{-1}ag, xk)\cdot\alpha(a, a^{-1})\alpha^{-1}(a, gx)\\
&\quad\cdot\ \alpha^{-1}(a, a^{-1})\alpha(a, gx)\chi(agx)\phi(a^{-1})\\
&= \frac{\phi(1)}{|H|^3}\frac{\chi(1)}{|G|}\sum_{a,k,h\in H}\sum_{y\in G}\Gamma\cdot\alpha^{-1}(a, a^{-1})\alpha(a, gx)\chi(agx)\phi(a^{-1}),
\end{aligned}$$

where

$$\begin{aligned}
\Gamma &= \alpha^{-1}(k^{-1}a, a^{-1}k)\alpha^{-1}(k, k^{-1})\alpha(k^{-1}a, g)\alpha(k, x)\alpha(k^{-1}, a^{-1}k)\\
&\quad\cdot\ \alpha(k^{-1}ag, xk)\alpha(a, a^{-1})\alpha^{-1}(a, gx).
\end{aligned}$$

We claim that Γ equals $\alpha(g,x)$. If then, we have our desired formula that

$$\begin{aligned}
&Y_{\chi\phi}(g)Y_{\chi\phi}(x)\\
&=\frac{\phi(1)}{|H|^3}\frac{\chi(1)}{|G|}\sum_{a,k,h\in H}\sum_{y\in G}\alpha(g,x)\cdot\alpha^{-1}(a,a^{-1})\alpha(a,gx)\chi(agx)\phi(a^{-1})\\
&=\phi(1)\chi(1)\frac{\alpha(g,x)}{|H|}\sum_{a\in H}\alpha^{-1}(a,a^{-1})\alpha(a,gx)\chi(agx)\phi(a^{-1})\\
&=\chi(1)\phi(1)\alpha(g,x)Y_{\chi\phi}(gx).
\end{aligned}$$

To prove the claim, we use the equalities

$$\alpha^{-1}(k^{-1}a,a^{-1}k)=\alpha(k^{-1},a)\alpha(a^{-1},k)\alpha^{-1}(a,a^{-1})\alpha^{-1}(k^{-1},k)$$

and

$$\alpha(k^{-1},a^{-1}k)=\alpha(k^{-1},k)\alpha^{-1}(a^{-1},k).$$

Then

$$\begin{aligned}
\Gamma&=\alpha(k^{-1},a)\alpha(a^{-1},k)\alpha^{-1}(a,a^{-1})\alpha^{-1}(k^{-1},k)\cdot\alpha^{-1}(k,k^{-1})\alpha(k^{-1}a,g)\\
&\quad\cdot\ \alpha(k,x)\cdot\alpha(k^{-1},k)\alpha^{-1}(a^{-1},k)\alpha(k^{-1}ag,xk)\alpha(a,a^{-1})\alpha^{-1}(a,gx)\\
&=\alpha(k^{-1},a)\alpha^{-1}(k,k^{-1})\alpha(k^{-1}a,g)\alpha(k,x)\alpha(k^{-1}ag,xk)\alpha^{-1}(a,gx).
\end{aligned}$$

Now by substituting some values of α in the above equation by

$$\alpha(k^{-1}a,g)\alpha(k^{-1}ag,xk)\ =\ \alpha(k^{-1}a,gxk)\alpha(g,xk),$$

and

$$\alpha^{-1}(a,gx)=\alpha^{-1}(ag,x)\alpha^{-1}(a,g)\alpha(g,x),$$

we obtain that

$$\begin{aligned}
\Gamma=&\ \alpha(k^{-1},a)\alpha^{-1}(k,k^{-1})\alpha(k^{-1}a,gxk)\alpha(g,xk)\\
&\cdot\alpha(k,x)\cdot\alpha^{-1}(ag,x)\alpha^{-1}(a,g)\alpha(g,x).
\end{aligned}$$

