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THE ALMOST SURE CONVERGENCE OF AANA SEQUENCES IN DOUBLE ARRAYS

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Abstract. For double arrays of constants $\{a_{ni},\ 1 \le i \le k_n,\ n \ge 1\}$ and sequences $\{X_n, n \ge 1\}$ of asymptotically almost negatively associated (AANA) random variables the almost sure convergence of $\sum_{i=1}^{k_n} a_{ni}X_i$ is derived.

1. Introduction

Recall that a finite sequence $\{X_1, \ldots, X_n\}$ is called negatively associated(NA) if for every finite disjoint subsets $A, B \subset \{1, \ldots, n\}$ and coordinatewise nondecreasing functions $f : R^A \to R$ and $g : R^B \to R$,

$$\mathrm{Cov}(f(X_i; i \in A), g(X_j; j \in B) \le 0,$$

whenever it exists and an infinite family of random variables is negatively associated if every finite subfamily is negatively associated (see [5]). By inspecting the proof of Matula's [7] maximal inequality for NA sequences Chandra and Ghosal discovered that one can also allow positive correlations provided they are small. Primarily motivated by this fact Chandra and Ghosal [2, 3] introduced the following dependence condition:

Definition 1.1. A sequence $\{X_n, n \ge 1\}$ of random variables is called asymptotically almost negatively associated(AANA) if there is a nonnegative sequence $q(m) \to 0$ such that

$$\begin{aligned} &\mathrm{Cov}(f(X_m), g(X_{m+1}, \ldots, X_{m+k})) \\ (1) \qquad &\le\ q(m)(\mathrm{Var}(f(X_m))\mathrm{Var}(g(X_{m+1}, \ldots, X_{m+k})))^{\frac{1}{2}} \end{aligned}$$

for all $m, k \ge 1$ and for all coordinatewise increasing continuous functions f and g whenever the right-hand side of (1) is finite.

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Notice that the family of AANA sequences contains NA (in particular, independent) sequences(with $q(m) = 0, \forall m \geq 1$) and some more sequences of random variables which are not much deviated from being negatively associated. Condition (1) is clearly satisfied if the $R_{2,2}$-measure of dependence(see [1]) between $\sigma(X_m)$ and $\sigma(X_{m+1}, X_{m+2}, \ldots)$ converges to zero.

The following is a non-trivial example of an AANA sequence constructed by Chandra and Ghosal [2]:

EXAMPLE 1.2. [2] Let $\{Y_n, n \geq 1\}$ be i.i.d. N(0,1) variables and define $X_n = (1 + a_n^2)^{-\frac{1}{2}}(Y_n + a_n Y_{n+1})$ where $a_n > 0$ and $a_n \to 0$. Then $\{X_n, n \geq 1\}$ is a sequence of AANA random variables. Note that $\{X_n, n \geq 1\}$ is not NA(indeed, is associated and 1-dependent). Chandra and Ghosal [2] derived maximal inequality for AANA random variables and obtained strong law of large numbers for AANA sequences by using this inequality. Chandra and Ghosal [3] also studied the almost sure convergence of weighted averages under AANA assumption. Kim and Ko [6] established Hǻjeck-Rènyi type inequality for AANA random variables and proved the Marcinkiewicz strong law of large numbers and integrability of supremum for AANA random variables by using this inequality.

In this paper we will derive an almost sure convergence for a triangular array of weighted sums of AANA random variables, which have not been established previously in the literature.

Finally, we will use the following concept in this paper. Let $\{X_n, n \geq 1\}$ be a sequence of random variables and X be a nonnegative random variable. If there exists a constant $C(0 < C < \infty)$ satisfying

$$\sup_{n \in N} P(\|X_n\| \geq t) \leq CP(X \geq t) \text{ for any } t \geq 0,$$

then $\{X_n, n \geq 1\}$ are said to be stochastically dominated by X (briefly $\{X_n, n \geq 1\} \prec X$).

