

## CONTROLLABILITY, OBSERVABILITY, AND REALIZABILITY OF MATRIX LYAPUNOV SYSTEMS

M. S. N. MURTY, B. V. APPA RAO, AND G. SURESH KUMAR

ABSTRACT. This paper presents necessary and sufficient conditions for complete controllability, complete observability and realizability associated with matrix Lyapunov systems under certain smoothness conditions.

### 1. Introduction

The importance of control theory in applied mathematics and its occurrence in several problems such as mechanics, electromagnetic theory, thermodynamics, artificial satellites etc., are well known. The main aim being to compel or control a given system to behave in some desired fashion. The main interest being to control the system automatically, with out direct human intervention.

In this paper we focus our attention to the first order matrix Lyapunov systems represented by

$$(1.1) \quad X'(t) = A(t)X(t) + X(t)B(t) + F(t)U(t)$$

$$(1.2) \quad Y(t) = C(t)X(t)$$

where  $X(t)$  is  $n \times n$  matrix,  $U(t)$  is  $m \times n$  input matrix called control and  $Y(t)$  is  $r \times n$  out put matrix. Here  $A(t), B(t), F(t)$  and  $C(t)$  are  $n \times n, n \times n, n \times m$  and  $r \times n$  matrices respectively and all of them are assumed to be continuous functions of  $t$ .

Many authors [2, 3] obtained controllability and observability criteria for similar systems of the type (1.1) and (1.2) with  $B(t) = 0$ .

---

Received November 18, 2004.

2000 Mathematics Subject Classification: 49K15, 93C15.

Key words and phrases: controllability, observability and realizability, Lyapunov systems and transition matrix.

In Section 2 we study some basic properties of Kronecker product of matrices and develop preliminary results by converting the given problem into a Kronecker product problem. The solution to the corresponding initial value problem obtained in terms of transition matrices of the systems  $X'(t) = A(t)X(t)$  and  $[X'(t)]^* = B^*(t)X^*(t)$  by using the standard technique of variation of parameters [4].

Section 3 deals with providing necessary and sufficient conditions for complete controllability and complete observability under certain smoothness conditions.

In Section 4 we develop realizability criteria and minimal realizability criteria, with zero initial state under more strengthened forms of controllability and observability developed in Section 3.

## 2. Preliminaries

In this section we present some properties and rules for Kronecker products and basic results related to matrix Lyapunov systems.

DEFINITION 2.1. [1] Let  $A \in C^{m \times n}$  and  $B \in C^{p \times q}$  then the Kronecker product of  $A$  and  $B$  written  $A \otimes B$  is defined to be the partitioned matrix

$$A \otimes B = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mn} \end{bmatrix}$$

is an  $mp \times nq$  matrix and is in  $C^{mp \times nq}$ .

DEFINITION 2.2. [1] Let  $A = [a_{ij}] \in C^{m \times n}$  we denote

$$\hat{A} = \text{Vec}A = \begin{bmatrix} A_{.1} \\ A_{.2} \\ \cdot \\ \cdot \\ A_{.n} \end{bmatrix}, \text{ where } A_{.j} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ a_{mj} \end{bmatrix} \quad (1 \leq j \leq n).$$

The Kronecker product has the following properties and rules [1]

- (1)  $(A \otimes B)^* = A^* \otimes B^*$  ( $A^*$  denotes the transpose of  $A$ )
- (2)  $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$
- (3) The mixed product rule  $(A \otimes B)(C \otimes D) = (AC \otimes BD)$  this rule holds good, provided the dimension of the matrices are such that the various expressions exist.