Due to easy computations that

$$\alpha(k^{-1},a)\alpha^{-1}(k,k^{-1})\alpha(k^{-1}a,gxk)\alpha(g,x)=\alpha^{-1}(gx,k)\alpha(ag,x)\alpha(a,g),$$

and

$$\alpha(g,xk)\alpha(k,x)=\alpha(gx,k)\alpha(g,x),$$

the claim follows immediately that

$$\begin{aligned}
\Gamma&=\alpha^{-1}(gx,k)\alpha(ag,x)\alpha(a,g)\cdot\alpha(gx,k)\alpha(g,x)\cdot\alpha^{-1}(ag,x)\alpha^{-1}(a,g)\\
&=\alpha(g,x).
\end{aligned}$$

□

COROLLARY 13. *Let $H = Z(G)$, and $g, x \in G$ satisfying $\alpha(a, gx) = \alpha(a, gx^y)$ for any $a \in H$, $y \in G$. Then $Y_{\chi\phi}(g)Y_{\chi\phi}(x) = \chi(1)\phi(1)\alpha(g,x)\frac{1}{|G|}\sum_{y\in G} Y_{\chi\phi}(gx^y)$.*

Proof. Since $H = Z(G)$, the formula in (4) shows that $Y_{\chi\phi}(g)Y_{\chi\phi}(x)$ equals

$$\begin{aligned}
&= \frac{\phi(1)}{|H|^3}\frac{\chi(1)}{|G|}\sum_{a,k,h\in H}\sum_{y\in G}\alpha^{-1}(k^{-1}a, a^{-1}k)\alpha^{-1}(k,k^{-1}) \\
&\quad\cdot\ \alpha(k^{-1}a, g)\alpha(k,x)\alpha(k^{-1}, a^{-1}k) \\
&\quad\cdot\ \alpha(k^{-1}ag, xk)\cdot\alpha(a,a^{-1})\alpha^{-1}(a, gx^y) \\
&\quad\cdot\ \alpha^{-1}(a,a^{-1})\alpha(a,gx^y)\chi(agx^y)\phi(a^{-1}) \\
&= \frac{\phi(1)}{|H|^3}\frac{\chi(1)}{|G|}\sum_{a,k,h\in H}\sum_{y\in G}\Gamma\cdot\alpha^{-1}(a,a^{-1})\alpha(a,gx^y)\chi(agx^y)\phi(a^{-1}),
\end{aligned}$$

where $\Gamma = \alpha^{-1}(k^{-1}a, a^{-1}k)\alpha^{-1}(k,k^{-1})\alpha(k^{-1}a,g)\alpha(k,x)\alpha(k^{-1},a^{-1}k)$
$$\cdot\ \alpha(k^{-1}ag, xk)\alpha(a,a^{-1})\alpha^{-1}(a,gx^y).$$

It is easy to see $\Gamma = \alpha(a,gx)\alpha(g,x)\alpha^{-1}(a,gx^y)\alpha(k,x)\alpha^{-1}(x,k)$ by Lemma 2. But since $k \in H = Z(G)$, $\alpha(k,x) = \alpha(x,k)$. And since $\alpha(a,gx) = \alpha(a,gx^y)$, it follows that $\Gamma = \alpha(g,x)$ and

$$Y_{\chi\phi}(g)Y_{\chi\phi}(x) = \chi(1)\phi(1)\alpha(g,x)\frac{1}{|G|}\sum_{y\in G} Y_{\chi\phi}(gx^y).$$

□

5. Subclass algebras of twisted group algebra

In this section, we investigate how α-spherical functions on G work over $Z(F^\alpha G)$ and over $C_{F^\alpha G}(F^\alpha H)$, where both $Z(F^\alpha G)$ and $C_{F^\alpha G}(F^\alpha H)$ are projective Schur algebras in $F^\alpha G$. By keeping the notations M_j ($j = 1, \ldots, s$), χ_j, $\mathcal{C}_j$ and $c_j^+ = \sum_{x\in\mathcal{C}_j} u_x$ as before, let f_j be the nonzero primitive central idempotent of $F^\alpha G$ such that each M_j lies over f_j (i.e., $M_j f_j \neq 0$), and let $\{g_j\}$ be a set of representatives of $\mathcal{C}_j$. We assume that $\alpha \in Z^2(G, F^*)$ is normal.