In the following statement, C stands for a constant whose value may vary from line to line.

2. Results

LEMMA 2.1. [6] *Let $\{X_n, n \geq 1\}$ be a sequence of asymptotically almost negatively associated*(AANA). *Then $\{f_n(X_n), n \geq 1\}$ is still a sequence of AANA random variables, where $f_n(\cdot), n = 1, 2, \ldots,$ are nondecreasing functions.*

THEOREM 2.2. [2] *Let $X_1, \ldots, X_n$ be mean zero, square integrable random variables such that* (1) *holds for $1 \leq m < k+m \leq n$ and for all coordinatewise nondecreasing continuous functions f and g whenever the right-hand side of* (1) *is finite. Let $A^2 = \sum_{m=1}^{n-1} q^2(m)$ and $\sigma_k^2 = EX_k^2$, $k \geq 1$. Then*

$$P\{\max_{1\leq k\leq n} |S_k| \geq \epsilon\} \leq 2\epsilon^{-2}(A + (1+A^2)^{1/2})^2 \sum_{k=1}^{n} \sigma_k^2. \tag{2}$$

The following theorem is the main result:

THEOREM 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with $EX_n = 0$ and $\{X_n, n \geq 1\}$ be stochastically bounded by a nonnegative random variable X with $EX^r < \infty$ for $0 < r < 2$. Assume $\{a_{ni}, 1 \leq i \leq k_n \uparrow, n \geq 1\}$ is an array of constants satisfying*

$$\sum_{i=1}^{k_n} |a_{ni} - a_{n,i+1}| = O(\frac{1}{k_n^{1/r}}), \tag{3}$$

where $a_{n,k_n+1} = 0$, and

$$B^2 = \sum_{m=1}^{\infty} q^2(m) < \infty. \tag{4}$$

Then

$$\sum_{i=1}^{k_n} a_{ni} X_i \to 0 \text{ a.s.} \tag{5}$$

Proof. Without loss of generality, we suppose that $a_{ni} \geq 0 (i \geq 1, n \geq 1)$. Otherwise we assume that $a_{ni_1}, \ldots, a_{ni_m}$ are nonnegative, while $a_{ni_{m+1}}, \ldots, a_{ni_{k_n}}$ are negative. It is easy to check that $\{a_{ni_j}, 1 \leq j \leq m\}$ and $\{a_{ni_j}, m+1 \leq j \leq k_n\}$ satisfy (3). Then we only have to consider $\sum_{j=1}^{m} a_{ni_j} X_{ni_j}$ and $\sum_{j=m+1}^{k_n} a_{ni_j} X_{ni_j}$ respectively.

(a) If $1 \leq r < 2$, let

$$X_i' = (-i^{1/r}) \vee (X_i \wedge i^{1/r}), \ X_i'' = X_i - X_i'.$$

Since X_i' and X_i'' are increasing functions of X_i both $\{X_n' - EX_n'\}$ and $\{X_n'' - EX_n''\}$ are also mean zero AANA sequences by Lemma 2.1. Let

$$S_n' = \sum_{i=1}^{k_n} a_{ni}(X_i' - EX_i'), \quad S_n'' = \sum_{i=1}^{k_n} a_{ni}(X_i'' - EX_i''),$$

$$A_k = \sum_{i=1}^{k} (X_i' - EX_i').$$

For fixed n, there exists $t \in N$ such that $2^t < k_n \leq 2^{t+1}$. Hence from (3) we easily get

