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# A CLASS OF MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY CONVOLUTION

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Abstract. For a given $p$-valent analytic function $g$ with positive coefficients in the open unit disk $\Delta$, we study a class of functions $f(z) = z^p - \sum_{n=m}^{\infty} a_n z^n$ $(a_n \geq 0)$ satisfying

$$\frac{1}{p}\Re\left(\frac{z(f*g)'(z)}{(f*g)(z)}\right) > \alpha \quad (0 \leq \alpha < 1; z \in \Delta).$$

Coefficient inequalities, distortion and covering theorems, as well as closure theorems are determined. The results obtained extend several known results as special cases.

## 1. Introduction

Let $\mathcal{A}(p,m)$ be the class of all $p$-valent analytic functions $f(z) = z^p + \sum_{n=m}^{\infty} a_n z^n$ defined on the open unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A} := \mathcal{A}(1,2)$. For two functions $f(z) = z^p + \sum_{n=m}^{\infty} a_n z^n$ and $g(z) = z^p + \sum_{n=m}^{\infty} b_n z^n$ in $\mathcal{A}(p,m)$, their convolution (or Hadamard product) is defined to be the function $(f*g)(z) := z^p + \sum_{n=m}^{\infty} a_n b_n z^n$.

Let $T(p,m)$ be the subclass of $\mathcal{A}(p,m)$ consisting of functions of the form

$$(1.1) \qquad f(z) = z^p - \sum_{n=m}^{\infty} a_n z^n \quad (a_n \geq 0 \text{ for } n \geq m)$$

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and let $T := T(1,2)$. A function $f(z) \in T(p,m)$ is called a function with negative coefficients. The subclass of $T(p,m)$ consisting of multivalent starlike (convex) functions of order $\alpha$ is denoted by $TS^*(p,m,\alpha)$ ($TC(p,m,\alpha)$). The classes $TS^*(\alpha) := TS^*(1,\ 2,\alpha)$ and $TC(\alpha) := TC(1,2,\alpha)$ were studied by Silverman [4]. In this article, we study the class $TS_g^*(p,m,\alpha)$ introduced in the following:

DEFINITION 1. Let $g(z) = z^p + \sum_{n=m}^{\infty} b_n z^n$ be a fixed function in $\mathcal{A}(p,m)$ with $b_n > 0$ $(n \geq m)$. The class $TS_g^*(p,m,\alpha)$ consists of functions $f(z)$ of the form (1.1) that satisfies

$$\frac{1}{p}\Re\left(\frac{z(f*g)'(z)}{(f*g)(z)}\right) > \alpha \quad (0 \leq \alpha < 1; z \in \Delta). \tag{1.2}$$

Several well-known subclasses of functions are special cases of our class for suitable choices of $g(z)$ when $p = 1$ and $m = 2$. For example, if $g(z) := z/(1-z)$, the class $TS_g^*(p,m,\alpha)$ is the class $TS^*(\alpha)$ of starlike functions with negative coefficients of order $\alpha$ introduced and studied by Silverman [4]. If $g(z) := z/(1-z)^2$, the class $TS_g^*(p,m,\alpha)$ is the class $TC(\alpha)$ of convex functions with negative coefficients of order $\alpha$ (See Silverman [4]). If $g(z) := \frac{z}{(1-z)^{\lambda+1}}$, $(\lambda > -1)$, $p = 1$, the class $TS_g^*(p,m,\alpha)$ reduces to the class

$$T_\lambda(\alpha) := \left\{ f \in T : \Re\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} > \alpha, \quad (z \in \Delta, \lambda > -1, \alpha < 1) \right\},$$

introduced and studied by Ahuja [1] where $D^\lambda$ denotes the Ruscheweyh derivatives of order $\lambda$. When $g(z) := z + \sum_{n=2}^{\infty} n^l z^n$, the class $TS_g^*(p,m,\alpha)$ is the class $TS_l^*(\alpha)$ where

$$TS_l^*(\alpha) := \left\{ f \in T : \Re\left(\frac{z(\mathcal{D}^l f(z))'}{\mathcal{D}^l f(z)}\right) > \alpha \right\}.$$

(Here $\mathcal{D}^l$ denotes the Salagean derivative of order $l$ [3]).

