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NON-LINEAR DIFFERENTIAL POLYNOMIALS SHARING 1-POINTS

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ABSTRACT. We prove two theorems on the uniqueness of nonlinear differential polynomials sharing 1-points, the first of which improves a recent result of Fang-Fang and Lin-Yi.

1. Introduction, definitions, and results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. Let k be a positive integer or infinity and $a \in \{\infty\} \cup \mathbb{C}$. We denote by $E_{k)}(a; f)$ the set of all a-points of f with multiplicities not exceeding k, where an a-point is counted according to its multiplicity. If for some $a \in \{\infty\} \cup \mathbb{C}$, $E_{\infty)}(a; f) = E_{\infty)}(a; g)$ we say that f, g share the value a CM (counting multiplicities).

In [4] the problem of uniqueness of meromorphic functions when two linear differential polynomials share the same 1-points was studied. Regarding the nonlinear differential polynomials the following question was asked in [4] : *What can be said if two nonlinear differential polynomials generated by two meromorphic functions share* 1 *CM?* Some works have already been done in this direction [1, 2, 7, 8]. Recently Fang-Fang [2] and Lin-Yi [8] proved the following result.

THEOREM A. [8] *Let f and g be two nonconstant meromorphic functions and $n(\geq 13)$ be an integer. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share the value* 1 *CM, then $f \equiv g$.*

In the paper we also investigate the uniqueness problem of meromorphic functions when two nonlinear differential polynomials share the value 1. We prove the following two theorems, the first of which improves Theorem A.

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THEOREM 1.1. *Let f and g be two nonconstant meromorphic functions and $n(\geq 13)$ be an integer. If $E_{3)}(1; f^n(f-1)^2 f') = E_{3)}(1; g^n(g-1)^2 g')$ then $f \equiv g$.*

THEOREM 1.2. *Let f and g be two nonconstant meromorphic functions and $n(\geq 14)$ be an integer. If $E_{3)}(1; f^n(f^3-1)f') = E_{3)}(1; g^n(g^3-1)g')$ then $f \equiv g$.*

Though for the standard notations and definitions of value distribution theory we refer [3], in the following definition we explain a notation used in the paper.

DEFINITION 1.1. Let f be a meromorphic function and $a \in \mathbb{C} \cup \{\infty\}$. For a positive integer p we denote by $N_p(r, a; f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq p$ and is counted p times if $m > p$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

LEMMA 2.1. *Let f and g be two nonconstant meromorphic functions. Then $f^n(f-1)^2 f' g^n(g-1)^2 g' \not\equiv 1$, where $n(\geq 7)$ is an integer.*

Proof. If possible let $f^n(f-1)^2 f' g^n(g-1)^2 g' \equiv 1$. Let z_0 be an 1-point of f with multiplicity $p(\geq 1)$. Then z_0 is a pole of g with multiplicity $q(\geq 1)$ such that $3p - 1 = (n+2)q + q + 1 \geq n + 4$ and so $p \geq \frac{n+5}{3}$.

Let z_1 be a zero of f with multiplicity $p(\geq 1)$ and it be a pole of g with multiplicity $q(\geq 1)$. Then $np + p - 1 = (n+3)q + 1$ i.e., $2q = (n+1)(p-q) - 2 \geq n - 1$ i.e., $q \geq \frac{n-1}{2}$. So $(n+1)p = (n+3)q + 2 \geq \frac{(n+3)(n-1)}{2} + 2$ and so $p \geq \frac{n+1}{2}$.

Since a pole of f is either a zero of $g(g-1)$ or a zero of g', we get

$$\begin{aligned}
\overline{N}(r, \infty; f) &\leq \overline{N}(r, 0; g) + \overline{N}(r, 1; g) + \overline{N}_0(r, 0; g') \\
&\leq \frac{2}{n+1} N(r, 0; g) + \frac{3}{n+5} N(r, 1; g) + \overline{N}_0(r, 0; g') \\
&\leq \left(\frac{2}{n+1} + \frac{3}{n+5}\right) T(r, g) + \overline{N}_0(r, 0; g'),
\end{aligned}$$

where $\overline{N}_0(r, 0; g')$ is the reduced counting function of those zeros of g' which are not the zeros of $g(g-1)$.