$$|S_n'| \leq C \frac{1}{(2^t)^{1/r}} \max_{1 \leq i \leq 2^{t+1}} |A_i|$$

by applying Abel transformation. Noticing that $\{X_i'' - EX_i'', n \geq 1\}$ is an AANA sequence by Lemma 2.1 and applying Theorem 2.2 (Theorem 1 in [2]), for $\forall \epsilon > 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} P(|S_n'| > \epsilon) \\
&\leq \sum_{t=1}^{\infty} P\left\{ \frac{1}{(2^t)^{1/r}} \max_{1 \leq i \leq 2^{t+1}} |A_i| > \frac{\epsilon}{C} \right\} \\
&\leq C \left(B + (1 + B^2)^{1/2} \right)^2 \sum_{t=1}^{\infty} \frac{1}{(2^t)^{2/r}} \sum_{i=1}^{2^{t+1}} EX_i'^2 \\
&\leq C \left(B + (1 + B^2)^{1/2} \right)^2 \left\{ \sum_{i=1}^{\infty} P\left(|X_i| \geq i^{1/r} \right) \right. \\
&\quad \left. + \sum_{i=1}^{\infty} i^{-2/r} EX_i^2 I\left(|X_i| \leq i^{1/r} \right) \right\} \\
&\leq C \left(B + (1 + B^2)^{1/2} \right)^2 \left\{ \sum_{i=1}^{\infty} P\left(X \geq i^{1/r} \right) + \sum_{i=1}^{\infty} \frac{EX^2 I(X \leq i^{1/r})}{i^{2/r}} \right\}
\end{aligned}$$

where C only depends on ϵ. It follows from the condition $EX^r < \infty$ that $\sum_{i=1}^{\infty} P(X \geq i^{1/r}) < \infty$ and $\sum_{i=1}^{\infty} i^{-2/r} EX^2 I(X \leq i^{1/r}) < \infty$ (see [Appendix]). Thus we have $\sum_{i=1}^{\infty} P(|S_n'| > \epsilon) < \infty$. By Borel-Cantelli lemma we conclude that

$$S_n' \to 0 \text{ a.s.}$$

On the other hand, since

$$\sum_{i=1}^{\infty} P\left(|X_i| \geq i^{1/r} \right) \leq C \sum_{i=1}^{\infty} P\left(X \geq i^{1/r} \right) < \infty$$

we have $P(|X_i| \geq i^{1/r}) \to 0$ as $i \to \infty$. From (3), we have

$$\begin{aligned}\left|\sum_{i=1}^{k_n} a_{ni}X_i''\right| &\leq \left(\max_{1\leq i\leq k_n}\left|\sum_{j=1}^{i} X_j''\right|\right)\left(\sum_{i=1}^{k_n}|a_{ni}-a_{n,i+1}|\right) \\ &\leq \frac{C}{k_n^{1/r}}\sum_{i=1}^{k_n}|X_i|I(|X_i|\geq i^{1/r}) \to 0 \text{ a.s.}\end{aligned} \tag{6}$$

by applying Abel transformation.

If $1 < r < 2$, since $\{X_n\} \prec X$ and $\sum_{i=1}^{\infty} i^{-1/r}EXI(X > i^{1/r}) < \infty$, we get that

$$\sum_{i=1}^{\infty}\frac{1}{i^{1/r}}E|X_i''| \leq C\sum_{i=1}^{\infty} i^{-1/r}EXI(X > i^{1/r}) < \infty. \tag{7}$$

By Kronecker lemma, we get

$$\frac{1}{k_n^{1/r}}\sum_{i=1}^{k_n}E|X_i''| \to 0.$$

If $r = 1$,

$$E|X_i''| \leq CEXI(X > i) \to 0 \text{ as } i \to \infty,$$

thus we have as well

$$\frac{1}{k_n}\sum_{i=1}^{k_n}E|X_i''| \to 0.$$

From (3) we obtain

$$\begin{aligned}|\sum_{i=1}^{k_n} a_{ni}EX_i''| &\leq \frac{C}{k_n^{1/r}}(\max_{1\leq i\leq k_n}|\sum_{j=1}^{i}EX_j''|) \\ &\leq \frac{C}{k_n^{1/r}}\sum_{i=1}^{k_n}E|X_i''| \to 0\end{aligned} \tag{8}$$

by applying Abel transformation. From (6) and (8) it follows that

$$S_n'' \to 0 \text{ a.s.}$$

Since $S_n = S_n' + S_n''$, we obtain (5) for $1 \leq r < 2$.

