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ASYMPTOTIC NUMBERS OF GENERAL 4-REGULAR GRAPHS WITH GIVEN CONNECTIVITIES

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Abstract. Let $g(n, l_1, l_2, d, t, q)$ be the number of general 4-regular graphs on n labelled vertices with $l_1 + 2l_2$ loops, d double edges, t triple edges and q quartet edges. We use inclusion and exclusion with five types of properties to determine the asymptotic behavior of $g(n, l_1, l_2, d, t, q)$ and hence that of $g(2n)$, the total number of general 4-regular graphs where l_1, l_2, d, t and $q = o(\sqrt{n})$, respectively. We show that almost all general 4-regular graphs are 2-connected. Moreover, we determine the asymptotic numbers of general 4-regular graphs with given connectivities.

1. Introduction

Let $g(n, l_1, l_2, d, t, q)$ be the number of general 4-regular graphs on n labelled vertices with $l_1 + 2l_2$ loops, d double edges, t triple edges and q quartet edges. A general 4-regular graph may have two loops on a vertex. We call it a double loop, and l_2 counts the number of double loops. Note that

$$n = \frac{2s + 2l_1 + 4l_2 + 4d + 6t + 8q}{4}, \tag{1.1}$$

where s is the number of single edges. Let $g(n)$ be the total number of general 4-regular graphs of order n. Wormald first gave the equation

$$g(n) = (1 + o(1))\frac{e^{15/4}}{(4!)^n}\frac{(4n)!}{2^{2n}\cdot(2n)!} \tag{1.2}$$

in [7], [8] by estimating the number of matrices with given row and column sums. McKay and Wormald also derived it in [9] using switching technique. In this paper, we establish the equation (1.2) using inclusion

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and exclusion with five types of properties and adapting the configurations from the equation

$$g(n,l_1,l_2,d,t,q) = (1+o(1)) \tag{1.3}$$
$$\frac{e^{-15/4}}{l_1!l_2!d!t!q!}\frac{(4n)!}{2^{2n}(2n)!(4!)^n}3^{l_1}\left(\frac{3}{2n}\right)^{l_2}\left(\frac{9}{2}\right)^{d}\left(\frac{9}{2n}\right)^{t}\left(\frac{9}{8n^2}\right)^{q}.$$

The formula (1.3) can give us not only the formula (1.2) but also more asymptotic information of general 4-regular graphs with given connectivities.

Inclusion and exclusion with two types of properties was used in [5] and [2] to find the asymptotic number of claw-free cubic graphs and that of general cubic graphs, respectively. Chae also derived an inequality of inclusion and exclusion on finitely many types of properties in [3] which is a generalization of inclusion and exclusion. It is summarized as follows(please refer to [3] for detailed explanation). Let U be the universal set of S_o elements. Let k be a positive integer and $k < s_k$. Suppose that $P_1^j,\ldots,P_{s_j}^j$ ($j=1,\ldots,k$) are subsets of U. The complement of a set C of U is denoted by $\overline{C}$. For any number k, $[k]$ denotes the set $\{1,2,\ldots,k\}$. For $0 \le l_j \le s_j$ ($j=1,\ldots,k$), define

$$S_{l_1,\ldots,l_k} = \sum\left|\bigcap_{i_1\in I_1} P_{i_1}^1 \cap\cdots\cap \bigcap_{i_k\in I_k} P_{i_k}^k\right|, \tag{1.4}$$

where the sum is over all l_j-subsets $I_j \subset [s_j]$ for $j=1,\ldots,k$. Now for $0\le l_j \le s_j$ ($j=1,\ldots,k$), let $N_{l_1,\ldots,l_k}$ be the number of elements in U that belongs to exactly l_j of the sets $\{P_i^j\}_{i=1}^{s_j}$ for $j=1,\ldots,k$. That is