(4) If  $A(t)$  and  $B(t)$  are matrices, then

$$(A \otimes B)' = A' \otimes B + A \otimes B' \quad (' = d/dt)$$

(5)  $\text{Vec}(AYB) = (B^* \otimes A)\text{Vec}Y$

(6) If  $A$  and  $B$  are matrices both of order  $n \times n$  then

(i)  $\text{Vec}(AX) = (I_n \otimes A)\text{Vec}X$

(ii)  $\text{Vec}(XA) = (A^* \otimes I_n)\text{Vec}X.$

Now by applying the  $\text{Vec}$  operator to the non-homogeneous controllable time varying matrix Lyapunov system (1.1) and also the output equation (1.2) and using the above properties we have

$$(2.1) \quad \psi'(t) = G(t)\psi(t) + (I_n \otimes F(t))\hat{U}(t)$$

$$(2.2) \quad \hat{Y}(t) = (I_n \otimes C(t))\psi(t)$$

where  $\psi(t) = \text{Vec}X(t)$ ,  $G(t) = (B^* \otimes I_n) + (I_n \otimes A)$ ,  $\hat{U}(t) = \text{Vec}U(t)$  and

$$\hat{Y}(t) = \text{Vec}Y(t).$$

Now we confine our attention to the corresponding homogeneous matrix system of (2.1) given by

$$(2.3) \quad \psi'(t) = G(t)\psi(t)$$

LEMMA 2.1. *Let  $\phi_1$  and  $\phi_2$  denote state transition matrices of the systems  $X'(t) = A(t)X(t)$  and  $(X^*(t))' = B^*(t)X^*(t)$  respectively. Then the matrix  $\phi(t, s)$  defined by*

$$(2.4) \quad \phi(t, s) = \phi_2(t, s) \otimes \phi_1(t, s)$$

*is the state transition matrix of (2.3) and every solution of (2.3) is of the form  $\psi(t) = \phi(t, s)C$  (where  $C$  is any constant vector of order  $n^2$ ).*

*Proof.* Consider

$$\begin{aligned} \phi'(t, s) &= (\phi_2' \otimes \phi_1) + (\phi_2 \otimes \phi_1') \\ &= B^* \phi_2 \otimes \phi_1 + \phi_2 \otimes A \phi_1 \\ &= (B^* \otimes I_n + I_n \otimes A)(\phi_2 \otimes \phi_1) \end{aligned}$$

Hence  $\phi' = G\phi$ .

Also

$$\phi(t, t) = \phi_2(t, t) \otimes \phi_1(t, t) = I_n \otimes I_n = I_{n^2}.$$

Hence  $\phi$  is the transition matrix of (2.3). Moreover it can be easily seen that  $\psi$  is a solution of (2.3) and every solution of (2.3) is of this form.  $\square$

**THEOREM 2.1.** *Let  $\phi = \phi_2 \otimes \phi_1$  be a transition matrix of (2.3), then the unique solution of (2.1), subject to the initial condition  $\psi(t_0) = \psi_0$ , is*

$$(2.5) \quad \psi(t) = \phi(t, t_0) \left[ \psi_0 + \int_{t_0}^t \phi(t_0, s) (I_n \otimes F(s)) \hat{U}(s) ds \right]$$

*Proof.* The proof is similar to the proof of Theorem 3.3 of [2].  $\square$

### 3. Controllability and observability

In this section we obtain necessary and sufficient conditions for controllability and observability of the systems (2.1) and (2.2). Unless otherwise stated  $I$  stands for  $I_n$ .

**DEFINITION 3.1.** A linear time varying system  $S_1$  given by (2.1) and (2.2) is said to be completely controllable (c.c.) if for  $t_0$ , any initial state  $\psi(t_0) = \psi_0$  and any given final state  $\psi_f$  there exists a finite time  $t_1 > t_0$  and a control  $\hat{U}(t)$ ,  $t_0 \leq t \leq t_1$ , such that  $\psi(t_1) = \psi_f$ .

**THEOREM 3.1.** *The system  $S_1$  is c.c. if and only if the  $n^2 \times n^2$  symmetric controllability matrix*

$$(3.1) \quad W(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_0, s) (I \otimes F(s)) (I \otimes F(s))^* \phi^*(t_0, s) ds,$$

where  $\phi$  is defined in (2.4), is nonsingular. In this case the control

$$(3.2) \quad \hat{U}(t) = -(I \otimes F(t))^* \phi^*(t_0, t) W^{-1}(t_0, t_1) \{ \psi_0 - \phi(t_0, t_1) \psi_f \}$$

defined on  $t_0 \leq t \leq t_1$ , transfers  $\psi(t_0) = \psi_0$  to  $\psi(t_1) = \psi_f$ .