(b) If $0 < r < 1$, let

$$X_n' = (-n^{1/r}) \vee (X_n \wedge n^{1/r}),\ \ X_n'' = X_n - X_n'.$$

We can show in the same way as we did in (a) that

$$\sum_{i=1}^{k_n} a_{ni}(X_i^{'} - EX_i^{'}) \to 0 \text{ a.s.}$$

and

$$\sum_{i=1}^{k_n} a_{ni}X_i^{''} \to 0 \text{ a.s.}$$

So it remains to show that

$$\sum_{i=1}^{k_n} a_{ni}EX_i^{'} \to 0.$$

From Appendix, we have

$$\sum_{i=1}^{\infty} \frac{1}{i^{1/r}} E|X_i^{'}|$$

$$\leq C\left\{\sum_{i=1}^{\infty} P(X \geq i^{1/r}) + \sum_{i=1}^{\infty} \frac{1}{i^{1/r}} EXI(X \leq i^{1/r})\right\} < \infty.$$

Consequently, by Kronecker lemma

$$\frac{1}{k_n^{1/r}} \sum_{i=1}^{k_n} E|X_i^{'}| \to 0 \text{ as } i \to \infty.$$

It follows that

$$\left|\sum_{i=1}^{k_n} a_{ni}EX_i^{'}\right| \leq \frac{C}{k_n^{1/r}}\left(\max_{1\leq i\leq k_n}\left|\sum_{j=1}^{i} EX_j^{'}\right|\right)$$

$$\leq \frac{C}{k_n^{1/r}} \sum_{i=1}^{k_n} E|X_i^{'}| = o(1).$$

Thus

$$\sum_{i=1}^{k_n} a_{ni}X_i \to 0 \text{ a.s.},$$

that is, (5) holds for $0 < r < 1$. The proof is completed. □

From Theorem 2.3, we get Marcinkiecz-type strong law of large number for AANA random variables:

COROLLARY 2.4. *Assume that* $0 < r < 2$ *and* $\{X, X_n, n \geq 1\}$ *is a sequence of identically distributed AANA random variables with* $EX = 0$. *Assume that* $E|X|^r < \infty$ *for* $0 < r < 2$ *and* (4) *holds. Then*

$$\frac{1}{n^{1/r}} \sum_{i=1}^{n} X_i \to 0 \text{ a.s.}$$

Proof. By Theorem 2.3, it is obvious that

$$\frac{1}{n^{1/r}} \sum_{i=1}^{n} X_i \to 0 \text{ a.s.}$$

□

REMARK. Theorem 2.3 generalizes Theorem 3 in [4] from the case of i.i.d sequences to AANA sequences, but also generalizes exponents from [1, 2) to (0,2).

3. Appendix

LEMMA A. *If* $\{X_n\}$ *is stochastically dominated by a nonnegative random variable* $X(\{X_n\} \prec X)$ *with* EX^r *for* $0 < r < 2$. *Then we have*

$$(a) \qquad \sum_{i=1}^{\infty} i^{-2/r} E(X_i^2 I\{|X_i|^r \leq i\}) < \infty,$$

$$(b) \qquad \begin{aligned} &\sum_{i=1}^{\infty} i^{-1/r} E(|X_i| I\{|X_i|^r \leq i\}) < \infty \quad \text{if} \quad 0 < r < 1, \\ &n^{-1/r} \sum_{i=1}^{n} E(|X_i| I\{|X_i|^r \leq i\}) \to 0 \quad \text{if} \quad 1 \leq r < 2. \end{aligned}$$

Proof. The proof is based on certain ideas in Chandra and Ghoshal [2]. Note that, for some $0 < r < 2$

$$(9) \qquad E|X|^r < \infty \Leftrightarrow \int_0^{\infty} y^{r-1} P\{|X| > y\} dy < \infty$$

and

$$(10) \qquad E|X|^r < \infty \Leftrightarrow \sum_{n=1}^{\infty} P\{|X|^r > n\} < \infty$$

since $\{X_n\}$ is stochastically dominated by a nonnegative random variable X.