LEMMA 14. *Each f_j forms $\frac{\chi_j(1)}{|G|}\sum_{k=1}^{s}\alpha^{-1}(g_k, g_k^{-1})\chi_j(g_k^{-1})c_k^+$ in $Z(F^\alpha G)$.*

Proof. The idempotent f_j equals $\frac{\chi_j(1)}{|G|}\sum_{g\in G_0}\alpha^{-1}(g,g^{-1})\chi_j(g^{-1})u_g$ ([10, vol. 3, (1.11.1)]). Since $\chi_j(g)=\chi_j(g_k)$ and $\alpha(g,g^{-1})=\alpha(g_k,g_k^{-1})$ for all $g\in\mathcal{C}_k$, we have

$$\begin{aligned} f_j &= \frac{\chi_j(1)}{|G|}\sum_{k=1}^{s}\sum_{g_k\in\mathcal{C}_k}\alpha^{-1}(g_k,g_k^{-1})\chi_j(g_k^{-1})u_{g_k} \\ &= \frac{\chi_j(1)}{|G|}\sum_{k=1}^{s}\alpha^{-1}(g_k,g_k^{-1})\chi_j(g_k^{-1})c_k^+, \end{aligned}$$

and this belongs to $Z(F^\alpha G)$ because c_k^+ constitutes a basis of $Z(F^\alpha G)$. □

THEOREM 15. *Over $Z(F^\alpha G)$, all $Z(F^\alpha G)f_j$ are irreducible modules with dimension 1, and each ω_{χ_j} is a linearly independent irreducible character corresponding to the simple module $Z(F^\alpha G)f_j$. In particular, $\omega_{\chi_j}(f_l)=\delta_{jl}$.*

Proof. Since $F^\alpha G$ is decomposed into $F^\alpha Gf_1\oplus\cdots\oplus F^\alpha Gf_s$, we have

$$Z(F^\alpha G)=Z(F^\alpha G)f_1\oplus\cdots\oplus Z(F^\alpha G)f_s=\bigoplus_{j=1}^{s}Z(F^\alpha G)f_j,$$

where $Z(F^\alpha G)f_j$ is a simple two sided ideal of $Z(F^\alpha G)$. Comparing dimensions, we have $s=\dim Z(F^\alpha G)=\sum_{j=1}^{s}\dim Z(F^\alpha G)f_j$, so $\dim Z(F^\alpha G)f_j=1$ for all j.

Since ω_{χ_j} is defined on $Z(F^\alpha G)$, ω_{χ_j} maps f_l to

$$\begin{aligned} \omega_{\chi_j}(f_l) &= \frac{\chi_l(1)}{|G|}\sum_{k=1}^{s}\alpha^{-1}(g_k,g_k^{-1})\chi_l(g_k^{-1})|\mathcal{C}_k|\frac{\chi_j(g_k)}{\chi_j(1)} \\ &= \frac{\chi_l(1)}{\chi_j(1)|G|}\sum_{g\in G_0}\alpha^{-1}(g,g^{-1})\chi_l(g^{-1})\chi_j(g) \\ &= \frac{\chi_l(1)}{\chi_j(1)}\langle\chi_j,\chi_l\rangle_{\alpha,G} \\ &= \frac{\chi_l(1)}{\chi_j(1)}\delta_{jl} \\ &= \delta_{jl}. \end{aligned}$$

Thus each ω_{χ_j} is a linearly independent irreducible character of $Z(F^\alpha G)$ corresponding to $Z(F^\alpha G)f_j$. □