*Proof.* Suppose that  $W(t_0, t_1)$  is nonsingular, then the control defined by (3.2) exists. Now substituting (3.2) in (2.5) with  $t = t_1$ , we have

$$\begin{aligned} & \psi(t_1) \\ &= \phi(t_1, t_0) \left[ \psi_0 - \int_{t_0}^{t_1} \phi(t_0, s) (I \otimes F(s)) (I \otimes F(s))^* \phi^*(t_0, s) W^{-1}(t_0, t_1) \right. \\ & \quad \left. \times \{ \psi_0 - \phi(t_0, t_1) \psi_f \} ds \right] \\ &= \phi(t_1, t_0) \phi(t_0, t_1) = \psi_f. \end{aligned}$$

Hence  $S_1$  is c.c.

Conversely suppose that  $S_1$  is c.c., then we have to show that  $W(t_0, t_1)$  is nonsingular.

Since  $W$  is symmetric we can construct the quadratic form

$$\begin{aligned}
 \alpha^* W \alpha &= \int_{t_0}^{t_1} \theta^*(s, t_0) \theta(s, t_0) ds \\
 (3.3) \qquad &= \int_{t_0}^{t_1} \|\theta\|_e^2 ds \geq 0
 \end{aligned}$$

where  $\alpha$  is an arbitrary constant column  $n^2$ -vector and  $\theta(s, t_0) = (I \otimes F(s))^* \phi^*(t_0, s) \alpha$ . From (3.3),  $W(t_0, t_1)$  is positive semi definite. Suppose there exists some  $\beta \neq 0$  such that  $\beta^* W(t_0, t_1) \beta = 0$ , then from equation (3.3) with  $\theta = \eta$  when  $\alpha = \beta$ , implies

$$\int_{t_0}^{t_1} \|\eta\|_e^2 ds = 0.$$

Using the properties of norms, we have

$$(3.4) \qquad \eta(s, t_0) = 0, \quad t_0 \leq t \leq t_1.$$

Since  $S_1$  is c.c. so there exists a control  $\hat{E}(t)$  making  $\psi(t_1) = 0$  if  $\psi(t_0) = \beta$ . Hence from (2.5) we have

$$\beta = - \int_{t_0}^{t_1} \phi(t_0, s) (I \otimes F(s)) \hat{E}(s) ds.$$

Consider

$$\begin{aligned}
 \|\beta\|_e^2 = \beta^* \beta &= - \int_{t_0}^{t_1} \hat{E}^*(s) (I \otimes F(s))^* \phi^*(t_0, s) \beta ds \\
 &= - \int_{t_0}^{t_1} \hat{E}^*(s) \eta(s, t_0) ds = 0.
 \end{aligned}$$

Hence  $\beta = 0$ , which is a contradiction to our supposition, thus  $W(t_0, t_1)$  is positive definite and is therefore nonsingular.  $\square$

**THEOREM 3.2.** *If, for a given  $\psi_f$ , there exists a constant column  $n^2$ -vector  $\gamma$  such that*

$$(3.5) \qquad W(t_0, t_1) \gamma = \psi_0 - \phi(t_0, t_1) \psi_f$$

then the control

$$\hat{U}(t) = -(I \otimes F(t))^* \phi^*(t_0, t) \gamma$$

transfers the system (2.1) from  $\psi(t_0) = \psi_0$  to  $\psi(t_1) = \psi_f$ .