$$N_{l_1,\ldots,l_k} = \sum\left|\bigcap_{i_1\in I_1} P_{i_1}^1 \cap \bigcap_{i_1\notin I_1} \overline{P_{i_1}^1} \cap\cdots\cap \bigcap_{i_k\in I_k} P_{i_k}^k \cap \bigcap_{i_k\notin I_k} \overline{P_{i_k}^k}\right|, \tag{1.5}$$

where the sum is again over all l_j-subsets $I_j \subset [s_j]$ for $j=1,\ldots,k$. The number $N_{l_1,\ldots,l_k}$ stands for the number of configurations what we need for counting general 4-regular graphs. In the following theorem [3], the bounds for $N_{l_1,\ldots,l_k}$ are obtained and they will be used to estimate the asymptotic number of general 4-regular graphs.

THEOREM 1. *There are values* $\{\overline{\alpha_i}\}$ *that gives us a lower bound of* $N_{l_1,\ldots,l_k}$.

$$\begin{aligned}
&\sum_{\substack{0\le v_1\le\overline{\alpha_1}\\ \vdots \\ 0\le v_k\le\overline{\alpha_k}}} (-1)^{v_1+\cdots+v_k}\prod_{i=1}^{k}\binom{l_i+v_i}{v_i}S_{l_1+v_1,\ldots,l_k+v_k}\\
(1.6)\qquad &\le\; N_{l_1,\ldots,l_k}\\
&\le \sum_{\substack{0\le v_1\le 2\alpha_1\\ \vdots \\ 0\le v_k\le 2\alpha_k}} (-1)^{v_1+\cdots+v_k}\prod_{i=1}^{k}\binom{l_i+v_i}{v_i}S_{l_1+v_1,\ldots,l_k+v_k},
\end{aligned}$$

where

$$\overline{\alpha_i} > \frac{(k-1)(s_i-l_i-1)-1}{k}.$$

By adapting configurations, $N_{l_1,\ldots,l_5}$ which is the number of configurations is calculated in section 2. In section 3, asymptotic number of 4-regular graphs is obtained that is not necessarily connected by using the number $N_{l_1,\ldots,l_5}$. And then in section 4, we have the asymptotic numbers of 4-regular graphs with given connectivities. For general graph theoretic terminology and notation we follow [4] and we assume the basic terminology developed in [6] for inclusion and exclusion.

2. Configurations

In this section, we extend an idea of Bollabás [1] for representing general 4-regular graphs. Let $V=\bigcup_{1\le i\le n}V_i$ be a partition of V into 4-subsets V_i for $i=1,\ldots,n$. A configuration is a perfect matching on this set of vertices. Therefore it is easy to see that the total number of configurations is

$$\frac{(4n)!}{2^{2n}(2n)!}. \tag{2.1}$$

Among a 1-factor, if a pair of vertices is matched entirely in a 4-subset V_i for some i, then it is called a loop(see Figure 1). But there might be two such loops in one 4-subset V_i for some i, because there are 4 vertices in a V_i for all i. In that case, it is called a double loop. If two vertices in a matched pair belong to different 4-subsets, we have a single edge. If there are two such pairs of vertices, *i.e.,* two matched pairs have two