Similar to the symbols M_j, f_j and χ_j over $F^\alpha G$, let N_i be an irreducible $F^\alpha H$-module which affords an α-character ϕ_i, and e_i be a primitive central idempotent of $F^\alpha H$ with $N_i e_i \neq 0$ for $i = 1, \ldots, t$. Then $F^\alpha G \cong \oplus \sum_{j=1}^{s} \mathrm{End}_F(M_j)$ and $F^\alpha H \cong \oplus \sum_{i=1}^{t} \mathrm{End}_F(N_i)$. Each M_j, viewed as a left $F^\alpha H$-module, is uniquely written by $M_j|_{F^\alpha H} = \oplus_{i=1}^{t} c_{ij} N_i$, where $c_{ij} = c_{\chi_j \phi_i}$ is the one satisfying $\chi_j|_H = \sum_{i=1}^{t} c_{ij}\phi_i$.

THEOREM 16. *Over* $C_{F^\alpha G}(F^\alpha H)$, *all* $e_i M_j$ $(i = 1, .., t; j = 1, .., s)$ *are irreducible modules with dimension* c_{ij}.

Proof. Each $e_i M_j$ is an irreducible $C_{F^\alpha G}(F^\alpha H)$-module by defining $q(e_i m) = e_i(qm) \in e_i M_j$ for $q \in C_{F^\alpha G}(F^\alpha H)$, $m \in M_j$. The fact $\dim(e_i M_j) = c_{ij}$ was proved in [9, Corollary 2.2] when $\alpha = 1$, and the proof can be modified to twisted group algebra. Indeed, $\mathrm{Hom}_{F^\alpha H}(F^\alpha H e_i, M_j)$ is an $C_{F^\alpha G}(F^\alpha H)$-module and is isomorphic to $e_i M_j$ under ψ to $\psi(e_i)$ for any $\psi \in \mathrm{Hom}_{F^\alpha H}(F^\alpha H e_i, M_j)$. Thus

$$\begin{aligned} e_i M_j \cong \mathrm{Hom}_{F^\alpha H}(F^\alpha H e_i, M_j) &\cong \mathrm{Hom}_{F^\alpha H}(N_i, \oplus_{k=1}^{t} c_{kj} N_k) \\ &= c_{ij} \mathrm{Hom}_{F^\alpha H}(N_i, N_i), \end{aligned}$$

so that $\dim(e_i M_j) = \dim(c_{ij}\mathrm{Hom}_{F^\alpha H}(N_i, N_i)) = c_{ij}$. □

Let $\mathcal{P}$ be the set of α-regular classes $\mathcal{D}_h$ in H with class sum $d_h^+ = \sum_{a \in \mathcal{D}_h} u_a$. And let $\mathcal{Q}$ be the set of α-H-regular classes $\mathcal{E}_y$ in G with $e_y^+ = \sum_{x \in \mathcal{E}_y} u_x$. Let S be an F-algebra generated by all α-H-regular class sums e_y^+, i.e., $S = \oplus_{y \in G_0} F e_y^+$. Since $\mathcal{Q}$ is a partition of G with $\mathcal{E}_y^{-1} = \mathcal{E}_{y^{-1}}$ and $\mathcal{E}_1 = \{1\}$, S is a projective Schur algebra over G in $F^\alpha G$. Moreover since e_y^+ constitutes a basis of $C_{F^\alpha G}(F^\alpha H)$ (see [10, v.2, (6.2.3)]), S is equal to $C_{F^\alpha G}(F^\alpha H)$ whose dimension is the number of α-H-regular classes in G.

Let ψ_{ij} $(i = 1, .., t; j = 1, .., s)$ be irreducible characters that correspond to S-modules $e_i M_j$ of dimension c_{ij} (Theorem 16), so $\deg \psi_{ij} = c_{ij} = \dim(e_i M_j)$.

THEOREM 17. *Let* Y_{ij}^* *be the* F*-linearly extended map of* $Y_{ij} = Y_{\chi_j \phi_i}$ *to* $F^\alpha G$ *where* $\phi_i \subset \chi_j|_H$. *Then* $Y_{ij}^*|_{Z(F^\alpha H)} = c_{ij}\omega_{\phi_i} = \psi_{ij}|_{Z(F^\alpha H)}$. *Thus over* $Z(F^\alpha H)$, Y_{ij}^* *is a character corresponding to* $e_i M_j$.