*Proof.* Substituting given control  $\hat{U}(t)$  into (2.5) with  $t = t_1$  gives

$$\begin{aligned} \psi(t_1) &= \phi(t_1, t_0) \left[ \psi_0 - \int_{t_0}^{t_1} \phi(t_0, s) (I \otimes F(s)) (I \otimes F(s))^* \phi^*(t_0, s) \gamma ds \right] \\ &= \phi(t_1, t_0) [\psi_0 - W(t_0, t_1) \gamma] \\ &= \phi(t_1, t_0) \phi(t_0, t_1) \psi_f = \psi_f. \end{aligned}$$

Hence we have our assertions. □

Define the transformation

$$(3.6) \quad z(t) = (I \otimes P(t)) \psi(t)$$

where  $P(t) (t \geq t_0)$  is continuous nonsingular square matrix of order  $n$ . The system  $S_2$  obtained from  $S_1$  by using the above transformation is said to be algebraically equivalent to  $S_1$ .

**THEOREM 3.3.** *If  $\phi(t, t_0)$  is the state transition matrix for  $S_1$  then*

$$(3.7) \quad \bar{\phi}(t, t_0) = (I \otimes P(t)) \phi(t, t_0) (I \otimes P^{-1}(t_0))$$

*is the state transition matrix for  $S_2$ .*

*Proof.* Given that  $\phi(t, t_0)$  is the state transition matrix for  $S_1$ . Then

$$\phi'(t, t_0) = G(t) \phi(t, t_0), \phi(t_0, t_0) = I_{n^2},$$

where  $G(t) = (B^*(t) \otimes I) + (I \otimes A(t))$ .

Clearly  $\bar{\phi}(t_0, t_0) = I_{n^2}$ . Differentiating (3.6) and using (2.1) gives

$$\begin{aligned} z'(t) &= (I \otimes P'(t)) \psi(t) + (I \otimes P(t)) [G(t) \psi(t) + (I \otimes F(t)) \hat{U}(t)] \\ &= [(I \otimes P'(t)) + (I \otimes P(t)) G(t)] \psi(t) \\ &\quad + (I \otimes P(t)) (I \otimes F(t)) \hat{U}(t) \\ &= [(I \otimes P'(t)) + (I \otimes P(t)) G(t)] (I \otimes P(t))^{-1} z(t) \\ (3.8) \quad &+ (I \otimes P(t)) (I \otimes F(t)) \hat{U}(t). \end{aligned}$$

Now we show that  $\bar{\phi}(t, t_0)$  is state transition matrix for (3.8).

Consider

$$\begin{aligned}
 \bar{\phi}'(t, t_0) &= (I \otimes P'(t))\phi(t, t_0)(I \otimes P^{-1}(t_0)) \\
 &\quad + (I \otimes P^{-1}(t_0))G(t)\phi(t, t_0)(I \otimes P^{-1}(t_0)) \\
 &= [(I \otimes P'(t)) + (I \otimes P(t))G(t)](I \otimes P(t))^{-1} \\
 &\quad \times (I \otimes P(t))\phi(t, t_0)(I \otimes P^{-1}(t_0)) \\
 &= [(I \otimes P'(t)) + (I \otimes P(t))G(t)](I \otimes P(t))^{-1}\bar{\phi}(t, t_0).
 \end{aligned}$$

Hence  $\bar{\phi}(t, t_0)$  is state transition matrix for  $S_2$ .  $\square$

**THEOREM 3.4.** *If  $S_1$  is c.c. then so is  $S_2$ .*

*Proof.* From (3.8) and (3.1) the controllability matrix for  $S_2$  is

$$\begin{aligned}
 (3.9) \quad &\bar{W}(t_0, t_1) \\
 &= \int_{t_0}^{t_1} \bar{\phi}(t_0, s)(I \otimes P(s))(I \otimes F(s))(I \otimes F(s))^*(I \otimes P(s))^*\bar{\phi}^*(t_0, s)ds \\
 &= (I \otimes P(t_0))W(t_0, t_1)(I \otimes P(t_0))^*
 \end{aligned}$$

Thus  $\bar{W}(t_0, t_1)$  is nonsingular since the matrices  $W(t_0, t_1)$ ,  $P(t_0)$  are nonsingular and from Theorem 3.1  $S_2$  is c.c.  $\square$

**DEFINITION 3.2.** The system  $S_1$  is completely observable(c.o) if for any time  $t_0$  and any initial state  $\psi(t_0) = \psi_0$  there exists a finite time  $t_1 > t_0$ , such that the knowledge of  $\hat{U}(t)$  and  $\hat{Y}(t)$  for  $t_0 \leq t \leq t_1$  suffices to determine  $\psi_0$  uniquely.