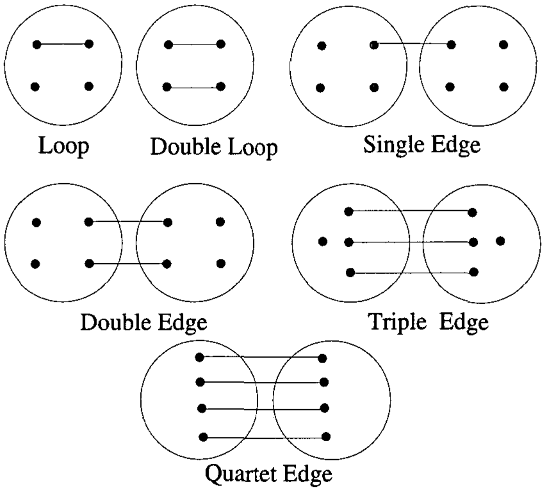

FIGURE 1

end-vertices in a 4-subset V_i for some i, and two other end-vertices in another 4-subset V_j for some j at the same time, it is called a double edge. If there are three of such pairs between two 4-subsets V_i and V_j for some i and j, it is called a triple edge. Similarly, if there are four of such pairs, it is called a quartet. If U in inclusion and exclusion with five types of properties is the universal set of $\frac{(4n)!}{2^{2n}(2n)!}$ elements, the number $N_{l_1,l_2,d,t,q}$ is the number of configurations with exactly l_1 loops, l_2 double loops, d double edges, t triple edges, and q quartet edges.

For $i = 1, \dots, n$, let A_i and B_i be the set of configurations which have a loop in V_i and a double loop in V_i, respectively. Assume $\binom{n}{2}$ pairs of $V_i's$ in the partition are ordered from 1 to $\binom{n}{2}$. Let D_j, T_j and Q_j be the set of configurations which have a double edge, a triple edge and a quartet edge in the j^{th} pair for $j = 1, \dots, \binom{n}{2}$, respectively. The corresponding notations in equation (1.4) of inclusion and exclusion to these notations are stated as follows:

$$A_i = P_i^1, \quad B_i = P_i^2, \quad D_j = P_j^3, \quad T_j = P_j^4 \quad \text{and} \quad Q_j = P_j^5,$$

where $1 \le i \le n = s_1 = s_2$ and $1 \le j \le \binom{n}{2} = s_3 = s_4 = s_5$. $S_{l_1,l_2,d,t,q}$ is needed for finding the bounds of $N_{l_1,l_2,d,t,q}$:

$$S_{l_1,l_2,d,t,q} = \binom{n}{l_1, l_2, 2d, 2t, 2q, n - l_1 - l_2 - 2d - 2t - 2q} \tag{2.2}$$

$$\binom{4}{2}^{l_1}\left(\binom{4}{2}/2\right)^{l_2}\left(\binom{4}{2}^2 2\right)^d\frac{(2d)!}{2^d d!}\left(\binom{4}{3}^2 6\right)^t\frac{(2t)!}{2^t t!}(4!)^q\frac{(2q)!}{2^q q!}$$
$$\left[\frac{[2(2n-l_1-2l_2-2d-3t-4q)]!}{2^{2n-l_1-2l_2-2d-3t-4q}(2n-l_1-2l_2-2d-3t-4q)!}\right].$$

Here we need a justification of this formula. Firstly we choose l_1, l_2, $2d$, $2t$, $2q$ and $n-l_1-l_2-2d-2t-2q$ labels from the n available. The number of ways to do this is

$$\binom{n}{l_1,l_2,2d,2t,2q,n-l_1-l_2-2d-2t-2q}.$$

It can be seen that the number of ways to form the adjacencies in a 4-subset V_i for each loop, double loop and in a pair of 4-subsets V_j, V_k for each double edge, triple and quartet for some i,j and k, is

$$\binom{4}{2}^{l_1}\left(\binom{4}{2}/2\right)^{l_2}\left(\binom{4}{2}^2 2\right)^d\frac{(2d)!}{2^d d!}\left(\binom{4}{3}^2 6\right)^t\frac{(2t)!}{2^t t!}(4!)^q\frac{(2q)!}{2^q q!}.$$