Proof. Let $b_g \in F^*$. The extended map $Y_{ij}^* : F^\alpha G \to F$ is determined by

$$\begin{aligned}Y_{ij}^*(\sum_{g\in G} b_g u_g) &= \sum_{g\in G} b_g Y_{ij}(g)\\ &= \sum_{g\in G, h\in H} b_g \frac{1}{|H|}\alpha^{-1}(h,h^{-1})\alpha(h,g)\chi_j(hg)\phi_i(h^{-1}).\end{aligned}$$

Since the restriction $\psi_{ij}|_{Z(F^\alpha H)}$ can be written as a linear combination of irreducible characters ω_{ϕ_i} on $Z(F^\alpha H)$ (Theorem 15), we may write

$$\psi_{ij}|_{Z(F^\alpha H)} = \sum_{k=1}^{t} b_{ijk}\ \omega_{\phi_k} \quad \text{for some } b_{ijk} \geq 0.$$

In terms of S-module e_iM_j and of $Z(F^\alpha H)$-module $Z(F^\alpha H)e_k$ that correspond to ψ_{ij} and ω_{ϕ_k} respectively (Theorems 15, 16), the equality can be interpreted by

$$e_iM_j|_{Z(F^\alpha H)} = \oplus_{k=1}^{t} b_{ijk} Z(F^\alpha H)e_k.$$

By multiplying e_i to both sides of the above equation, it follows that

$$e_iM_j|_{Z(F^\alpha H)} = e_ie_iM_j|_{Z(F^\alpha H)} = \oplus_{k=1}^{t} b_{ijk} Z(F^\alpha H)e_ie_k = b_{iji}Z(F^\alpha H)e_i.$$

Comparing dimensions of both sides, we have $b_{iji} = c_{ij}$ and $b_{ijk} = 0$ for $i \neq k$, for $\dim(Z(F^\alpha H)e_i) = 1$. Thus $\psi_{ij}|_{Z(F^\alpha H)} = c_{ij}\omega_{\phi_i}$ and $e_iM_j|_{Z(F^\alpha H)} = c_{ij}Z(F^\alpha H)e_i$.

On the other hand, since $d_h^+ = \sum_{a\in\mathcal{D}_h} u_a$ forms a basis of $Z(F^\alpha H)$, $Z(F^\alpha H)$ is a projective Schur subalgebra whose dimension is the number of α-regular classes in H. Then since Y_{ij} is an H-class function (Theorem 7), it follows that

$$\begin{aligned}&Y_{ij}^*(d_h^+)\\ &= |\mathcal{D}_h|Y_{ij}(h)\\ &= \frac{|\mathcal{D}_h|}{|H|}\sum_{a\in H}\alpha^{-1}(a,a^{-1})\alpha(a,h)\chi_j(ah)\phi_i(a^{-1})\\ &= \frac{|\mathcal{D}_h|}{|H|}\sum_{a\in H}\alpha^{-1}(a,a^{-1})\alpha(a,h)\Big(c_{ij}\phi_i(ah) + \sum_{l\neq i} c_{lj}\phi_l(ah)\Big)\phi_i(a^{-1})\\ &= \frac{|\mathcal{D}_h|}{|H|}\sum_{a\in H}\alpha^{-1}(a,a^{-1})\alpha(a,h)c_{ij}\phi_i(ah)\phi_i(a^{-1}) = c_{ij}|\mathcal{D}_h|\frac{\phi_i(h)}{\phi_i(1)}\\ &= c_{ij}\omega_{\phi_i}(d_h^+),\end{aligned}$$

where the fourth equality is due to the orthogonality relation in (2). □

We observe $\deg \omega_{\phi_i} = 1$, because $c_{ij} = \deg \psi_{ij} = c_{ij} \deg \omega_{\phi_i}$. When $\alpha = 1$, the equality $\psi_{ij}|_{Z(F^\alpha H)} = c_{ij}\omega_{\phi_i}$ was proved in [9, Theorem 3.1].