(a) The proof of (a),

$$\begin{aligned}
&\sum_{i=1}^{\infty} i^{-2/r} E(X_i^2 I\{|X_1|^p \le i\}) \\
&\le C \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} k^{\frac{-2}{r}-1} E(X_i^2 I\{|X_i|^r \le i\}) \\
&\le C \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} k^{\frac{-2}{r}-1} \int_0^{j^{\frac{1}{r}}} yP(\{|X_i| > y\})dy \\
&\le C \sum_{k=1}^{\infty} \sum_{i=1}^{k} k^{\frac{-2}{r}-1} \sum_{n=1}^{i} \int_{(n-1)^{\frac{1}{r}}}^{n^{\frac{1}{r}}} yP\{|X_i| > y\}dy \\
&\le C \sum_{k=1}^{\infty} \sum_{n=1}^{k} k^{\frac{-2}{r}} \int_{(n-1)^{\frac{1}{r}}}^{n^{\frac{1}{r}}} y \left(k^{-1} \sum_{i=1}^{k} P\{|X| > y\} \right) dy \\
&\le C \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} k^{\frac{-2}{r}} \int_{(n-1)^{\frac{1}{r}}}^{n^{\frac{1}{r}}} yP\{|X| > y\}dy \\
&\le C \sum_{n=1}^{\infty} n^{1-\frac{2}{r}} \int_{(n-1)^{\frac{1}{r}}}^{n^{\frac{1}{r}}} yP\{|X| > y\}dy \\
&\le C \sum_{n=1}^{\infty} \int_{(n-1)^{\frac{1}{r}}}^{n^{\frac{1}{r}}} y^{r-1} P\{|X| > y\}dy \\
&\le CE|X|^r \\
&< \infty.
\end{aligned}$$

(b) The proof of (b), in the case $0 < r < 1$, is similar to that of (a). Now let $1 \le r < 2$ and fix $N \ge 1$. For $n > N$, we have

$$\begin{aligned}
&n^{-1/r} \sum_{i=1}^{n} E(|X_i| I\{|X_i|^r > i\}) \\
&= n^{-1/r} \sum_{i=N+1}^{N} \int_{1/r}^{\infty} P\{|X_i| > y\}dy
\end{aligned}$$

$$
\begin{aligned}
&+ n^{-1/r}\sum_{i=1}^{\infty}\int_{1/r}^{\infty} P\{|X_i| > y\}dy \\
&+ n^{-1/r}\sum_{i=1}^{n} i^{1/r} P\{|X_i| > y\} \\
&= I + II + III.
\end{aligned}
$$

Obviously, the first term (I) on the right side goes to 0 as $n \to \infty$. The third term (III) converges to 0 by (2.10), (2.11), and Kronecker's lemma. Finally, the second term (II) is at most

$$
\begin{aligned}
& n^{-1/r}\sum_{i=N}^{n}\sum_{k=i}^{\infty}\int_{k^{1/r}}^{(k+1)^{1/r}} P\{|X_i| > y\}dy \\
\le\ & n^{-1/r}\sum_{k=N}^{\infty}\int_{k^{1/r}}^{(k+1)^{1/r}}\sum_{i=1}^{m(k,n)} P\{|X_i| > y\}dy \\
\le\ & n^{-1/r}\sum_{k=N}^{\infty}\left((\min(k,n))^{1/r}\right)^{(r-1)+1}\int_{k^{1/r}}^{(k+1)^{1/r}} P\{|X_i| > y\}dy \\
\le\ & \sum_{k=N}^{\infty}\int_{k^{1/r}}^{(k+1)^{1/r}} y^{r-1}P\{|X| > y\}dy \to 0 \text{ as } N \to \infty.
\end{aligned}
$$

The proof of (c) follows from (a) and an interchange of summation signs. □

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