A function $f \in \mathcal{A}(p,m)$ is $\beta$-Pascu convex of order $\alpha$ if

$$\frac{1}{p}\Re\left(\frac{(1-\beta)zf'(z) + \frac{\beta}{p}z(zf'(z))'}{(1-\beta)f(z) + \frac{\beta}{p}zf'(z)}\right) > \alpha \quad (\beta \geq 0;\ 0 \leq \alpha < 1).$$

We denote by $TPC(p,m,\alpha,\beta)$ the subclass of $T(p,m)$ consisting of $\beta$-Pascu convex functions of order $\alpha$. Clearly $TS^*(\alpha)$ and $TC(\alpha)$ are special cases of $TPC(1,2,\alpha,\beta)$.

In this paper, we obtain the coefficient inequalities, distortion and covering theorems, as well as closure theorems for functions in the class

$TS_g^*(p,m,\alpha)$. Several known results are easily deduced from ours, for example, results for the classes $T_\lambda(\alpha)$ and $TS_l^*(\alpha)$. Additionally, we present results for the $\alpha$-Pascu convex functions that unifies corresponding results for $TS^*(\alpha)$ and $TC(\alpha)$.

## 2. The class $TS_g^*(p,m,\alpha)$

We first prove a necessary and sufficient condition for functions to be in $TS_g^*(p,m,\alpha)$ in the following:

THEOREM 1. *A function $f(z)$ given by (1.1) is in $TS_g^*(p,m,\alpha)$ if and only if*

$$\sum_{n=m}^{\infty}(n-p\alpha)a_nb_n \le p(1-\alpha). \tag{2.1}$$

*Proof.* If $f \in TS_g^*(p,m,\alpha)$, then (2.1) follows from (1.2) by letting $z \to 1-$ through real values. To prove the converse, assume that (2.1) holds. Then by making use of (2.1), we obtain

$$\left|\frac{z(f*g)'(z)-p(f*g)(z)}{(f*g)(z)}\right| \le \frac{\sum_{n=m}^{\infty}(n-p)a_nb_n}{1-\sum_{n=m}^{\infty}a_nb_n} \le p(1-\alpha)$$

or $f \in TS_g^*(p,m,\alpha)$. □

COROLLARY 1. *A function $f(z)$ given by (1.1) is in $TPC(p,m,\alpha,\beta)$ if and only if*

$$\sum_{n=m}^{\infty}(n-p\alpha)[(1-\beta)p+\beta n]a_n \le p^2(1-\alpha).$$

As an immediate application of Theorem 1, we obtain the following:

THEOREM 2. *Let $f(z)$ be given by (1.1). If $f \in TS_g^*(p,m,\alpha)$, then*

$$a_n \le \frac{p(1-\alpha)}{(n-p\alpha)b_n}$$

*with equality only for functions of the form*

$$f_n(z) = z^p - \frac{p(1-\alpha)}{(n-p\alpha)b_n}z^n.$$

*Proof.* If $f \in TS_g^*(p,m,\alpha)$, then, by making use of (2.1), we obtain

$$(n-p\alpha)a_nb_n \le \sum(n-p\alpha)a_nb_n \le p(1-\alpha)$$

or

$$a_n \leq \frac{p(1-\alpha)}{(n-p\alpha)b_n}.$$

Clearly for $f_n(z) = z^p - \frac{p(1-\alpha)}{(n-p\alpha)b_n} z^n \in TS_g^*(p,m,\alpha)$, we have

$$a_n = \frac{p(1-\alpha)}{(n-p\alpha)b_n}.$$

□

COROLLARY 2. *Let $f(z)$ be given by (1.1). If $f \in TPC(p,m,\alpha,\beta)$, then*

$$a_n \leq \frac{p^2(1-\alpha)}{(n-p\alpha)[(1-\beta)p+\beta n]}$$

*with equality only for functions of the form*

$$f_n(z) = z^p - \frac{p^2(1-\alpha)}{(n-p\alpha)[(1-\beta)p+\beta n]} z^n.$$

By making use of Theorem 1, we obtain the following growth estimate for functions in the class $TS_g^*(p,m,\alpha)$.