Let $S = C_{F^\alpha G}(F^\alpha H)$. For the projective Schur subalgebra $Z(F^\alpha H)$ of S, an irreducible character ω_{ϕ_i} was defined over $Z(F^\alpha H)$ in Theorem 15 that $\omega_{\phi_i}(d_h^+) = |\mathcal{D}_h| \frac{\phi_i(h)}{\phi_i(1)}$ for $d_h^+ = \sum_{a \in \mathcal{D}_h} u_a$. And its induced character $\omega_{\phi_i}^S$ on S is defined in the following way. First we let

$$\zeta_S : S \to S, \quad \zeta_S(z) = \sum_{\mathcal{E}_y \in \mathcal{Q}} \frac{\mathrm{lcm}_{y \in G}|\mathcal{E}_y|}{|\mathcal{E}_y|} e_y^+ z e_{y^{-1}}^+ \quad \text{for} \quad z \in S,$$

where $\mathcal{Q}$ consists of α-H-regular classes $\mathcal{E}_y$ in G with class sum e_y^+ ([15, Section 1]). Similarly, let $\zeta_{Z(F^\alpha H)} : Z(F^\alpha H) \to Z(F^\alpha H)$ be defined in the same manner as is determined ζ_S. Let $d_0 = \zeta_{Z(F^\alpha H)}(d_1^+)$. Then the map $\omega_{\phi_i}^S$ is constructed by

$$\omega_{\phi_i}^S : S \to F^* , \quad \omega_{\phi_i}^S = \frac{\mathrm{lcm}_{h \in H}|\mathcal{D}_h|}{\mathrm{lcm}_{y \in G}|\mathcal{E}_y|} \frac{1}{\omega_{\phi_i}(d_0)} \widetilde{\omega}_{\phi_i} \cdot \zeta_S,$$

where $\widetilde{\omega}_{\phi_i} : S \to F^*$ is the extension of ω_{ϕ_i} (i.e., for any $z \in S$, $\widetilde{\omega}_{\phi_i}(z) = \omega_{\phi_i}(z)$ if z is a class sum in $\mathcal{P}$ and $\widetilde{\omega}_{\phi_i} = 0$ otherwise) (see [15, Section 3]).

THEOREM 18. *If $\omega_{\phi_i}^S$ on S is irreducible then $\widetilde{\omega}_{\phi_i} = Y_{ij}^*|_S$ for some j.*

Proof. Since $\psi_{ij}|_{Z(F^\alpha H)} = c_{ij}\omega_{\phi_i}$ by Theorem 17, the reciprocity theorem ([15, Satz 9] or [9, Theorem 3.1]) shows that $\omega_{\phi_i}^S = \sum_{j=1}^s c_{ij}\psi_{ij}$ for all $1 \le i \le t$.

Since $\omega_{\phi_i}^S$ is irreducible, there is $1 \le k \le s$ such that $c_{ik} = 1$ and $c_{il} = 0$ for all $l \ne k$. Moreover we can observe that each $\mathcal{E}_y \in \mathcal{Q}$ is contained in $\mathcal{D}_a \in \mathcal{P}$ for some $a \in H$, where $\mathcal{P}$ [resp. $\mathcal{Q}$] means the set of all α-regular classes $\mathcal{D}_h$ in H [resp. α-H-regular classes $\mathcal{E}_g$ in G] with class sum d_h^+ [resp. e_g^+]. In fact, we suppose contrary that $\mathcal{E}_y$ $(y \in G)$ is not contained in any $\mathcal{D}_a \in \mathcal{P}$. Then $\mathcal{E}_y \cap H = \emptyset$, otherwise if $b \in \mathcal{E}_y \cap H$ then $\mathcal{E}_y = \mathcal{E}_b = \{cbc^{-1} | c \in H\}$ might be a conjugacy class $\mathcal{D}_b$ in H, which is a contradiction. Due to [9, Proposition 3.4], it follows that $\sum_l \chi_l(1)\psi_{il}(e_y^+) = 0$, where the sum is taken over all l such that $c_{il} \ne 0$. But since $c_{ik} = 1$ and $c_{il} = 0$ for all $l \ne k$, we have $0 = \chi_k(1)\psi_{ik}(e_y^+)$, so $\psi_{ik}(e_y^+) = 0$ for all $e_y^+ \in S$. This yields a contradiction that ψ_{ik} is an irreducible character of S which is generated by all e_y^+.