By the second fundamental theorem we obtain

$$\begin{aligned}
& T(r,f) \\
\leq\ & \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,1;f) - \overline{N}_0(r,0;f') + S(r,f) \\
\leq\ & \frac{2}{n+1}N(r,0;f) + \frac{3}{n+5}N(r,1;f) + \left(\frac{2}{n+1} + \frac{3}{n+5}\right)T(r,g) \\
& +\overline{N}_0(r,0;g') - \overline{N}_0(r,0;f') + S(r,f)
\end{aligned}$$

i.e.,

(2.1)

$$\begin{aligned}
& \left(1 - \frac{2}{n+1} - \frac{3}{n+5}\right)T(r,f) \\
\leq\ & \left(\frac{2}{n+1} + \frac{3}{n+5}\right)T(r,g) + \overline{N}_0(r,0;g') - \overline{N}_0(r,0;f') + S(r,f).
\end{aligned}$$

Similarly we get

(2.2)

$$\begin{aligned}
& \left(1 - \frac{2}{n+1} - \frac{3}{n+5}\right)T(r,g) \\
\leq\ & \left(\frac{2}{n+1} + \frac{3}{n+5}\right)T(r,f) + \overline{N}_0(r,0;f') - \overline{N}_0(r,0;g') + S(r,g).
\end{aligned}$$

Adding (2.1) and (2.2) we get

$$\left(1 - \frac{4}{n+1} - \frac{6}{n+5}\right)\{T(r,f) + T(r,g)\} \leq S(r,f) + S(r,g),$$

which is a contradiction. This proves the lemma. □

LEMMA 2.2. [9] *Let f be a nonconstant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \cdots + a_n f^n$, where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$. Then*

$$T(r,P(f)) = nT(r,f) + S(r,f).$$

LEMMA 2.3. *Let $F = f^{n+1}\left(\frac{f^2}{n+3} - \frac{2f}{n+2} + \frac{1}{n+1}\right)$ and $G = g^{n+1}(\frac{g^2}{n+3} - \frac{2g}{n+2} + \frac{1}{n+1})$, where $n(\geq 5)$ is an integer. Then $F' \equiv G'$ implies $F \equiv G$.*

Proof. Let $F' \equiv G'$. Then $F \equiv G + c$, where c is a constant. If possible, let $c \neq 0$. Then by the second fundamental theorem we get

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,0;F) + \overline{N}(r,c;F) + \overline{N}(r,\infty;F) + S(r,F) \\ &\leq \overline{N}(r,0;f) + 2T(r,f) + \overline{N}(r,\infty;f) + \overline{N}(r,0;G) + S(r,F) \\ &\leq 4T(r,f) + \overline{N}(r,0;g) + 2T(r,g) + S(r,F) \\ &\leq 4T(r,f) + 3T(r,g) + S(r,F). \end{aligned}$$

Since by Lemma 2.2 $T(r,F) = (n+3)T(r,f) + S(r,f)$, it follows that

$$(2.3) \qquad (n+3)T(r,f) \leq 4T(r,f) + 3T(r,g) + S(r,f).$$

Similarly we get

$$(2.4) \qquad (n+3)T(r,g) \leq 3T(r,f) + 4T(r,g) + S(r,g).$$

Adding (2.3) and (2.4) we obtain

$$(n-4)\{T(r,f) + T(r,g)\} \leq S(r,f) + S(r,g),$$

which is a contradiction. So $c = 0$ and the lemma is proved. □

LEMMA 2.4. *Let F and G be given as in Lemma 2.3. Then $F \equiv G$ implies $f \equiv g$.*

The proof of the lemma can be found in [8].

LEMMA 2.5. [6] *If f, g are nonconstant meromorphic functions and $E_{3)}(1;f) = E_{3)}(1;g)$ then one of the following cases holds :*

(i) $T(r,f) + T(r,g) \leq 2\{N_2(r,0;f) + N_2(r,0;g) + N_2(r,\infty;f) + N_2(r,\infty;g)\} + S(r,f) + S(r,g)$;

(ii) $f \equiv g$;

(iii) $fg \equiv 1$.

LEMMA 2.6. [5] *Let f be a nonconstant meromorphic function and k be a positive integer. Then*

$$N_2(r,0;f^{(k)}) \leq k\overline{N}(r,\infty;f) + N_{2+k}(r,0;f) + S(r,f).$$

LEMMA 2.7. *Let F and G be given as in Lemma 2.3 and $a = \frac{n+3}{n+2} + i\sqrt{\frac{n+3}{n+1}} \cdot \frac{1}{n+2}$. Then*

(i) $T(r,F) \leq T(r,F') + N(r,0;f) + N(r,a;f) + N(r,\bar{a};f) - 2N(r,1;f) - N(r,0;f') + S(r,f)$;

(ii) $T(r,G) \leq T(r,G') + N(r,0;g) + N(r,a;g) + N(r,\bar{a};g) - 2N(r,1;g) - N(r,0;g') + S(r,g)$;