THEOREM 3. *If $f \in TS_g^*(p,m,\alpha)$, then*

$$r^p - \frac{p(1-\alpha)}{(m-p\alpha)b_m} r^m \leq |f(z)| \leq r^p + \frac{p(1-\alpha)}{(m-p\alpha)b_m} r^m, \quad |z| = r < 1,$$

*provided $b_n \geq b_m$ $(n \geq m)$. The result is sharp with equality for*

$$f(z) = z^p - \frac{p(1-\alpha)}{(m-p\alpha)b_m} z^m \tag{2.2}$$

*at $z = r$ and $z = re^{\frac{i\pi(2k+1)}{m-p}}$ $(k \in \mathbb{Z})$.*

*Proof.* Let $|z| = r$. Since $f(z) = z^p - \sum_{n=m}^{\infty} a_n z^n$, we have

$$|f(z)| \leq r^p + \sum_{n=m}^{\infty} a_n r^m$$

$$\leq r^p + r^m \sum_{n=m}^{\infty} a_n. \tag{2.3}$$

Since for $n \geq m$,

$$(m-p\alpha)b_m \leq (n-p\alpha)b_n,$$

using (2.1) yields

$$b_m(m-p\alpha)\sum_{n=m}^{\infty} a_n \le \sum_{n=m}^{\infty}(n-p\alpha)a_n b_n \le p(1-\alpha)$$

or

$$\sum_{n=m}^{\infty} a_n \le \frac{p(1-\alpha)}{(m-p\alpha)b_m}. \tag{2.4}$$

This together with (2.3) shows that

$$|f(z)| \le r^p + r^m \frac{p(1-\alpha)}{(m-p\alpha)b_m}$$

and similarly we have

$$|f(z)| \ge r^p - r^m \frac{p(1-\alpha)}{(m-p\alpha)b_m}.$$

□

Let $\frac{p(1-\alpha)}{(m-p\alpha)b_m} < 1$. By letting $r \to 1-$ in Theorem 3, we see that functions $f \in TS_g^*(p,m,\alpha)$ map the unit disk $\Delta$ onto regions that contained the disk $|w| < 1 - \frac{p(1-\alpha)}{(m-p\alpha)b_m}$.

COROLLARY 3. *If $f \in TPC(p,m,\alpha,\beta)$, then*

$$\begin{aligned} r^p - &\frac{p^2(1-\alpha)}{(m-p\alpha)[(1-\beta)p+\beta m]} r^m \\ \le\ &|f(z)| \\ \le\ &r^p + \frac{p^2(1-\alpha)}{(m-p\alpha)[(1-\beta)p+\beta m]} r^m, \quad |z| = r < 1. \end{aligned}$$

*The result is sharp for*

$$f(z) = z^p - \frac{p^2(1-\alpha)}{(m-p\alpha)[(1-\beta)p+\beta m]} z^m. \tag{2.5}$$

We now prove the distortion theorem for the functions in $TS_g^*(p,m,\alpha)$ in the following:

THEOREM 4. *If $f \in TS_g^*(p,m,\alpha)$, then*

$$\begin{aligned} pr^{p-1} - \frac{mp(1-\alpha)}{(m-p\alpha)b_m} r^{m-1} &\le |f'(z)| \\ &\le pr^{p-1} + \frac{mp(1-\alpha)}{(m-p\alpha)b_m} r^{m-1}, \quad |z| = r < 1, \end{aligned}$$

*provided $b_n \geq b_m$. The result is sharp for $f(z)$ given by (2.2).*

*Proof.* For a function $f \in TS_g^*(p, m, \alpha)$, it follows from (2.1) and (2.4) that

$$\sum_{n=m}^{\infty} n a_n \leq \frac{mp(1-\alpha)}{(m-p\alpha)b_m}.$$

Since the remaining part of the proof is similar to the proof of Theorem 3, we omit the details. □

COROLLARY 4. *If $f \in TPC(p, m, \alpha, \beta)$, then*

$$\begin{aligned} & pr^{p-1} - \frac{mp^2(1-\alpha)}{(m-p\alpha)[(1-\beta)p+\beta m]} r^{m-1} \\ & \leq |f'(z)| \\ & \leq pr^{p-1} + \frac{mp^2(1-\alpha)}{(m-p\alpha)[(1-\beta)p+\beta m]} r^{m-1} \end{aligned}$$

*where $|z| = r < 1$. The result is sharp for $f(z)$ given by (2.5).*

We shall now prove the following closure theorems for the class $TS_g^*(p, m, \alpha)$.