**THEOREM 3.5.** *The system  $S_1$  is c.o. if and only if the symmetric observability matrix*

$$(3.10) \quad V(t_0, t_1) = \int_{t_0}^{t_1} \phi^*(s, t_0)(I \otimes C(s))^*(I \otimes C(s))\phi(s, t_0)ds$$

*is nonsingular.*

*Proof.* Suppose that  $V(t_0, t_1)$  is nonsingular. Assuming  $\hat{U}(t) = 0$ ,  $t_0 \leq t \leq t_1$ , we have

$$\hat{Y}(t) = (I \otimes C(t))\psi(t).$$

Since from  $\psi(t) = \phi(t, t_0)\psi_0$ , we have

$$(3.11) \quad \hat{Y}(t) = (I \otimes C(t))\phi(t, t_0)\psi_0.$$

Multiplying (3.11) on the left by  $\phi^*(t, t_0)(I \otimes C(t))^*$  and integrating from  $t_0$  to  $t_1$  we obtain

$$\int_{t_0}^{t_1} \phi^*(s, t_0)(I \otimes C(s))^* \hat{Y}(s) ds = V(t_0, t_1) \psi_0,$$

$$\Rightarrow \psi_0 = V^{-1}(t_0, t_1) \int_{t_0}^{t_1} \phi^*(s, t_0)(I \otimes C(s))^* \hat{Y}(s) ds.$$

Hence  $S_1$  is c.o.

Conversely suppose that  $S_1$  is c.o., then we prove that  $V(t_0, t_1)$  is nonsingular. Since  $V(t_0, t_1)$  is symmetric, we can construct the quadratic form

$$(3.12) \quad \alpha^* V \alpha = \int_{t_0}^{t_1} [(I \otimes C(s)) \phi(s, t_0) \alpha]^* [(I \otimes C(s)) \phi(s, t_0) \alpha] ds$$

$$= \int_{t_0}^{t_1} \|\eta(s, t_0)\|_e^2 ds \geq 0,$$

where  $\alpha$  is an arbitrary column  $n^2$ -vector and  $\eta(s, t_0) = (I \otimes C(s)) \phi(s, t_0) \alpha$ . From (3.12)  $V(t_0, t_1)$  is positive semi definite. Suppose there exists a  $\beta$  such that  $\beta^* V \beta = 0$ . From equation (3.12) with  $\eta = \theta$  when  $\alpha = \beta$ , then implies

$$\int_{t_0}^{t_1} \|\theta(s, t_0)\|_e^2 ds = 0$$

$$\Rightarrow \theta(s, t_0) = 0, t_0 \leq s \leq t_1$$

$$\Rightarrow (I \otimes C(s)) \theta(s, t_0) \beta = 0, t_0 \leq s \leq t_1.$$

From (3.11), this implies that when  $\psi_0 = \beta$ , the out put is identically zero throughout the interval, so that  $\psi_0$  cannot be determined in this case from a knowledge of  $\hat{Y}(t)$ . This contradicts the supposition that  $S_1$  is c.o. Hence  $V(t_0, t_1)$  is positive definite, and therefore nonsingular.  $\square$

#### 4. Realizability

In this section we discuss realizability and minimal realizability criteria for the systems (2.1) and (2.2) with zero initial state.



The output corresponding to zero initial state is given by

$$\begin{aligned}\hat{Y}(t) &= (I \otimes C(t)) \int_{t_0}^t \phi(t, s)(I \otimes F(s))\hat{U}(s)ds \\ &= \int_{t_0}^t K(t, s)\hat{U}(s)ds,\end{aligned}$$

where  $\phi$  is defined by (2.4). The matrix

$$(4.1) \quad K(t, s) = (I \otimes C(t))\phi(t, s)(I \otimes F(s))$$

is called the weighting pattern matrix.