Proof. We prove (i) because (ii) is similar. By the first fundamental theorem and Lemma 2.2 we get

$$\begin{aligned} T(r,F) &= T(r,1/F)+O(1) \\ &= N(r,0;F)+m(r,1/F)+O(1) \\ &\le N(r,0;F)+m(r,F'/F)+m(r,0;F')+O(1) \\ &= T(r,F')+N(r,0;F)-N(r,0;F')+S(r,F) \\ &= T(r,F')+(n+1)N(r,0;f)+N(r,a;f)+N(r,\bar{a};f) \\ &\quad -nN(r,0;f)-2N(r,1;f)-N(r,0;f')+S(r,f) \\ &= T(r,F')+N(r,0;f)+N(r,a;f)+N(r,\bar{a};f) \\ &\quad -2N(r,1;f)-N(r,0;f')+S(r,f). \end{aligned}$$

This proves the lemma. □

LEMMA 2.8. *Let f and g be two nonconstant meromorphic functions. Then $f^n(f^3-1)f'g^n(g^3-1)g' \not\equiv 1$, where n is a positive integer.*

Proof. If possible let $f^n(f^3-1)f'g^n(g^3-1)g' \equiv 1$. Let z_0 be a 1-point of f with multiplicity p. Then z_0 is a pole of g with multiplicity q, say, such that $2p-1=(n+4)q+1 \ge n+5$ i.e., $p \ge \frac{n+6}{2}$. Hence $\Theta(1;f) \ge 1-\frac{2}{n+6}$. Similarly we can show that $\Theta(\omega;f) \ge 1-\frac{2}{n+6}$ and $\Theta(\omega^2;f) \ge 1-\frac{2}{n+6}$, where ω is the imaginary cube root of unity. Therefore

$$\Theta(1;f)+\Theta(\omega;f)+\Theta(\omega^2;f) \ge 3-\frac{6}{n+6} > 2,$$

a contradiction. This proves the lemma. □

LEMMA 2.9. Let $F_1 = f^{n+1}\left(\frac{f^3}{n+4}-\frac{1}{n+1}\right)$ and $G_1 = g^{n+1}\left(\frac{g^3}{n+4}-\frac{1}{n+1}\right)$, *where $n(\ge 2)$ is an integer. If $F_1 \equiv G_1$ then $f \equiv g$.*

Proof. Let $h=g/f$. If possible, suppose that h is nonconstant. Since $F_1 \equiv G_1$, it follows that

$$f^3 \equiv \frac{n+4}{n+1}\cdot\frac{h^{n+1}-1}{h^{n+4}-1}.$$

Since f^3 has no simple pole, it follows that $h-u_k=0$ has no simple root for $k=1,2,\ldots,n+3$, where $u_k=\exp\left(\frac{2\pi ik}{n+4}\right)$. Hence $\Theta(u_k;h)\ge 1/2$ for $k=1,2,\ldots,n+3$, which is impossible. Therefore h is a constant. If $h\ne 1$, it follows that f is a constant, which is not the case. So $h=1$ and hence $f\equiv g$. This proves the lemma. □

LEMMA 2.10. *Let F_1 and G_1 be defined as in Lemma 2.9. Then*

$$\text{(i) } T(r,F_1) \le T(r,F_1') + N(r,0;f) + N(r,\frac{n+4}{n+1};f^3) - N(r,1;f^3) - N(r,0;f') + S(r,f),$$
$$\text{(ii) } T(r,G_1) \le T(r,G_1') + N(r,0;g) + N(r,\frac{n+4}{n+1};g^3) - N(r,1;g^3) - N(r,0;g') + S(r,g).$$

The lemma can be proved in the line of the proof of Lemma 2.7.

LEMMA 2.11. *Let F_1 and G_1 be defined as in Lemma 2.9, where $n(\ge 5)$ is an integer. Then $F_1' \equiv G_1'$ implies $F_1 \equiv G_1$.*

The proof is similar to that of Lemma 2.3.

3. Proofs of the Theorems

In this section we present the proofs of the main results.