THEOREM 5. *Let $\lambda_k \geq 0$ for $k = 1, 2, \ldots, l$ and $\sum_{k=1}^{l} \lambda_k \leq 1$. If the functions $F_k(z)$ defined by*

$$F_k(z) = z^p - \sum_{n=m}^{\infty} f_{n,k} z^n \tag{2.6}$$

*are in the class $TS_g^*(p, m, \alpha)$ for every $k = 1, 2, ..., l$, then the function $f(z)$ defined by*

$$f(z) = z^p - \sum_{n=m}^{\infty} \left( \sum_{k=1}^{l} \lambda_k f_{n,k} \right) z^n$$

*is in the class $TS_g^*(p, m, \alpha)$.*

*Proof.* Since $F_k(z) \in TS_g^*(p, m, \alpha)$, it follows from Theorem 2.1 that

$$\sum_{n=m}^{\infty} (n - p\alpha) f_{n,k} b_n \leq p(1-\alpha) \tag{2.7}$$

for every $k = 1, 2, \dots, l$. Hence

$$\begin{aligned}\sum_{n=m}^{\infty}(n-p\alpha)\left(\sum_{k=1}^{l}\lambda_k f_{n,k}\right)b_n &= \sum_{k=1}^{l}\lambda_k\left(\sum_{n=m}^{\infty}(n-p\alpha)f_{n,k}b_n\right)\\ &\leq p(1-\alpha)\sum_{k=1}^{l}\lambda_k\\ &\leq p(1-\alpha).\end{aligned}$$

By Theorem 1, it follows that $f(z) \in TS_g^*(p, m, \alpha)$. □

COROLLARY 5. *The class $TS_g^*(p, m, \alpha)$ is closed under convex linear combinations.*

THEOREM 6. *Let $F_p(z) := z^p$ and $F_n(z) := z^p - \frac{p(1-\alpha)}{(n-p\alpha)b_n} z^n$ for $n = m, m+1, \dots$. The function $f(z) \in TS_g^*(p, m, \alpha)$ if and only if $f(z)$ can be expressed in the form*

$$f(z) = \lambda_p z^p + \sum_{n=m}^{\infty} \lambda_n F_n(z) \tag{2.8}$$

*where $\lambda_n \geq 0$ for $n = p, m, m+1, \dots$ and $\lambda_p + \sum_{n=m}^{\infty} \lambda_n = 1$.*

*Proof.* If the function $f(z)$ is expressed in the form given by (2.8), then

$$f(z) = z^p - \sum_{n=m}^{\infty} \frac{\lambda_n p(1-\alpha)}{(n-p\alpha)b_n} z^n$$

and for this function, we have

$$\sum_{n=m}^{\infty}(n-p\alpha)\frac{\lambda_n p(1-\alpha)b_n}{(n-p\alpha)b_n} = \sum_{n=m}^{\infty} p(1-\alpha)\lambda_n = p(1-\alpha)(1-\lambda_p) \leq p(1-\alpha).$$

By Theorem 1, we have $f(z) \in TS_g^*(p, m, \alpha)$.

Conversely, let $f(z) \in TS_g^*(p, m, \alpha)$. From Theorem 2, we have

$$a_n \leq \frac{p(1-\alpha)}{(n-p\alpha)b_n} \quad \text{for} \quad n = m, m+1, \dots.$$

Therefore by taking

$$\lambda_n := \frac{(n-p\alpha)a_n b_n}{p(1-\alpha)} \quad \text{for} \quad n = m, m+1, \dots$$

and

$$\lambda_p := 1 - \sum_{n=m}^{\infty} \lambda_n,$$

we see that $f(z)$ is of the form given by (2.8). □

THEOREM 7. *Let $h(z) = z^p + \sum_{n=m}^{\infty} h_n z^n$ with $h_n > 0$.*

(i) *Let $(n - p\alpha)b_n \geq (1-\alpha)nh_n$ and*

$$\beta := \inf_{n \geq m} \left[ \frac{(n - p\alpha)b_n - (1-\alpha)nh_n}{(n - p\alpha)b_n - (1-\alpha)ph_n} \right].$$