For a given  $K(t, s)$  the realization problem is to find a  $\{G(t), (I \otimes F(t)), (I \otimes C(t))\}$  such that (4.1) is satisfied. The minimality of a realization is  $G(t)$  has least possible dimension.

**THEOREM 4.1.** *A realization exists for a matrix  $K(t, s)$  if and only if it can be expressed in the form*

$$(4.2) \quad K(t, s) = L(t)M(s),$$

where  $L$  and  $M$  are matrices having finite dimensions.

*Proof.* Suppose  $K$  possesses a realization, then (4.1) exists and

$$\begin{aligned}K(t, s) &= (I \otimes C(t))\phi(t, s)(I \otimes F(s)) \\ &= (I \otimes C(t))\{\phi_2(t, s) \otimes \phi_1(t, s)\}(I \otimes F(s)) \\ &= (I \otimes C(t))\{X_2(t)X_2^{-1}(s) \otimes X_1(t)X_1^{-1}(s)\}(I \otimes F(s)) \\ &= (I \otimes C(t))(X_2(t) \otimes X_1(t))(X_2^{-1}(s) \otimes X_1^{-1}(s))(I \otimes F(s)) \\ &= L(t)M(s).\end{aligned}$$

Where  $X_1$  and  $X_2$  are fundamental matrices of  $X'(t) = A(t)X(t)$  and  $[X'(t)]^* = B^*(t)X^*(t)$  respectively,  $L(t) = (I \otimes C(t))(X_2(t) \otimes X_1(t))$  and  $M(s) = (X_2^{-1}(s) \otimes X_1^{-1}(s))(I \otimes F(s))$ . So (4.2) is certainly a necessary condition.

Conversely, if (4.2) holds  $K(t, s) = L(t)M(s) = L(t)I_{n^2}M(s)$  this implies that  $\phi(t, s) = I_{n^2}$ , then a realization of  $K(t, s)$  is  $\{0_{n^2}, M(t), L(t)\}$ , where  $0_{n^2}$  denotes an  $n^2 \times n^2$  zero matrix.  $\square$

**THEOREM 4.2.** *A realization  $R = \{G(t), (I \otimes F(t)), (I \otimes C(t))\}$  of  $K(t, s)$  is minimal if and only if it is c.c. and c.o.*

*Proof.* Suppose that  $R = \{G(t), (I \otimes F(t)), (I \otimes C(t))\}$  is minimal. We assume that the pair  $[G(t), (I \otimes F(t))]$  is not c.c. and show that  $R$  is

not minimal. A similar argument applies if  $[G(t), (I \otimes C(t))]$  is assumed not c.o.

Now suppose that  $G(t)$  is  $n^2 \times n^2$  and that the controllability matrix  $W(t_0, t_1)$  in (3.1) has rank  $p < n^2$ . From the proof of Theorem 3.1,  $W(t_0, t_1)$  is positive semi definite, so there exists a nonsingular matrix  $T$  such that

$$(4.3) \quad TWT^* = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}.$$

Consider the algebraic equivalence transformation (3.6) with  $I \otimes P(t) = T\phi(t_0, t)$ . Then the corresponding realization under this transformation  $R$  becomes

$$\bar{R} = \{0_{n^2}, (I \otimes P(t))(I \otimes F(t)), (I \otimes C(t))(I \otimes P^{-1}(t))\}.$$

From (3.9), the controllability matrix associated with  $\bar{R}$  is

$$(4.4) \quad \begin{aligned} \bar{W} &= (I \otimes P(t_0))W(t_0, t_1)(I \otimes P(t_0))^* \\ &= TWT^* \end{aligned}$$

and since  $\phi = I_{n^2}$  for  $R$ , we also have from (3.1)

$$(4.5) \quad \bar{W} = \int_{t_0}^{t_1} (I \otimes P(s))(I \otimes F(s))(I \otimes F(s))^*(I \otimes P(s))^* ds.$$

From (4.3), (4.4), and (4.5), it follows that

$$(I \otimes P(t))(I \otimes F(s)) = \begin{bmatrix} \beta \\ 0 \end{bmatrix},$$

where  $\beta$  has dimension  $p \times mn$ . This implies that  $\{0_p, \beta, \gamma\}$ , where  $\gamma$  is  $nr \times p$ , is a realization of  $K$ , and this contradicts the minimality of  $R$ .