Thus, for each $y \in G$, there is $a \in H$ such that $\mathcal{E}_y \subseteq \mathcal{D}_a$ in $\mathcal{P}$. So we have

$$\widetilde{\omega}_{\phi_i}(e_y^+) = \omega_{\phi_i}(c_a^+)\frac{|\mathcal{E}_y|}{|\mathcal{D}_a|} = |\mathcal{D}_a|\frac{\phi_i(a)}{\phi_i(1)}\frac{|\mathcal{E}_y|}{|\mathcal{D}_a|} = |\mathcal{E}_y|\frac{\phi_i(a)}{\phi_i(1)} = |\mathcal{E}_y|\frac{\phi_i(y)}{\phi_i(1)}$$

(refer to [9, (1.5)] or [15, Section 3]), because y and a are H-conjugate.

Moreover since y and hy (for any $h \in H$) belong to H, we obtain

$$\chi_k(hy) = \chi_k|_H(hy) = \sum_{i=1}^{t} c_{ik}\phi_i(hy) = c_{ik}\phi_i(hy) = \phi_i(hy).$$

But since Y_{ik} is an H-class function, it follows that

$$\begin{aligned} Y_{ik}^*|_S(e_y^+) &= Y_{ik}^*(\sum_{x\in\mathcal{E}_y} u_x) = |\mathcal{E}_y|Y_{ik}(y) \\ &= \frac{|\mathcal{E}_y|}{|H|}\sum_{h\in H}\alpha^{-1}(h,h^{-1})\alpha(h,y)\chi_k(hy)\phi_i(h^{-1}) \\ &= \frac{|\mathcal{E}_y|}{|H|}\sum_{h\in H}\alpha^{-1}(h,h^{-1})\alpha(h,y)\phi_i(hy)\phi_i(h^{-1}) \\ &= |\mathcal{E}_y|\frac{\phi_i(y)}{\phi_i(1)} \end{aligned}$$

by (2). Hence this proves $Y_{ij}^*|_S = \widetilde{\omega}_{\phi_i}$. □

It would be nice if we know any explicit relations of Y_{ij}^* on S to $\omega_{\phi_i}^S$.

THEOREM 19. *If $c_{ij} \neq 0$, then $e_i f_j = \frac{\phi_i(1)\chi_j(1)}{|G|}\sum_{g\in G_0}\alpha(g,g^{-1})Y_{ij}(g^{-1})u_g = f_j e_i$ is a distinct block idempotent of S.*

Proof. It is easy to see that

$$\begin{aligned} &\frac{|G|}{\phi_i(1)\chi_j(1)}f_j e_i \\ &= \frac{1}{|H|}\sum_{x\in G}\alpha^{-1}(x,x^{-1})\chi_j(x^{-1})u_x\sum_{h\in H}\alpha^{-1}(h,h^{-1})\phi_i(h^{-1})u_h \\ &= \sum_{x\in G}\frac{1}{|H|}\sum_{h\in H}\alpha^{-1}(x,x^{-1})\alpha^{-1}(h,h^{-1})\alpha(x,h)\chi_j(x^{-1})\phi_i(h^{-1})u_{xh} \end{aligned}$$