*If $f \in TS_g^*(p, m, \alpha)$, then $f \in TS_h^*(p, m, \beta)$.*

(ii) *If $f \in TS_g^*(p, m, \alpha)$, then $f \in TS_h^*(p, m, \beta)$ in $|z| < r(\alpha, \beta)$, where*

$$r(\alpha, \beta) := \min \left\{ 1, \inf_{n \geq m} \left[ \frac{(n - p\alpha)}{(n - p\beta)} \frac{(1-\beta)}{(1-\alpha)} \frac{b_n}{h_n} \right]^{\frac{1}{n-p}} \right\}.$$

*Proof.* (i) From the definition of $\beta$, it follows that

$$\beta \leq \frac{(n - p\alpha)b_n - (1-\alpha)nh_n}{(n - p\alpha)b_n - (1-\alpha)ph_n}$$

or

$$\frac{(n - p\beta)h_n}{1-\beta} \leq \frac{(n - p\alpha)b_n}{1-\alpha}$$

and therefore, in view of (2.1),

$$\sum_{n=m}^{\infty} \frac{(n - p\beta)}{p(1-\beta)} a_n h_n \leq \sum_{n=m}^{\infty} \frac{(n - p\alpha)}{p(1-\alpha)} a_n b_n \leq 1.$$

This completes the proof of (i).

(ii) It is easy to see that $f$ satisfies

$$\frac{1}{p} \Re \left( \frac{z(f * h)'(z)}{(f * h)(z)} \right) > \beta \quad (|z| < r)$$

if and only if

$$\sum_{n=m}^{\infty} (n - p\beta) a_n h_n r^{n-p} \leq p(1-\beta). \tag{2.9}$$

From the definition of $r(\alpha, \beta)$, we have

$$\frac{(n - p\beta)}{p(1-\beta)} h_n r^{n-p} \leq \frac{(n - p\alpha)}{p(1-\alpha)} b_n \tag{2.10}$$

and the result now follows from (2.10), (2.9) and (2.1). □

Theorem 7 contains several results. For example, when $p = 1$, $m = 2$, $h(z) = z/(1-z)$ and $g(z) = z/(1-z)^2$, the class $TS_g^*(1, 2, \alpha)$ consists of convex functions of order $\alpha$ in $T$. Theorem 7(i) yields the order of starlikeness, i.e., $\beta = 2/(3-\alpha)$. Similarly, when $p = 1$, $m = 2$, $h(z) = z/(1-z)^2$, $g(z) = z/(1-z)$, and $\beta = 0$, we get the radius of convexity for starlike functions of order $\alpha$ in $T$. These results were proved by Silverman [4].

We now prove that the class $TS_g^*(p, m, \alpha)$ is closed under convolution with certain functions and give an application of this result to show that the class $TS_g^*(p, m, \alpha)$ is closed under the familiar Bernardi integral operator.

THEOREM 8. *Let* $h(z) = z^p + \sum_{n=m}^{\infty} h_n z^n$ *be analytic in* $\Delta$ *with* $0 \leq h_n \leq 1$. *If* $f(z) \in TS_g^*(p, m, \alpha)$, *then* $(f * h)(z) \in TS_g^*(p, m, \alpha)$.

*Proof.* The result follows by a straight forward application of Theorem 1. □

The generalized Bernardi integral operator is defined by

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)dt \quad (c > -1; z \in \Delta). \tag{2.11}$$

Since

$$F(z) = f(z) * \left( z^p + \sum_{n=m}^{\infty} \frac{c+p}{c+n} z^n \right),$$

we have the following:

COROLLARY 6. *If* $f(z) \in TS_g^*(p, m, \alpha)$, *then* $F(z)$ *given by* (2.11) *is also in* $TS_g^*(p, m, \alpha)$.

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