Conversely assume that  $R$  is c.c. and c.o. and show that there can not exist a realization  $R_1$  of  $K$  having order  $n_1 < n^2$ .

Now assume that such a realization exists and is of the form

$$R_1 = \{0_{n_1}, (I \otimes F_1(t)), (I \otimes C_1(t))\}.$$

Now taking the transformation (3.6) with  $(I \otimes P(t)) = (X_2(t) \otimes X_1(t))^{-1}$  then  $R$  becomes  $\bar{R} = \{0_{n^2}, I \otimes \bar{F}(t), I \otimes \bar{C}(t)\}$  and remains c.c. and c.o. Since

$$K(t, s) = (I \otimes C_1(t))(I \otimes F_1(s)) = (I \otimes \bar{C}(t))(I \otimes \bar{F}(s)),$$

multiplying the second and third term in the above equation on the left and right by  $(I \otimes \bar{C}(t))^*$  and  $(I \otimes \bar{F}(s))^*$  respectively and integrating with respect to  $t$  and  $s$  gives

$$(4.6) \quad W_1 W_2 = \bar{V} \bar{W},$$

where

$$W_1 = \int_{t_0}^{t_1} (I \otimes \bar{C}(t))^* (I \otimes C_1(t)) dt, \quad W_2 = \int_{t_0}^{t_1} (I \otimes F_1(s)) (I \otimes \bar{F}(s))^* ds$$

and  $\bar{V}$  and  $\bar{W}$  are the observability and controllability matrices for  $\bar{R}$ . By assumption  $\bar{V}$  and  $\bar{W}$  have rank  $n^2$ , so  $\bar{V} \bar{W}$  also has rank  $n^2$ . However,  $W_1$  and  $W_2$  have dimensions  $n^2 \times n_1$  and  $n_1 \times n^2$  respectively, so that  $\text{rank}(W_1 W_2) \leq n_1$ . From (4.6) we have  $n_1 \geq n^2$ , and hence  $R$  is minimal.  $\square$

**THEOREM 4.3.** *If  $R = \{G(t), I \otimes F(t), I \otimes C(t)\}$  is a minimal realization of  $K(t, s)$ , then  $\bar{R} = \{\bar{G}(t), I \otimes \bar{F}(t), I \otimes \bar{C}(t)\}$  is also a minimal realization if and only if*

$$\begin{aligned} \bar{G}(t) &= [(I \otimes P'(t)) + (I \otimes P(t))G(t)](I \otimes P(t))^{-1}, \\ \bar{F}(t) &= P(t)F(t), \quad \bar{C}(t) = C(t)P^{-1}(t), \end{aligned}$$

where  $P(t)$  is continuous and nonsingular.

*Proof.* Proof follows along similar lines to that of Theorem 4.2.  $\square$

### References

- [1] Graham. Alexander, *Kronecker products and matrix calculus; with applications*, Ellis Horwood Ltd. England, 1981.
- [2] S. Barnett, *Introduction to mathematical control theory*, Clarenton Press, Oxford, 1975.
- [3] R. Bellman, *Introduction to the mathematical theory of control processes*, Mathematics in Science and Engineering Vol. 40, Academic press, New York, 1967.
- [4] R. H. Cole, *The theory of ordinary differential equations*, Appleton-Century-Crofts Meredith Corp., New York, 1968.
- [5] M. S. N. Murthy and B. V. Apparao, *Kronecker product boundary value problems—existence and uniqueness*, J. Indian Acad. Math. **25** (2003), no. 2, 383–394.

DEPARTMENT OF APPLIED MATHEMATICS, ACHARYA NAGARJUNA UNIVERSITY, POST GRADUATE CENTRE, NUZVID-521201, ANDHRAPRADESH, INDIA  
*E-mail:* drmsn2002@yahoo.com