Proof of Theorem 1.1. Let F and G be defined as in Lemma 2.3. If possible, suppose that

$$\begin{aligned} T(r,F') + T(r,G') \le\ & 2\{N_2(r,0;F') + N_2(r,0;G') + N_2(r,\infty;F') \\ & + N_2(r,\infty;G')\} + S(r,F') + S(r,G'). \end{aligned}$$

Then by Lemmas 2.2, 2.6 and 2.7 we get

$$\begin{aligned} & T(r,F) + T(r,G) \\ \le\ & T(r,F') + N(r,0;f) + N(r,a;f) + N(r,\bar{a};f) - 2N(r,1;f) \\ & -N(r,0;f') + T(r,G') + N(r,0;g) + N(r,a;g) + N(r,\bar{a};g) \\ & -2N(r,1;g) - N(r,0;g') + S(r,f) + S(r,g) \\ \le\ & 2\left\{N_2(r,0;F') + N_2(r,0;G') + N_2(r,\infty;F') + N_2(r,\infty;G')\right\} \\ & +N(r,0;f) + N(r,a;f) + N(r,\bar{a};f) - 2N(r,1;f) - N(r,0;f') \\ & +N(r,0;g) + N(r,a;g) + N(r,\bar{a};g) - 2N(r,1;g) - N(r,0;g') \\ & +S(r,f) + S(r,g) \\ \le\ & 4\overline{N}(r,0;f) + 2N(r,0;(f-1)^2) + 2N_2(r,0;f') + 4\overline{N}(r,0;g) \\ & +2N(r,0;(g-1)^2) + 2N_2(r,0;g') + 4\overline{N}(r,\infty;f) + 4\overline{N}(r,\infty;g) \\ & +N(r,0;f) + N(r,a;f) + N(r,\bar{a};f) - 2N(r,1;f) - N(r,0;f') \\ & +N(r,0;g) + N(r,a;g) + N(r,\bar{a};g) - 2N(r,1;g) - N(r,0;g') \\ & +S(r,f) + S(r,g) \end{aligned}$$

$$\begin{aligned}&\le 11T(r,f)+2N(r,1;f)+T(r,f')+11T(r,g)+2N(r,1;g)\\&\quad +T(r,g')+S(r,f)+S(r,g)\\&\le 15T(r,f)+15T(r,g)+S(r,f)+S(r,g).\end{aligned}$$

So by Lemma 2.2 we get

$$(n-12)\left\{T(r,f)+T(r,g)\right\}\le S(r,f)+S(r,g),$$

which is a contradiction.

Hence by Lemma 2.5 either $F'\equiv G'$ or $F'G'\equiv 1$. Since by Lemma 2.1 $F'G'\not\equiv 1$, it follows by Lemma 2.3 and Lemma 2.4 $f\equiv g$. This proves the theorem. □

Proof of Theorem 1.2. Let F_1 and G_1 be defined as in Lemma 2.9. If possible suppose that

$$\begin{aligned}&T(r,F_1')+T(r,G_1')\\&\le 2\left\{N_2(r,0;F_1')+N_2(r,0;G_1')+N_2(r,\infty;F_1')+N_2(r,\infty;G_1')\right\}\\&\quad +S(r,F_1')+S(r,G_1').\end{aligned}$$

Then by Lemmas 2.2, 2.6 and 2.10 we get

$$\begin{aligned}&T(r,F_1)+T(r,G_1)\\&\le 2\left\{N_2(r,0;F_1')+N_2(r,0;G_1')+N_2(r,\infty;F_1')+N_2(r,\infty;G_1')\right\}\\&\quad +N(r,0;f)+N(r,\frac{n+4}{n+1};f^3)-N(r,0;f')-N(r,1;f^3)\\&\quad +N(r,0;g)+N(r,\frac{n+4}{n+1};g^3)-N(r,1;g^3)-N(r,0;g')+S(r,f)\\&\quad +S(r,g)\\&\le 4N(r,0;f)+2N_2(r,1;f^3)+2N_2(r,0;f')+4N(r,0;g)\\&\quad +2N_2(r,1;g^3)+2N_2(r,0;g')+4\overline{N}(r,\infty;f)+4\overline{N}(r,\infty;g)\\&\quad +N(r,0;f)+N(r,\frac{n+4}{n+1};f^3)-N(r,1;f^3)-N(r,0;f')\\&\quad +N(r,0;g)+N(r,\frac{n+4}{n+1};g^3)-N(r,1;g^3)-N(r,0;g')\\&\quad +S(r,f)+S(r,g)\\&\le 17T(r,f)+17T(r,g)+S(r,f)+S(r,g)\end{aligned}$$

and so by Lemma 2.2 we get

$$(n-13)\left\{T(r,f)+T(r,g)\right\}\le S(r,f)+S(r,g),$$

which is a contradiction.

Hence by Lemma 2.5 either $F_1' \equiv G_1'$ or $F_1'G_1' \equiv 1$. Since by Lemma 2.8 $F_1'G_1' \not\equiv 1$, it follows by Lemmas 2.9 and 2.11 that $f \equiv g$. This proves the theorem. □

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