$$= \sum_{g\in G} \frac{1}{|H|}\sum_{h\in H}\alpha^{-1}(gh^{-1}, hg^{-1})\alpha^{-1}(h, h^{-1})\alpha(gh^{-1}, h)\chi_j(hg^{-1})\phi_i(h^{-1})u_g$$

$$= \sum_{g\in G} \alpha^{-1}(g, g^{-1})\frac{1}{|H|}\sum_{h\in H}\alpha^{-1}(h, h^{-1})\alpha(h, g^{-1})\chi_j(hg^{-1})\phi_i(h^{-1})u_g$$

$$= \sum_{g\in G} \alpha^{-1}(g, g^{-1})Y_{\chi\phi}(g^{-1})u_g.$$

Since $Z(F^\alpha H)$ and $Z(F^\alpha G)$ are contained in $Z(S)$, the central idempotents e_i and f_j belong to $Z(S)$ and $e_i f_j$ is a central primitive orthogonal idempotent of S. □

References

[1] C. Apostolopoulos, M. Van den Bergh, and F. Van Oystaeyen, *On Schur rings of group rings of finite groups*, Comm. Algebra **20** (1992), no. 7, 2139–2152.

[2] E. Bannai and T. Ito, *Algebraic combinatorics I*, Association schemes, The Benjamin/Cummings Publishing Co., Inc., Menlo Park, CA. 1984.

[3] L. Delvaux and E. Nauwelaerts, *Applications of Frobenius algebras to representation theory of Schur algebras*, J. Algebra **199** (1998), no. 2, 591–617.

[4] W. Feit, *Characters of finite groups*, W. A. Benjamin, Inc., New York-Amsterdam, 1967.

[5] P. X. Gallagher, *Functional equation for spherical functions on finite groups*, Math. Z. **141** (1975), 77–81.

[6] I. M. Gel'fand, *Spherical functions in symmetric Riemann spaces*, Doklady Akad. Nauk SSSR **70** (1950), 5–8.

[7] R. Godement, *A theory of spherical functions*, I. Trans. Amer. Math. Soc. **73** (1952), 496–556.

[8] I. M. Isaacs, *Character theory of finite groups*, Dover Publications, Inc., New York, 1994.

[9] J. Karlof, *The subclass algebra associated with a finite group and subgroup*, Trans. Amer. Math. Soc. **207** (1975), 329–341.

[10] G. Karpilovsky, *Group representations*, Vol.2, Vol.3, North-Holland, 1993, 1994.

[11] K. H. Leung and S. H. Man, *On Schur rings over cyclic groups*, Israel J. Math. **106** (1998), no. 1, 251–267.

[12] ———, *On Schur rings over cyclic groups II*, J. Algebra **183** (1996), no. 2, 273–285.

[13] S. L. Ma, *On association schemes, Schur rings, strongly regular graphs and partial difference sets*, Ars Combin. **27** (1989), 211–220.

[14] M. Muzychuk, *The structure of rational Schur rings over cyclic groups*, European J. Combin **14** (1993), no. 5, 479–490.

[15] F. Roesler, *Darstellungstheorie von Schur-Algebren*, Math. Z. **125** (1972), 32–58.

[16] I. Schur, *Sur Theorie der einfact transitiven Permutationsgruppen*, Sitzungsber, Preues. Akad. Wiss. Berlin, Phys.-Math. Kl. (1933), 589–623.

[17] O. Tamaschke, *On the theory of Schur-rings*, Ann. Mat. Pura Appl. (4) **81** (1969), no. 4. 1–43.

[18] ———, *On Schur-rings which define a proper character theory on finite groups*, Math. Z. **117** (1970), 340–360.

[19] D. Travis, *Spherical functions on finite groups*, J. Algebra **29** (1974), 65–76.

[20] H. Wielandt, *Zur Theorie der einfach transitiven Permutationsgruppen II*, Math. Z. **52** (1949), 384–393.

[21] E. P. Wigner, *On representations of certain finite groups*, Amer. J. Math. **63** (1941), 57–63.

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