The last quantity in the formula represent the number of ways for a 1-factor of

$$2(2n-l_1-2l_2-2d-3t-4q)$$

vertices. On substituting equation (2.2) into (1.6) and simplifying we have

(2.3)

$$\begin{aligned}N_{l_1,l_2,d,t,q}=&\frac{1}{l_1!l_2!d!t!q!}\frac{(4n)!}{2^{2n}(2n)!}\sum(-1)^{v_1+\cdots+v_5}\\&\frac{1}{v_1!v_2!v_3!v_4!v_5!}\frac{n!}{(n-k_1)!}\frac{(4n-2k_2)!}{(4n)!}\frac{(2n)!}{(2n-k_2)!}(32^2)^{l_1+v_1}\\&(32^2)^{l_2+v_2}(3^22^4)^{d+v_3}(32^7)^{t+v_4}(32^6)^{q+v_5},\end{aligned}$$

where $k_1=l_1+l_2+2d+2t+2q+v_1+v_2+2v_3+2v_4+2v_5$, and $k_2=l_1+2l_2+2d+3t+4q+v_1+2v_2+2v_3+3v_4+4v_5$.

Therefore we have:

THEOREM 2. *For l_1, l_2, d, t and $q=o(\sqrt{n})$, respectively,*

$$\begin{aligned}N_{l_1,l_2,d,t,q}=&(1+o(1))\frac{e^{-15/4}}{l_1!l_2!d!t!q!}\frac{(4n)!}{2^{2n}(2n)!}\\&\left(\frac{3}{2}\right)^{l_1}\left(\frac{3}{16n}\right)^{l_2}\left(\frac{9}{4}\right)^d\left(\frac{3}{4n}\right)^t\left(\frac{3}{64n^2}\right)^q.\end{aligned}\qquad(2.4)$$

Proof. The proof of this theorem can be obtained from the direct simplification of equation (2.3) by using $(n)_k/n^k = 1 + o(1)$. □

COROLLARY 1. *For any* l_1, l_2, d, t *and* q,

$$(2.5)\quad \begin{aligned} N_{l_1,l_2,d,t,q} =& O(1)\frac{1}{l_1!l_2!d!t!q!}\frac{(4n)!}{2^{2n}\cdot(2n)!} \\ & \left(\frac{3}{2}\right)^{l_1}\left(\frac{3}{16n}\right)^{l_2}\left(\frac{9}{4}\right)^{d}\left(\frac{3}{4n}\right)^{t}\left(\frac{3}{64n^2}\right)^{q}. \end{aligned}$$

Proof. By Stirling's formula, $n! = (1+o(1))\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, we get $\frac{n!}{(n-k)!} = \left(\frac{n}{n-k}\right)^{n-k+1/2}\frac{n^k}{e^k}$. Therefore the three fractions containing factorial in the equation (2.3) can be simplified to

$$(2.6)\quad \frac{\left(\frac{n}{n-k_1}\right)^{n-k_1+1/2}\frac{1}{e^{k_1}-e^{k_2}}\left(\frac{2n}{2n-k_2}\right)^{2n-k_2+1/2}}{\left(\frac{4n}{4n-2k_2}\right)^{4n-2k_2+1/2}}\frac{n^{k_1}2n^{k_2}}{4n^{2k_2}}.$$

It is enough to show that

$$(2.7)\quad \frac{\left(\frac{n}{n-k_1}\right)^{n-k_1+1/2}\frac{1}{e^{k_1}-e^{k_2}}\left(\frac{2n}{2n-k_2}\right)^{2n-k_2+1/2}}{\left(\frac{4n}{4n-2k_2}\right)^{4n-2k_2+1/2}} = O(1),$$

since the term $\frac{n^{k_1}2n^{k_2}}{4n^{2k_2}}$ will be calculated in the sum of the equation (2.3) as follows:

$$(2.8)\quad \begin{aligned} &\sum(-1)^{v_1+\cdots+v_5}\frac{1}{v_1!v_2!v_3!v_4!v_5!}(32^2)^{l_1+v_1}(32^2)^{l_2+v_2} \\ &\quad (3^2 2^4)^{d+v_3}(32^7)^{t+v_4}(32^6)^{q+v_5}\frac{n^{k_1}2n^{k_2}}{4n^{2k_2}} \\ &= O(1)e^{-\frac{3}{16n}}e^{-\frac{3}{4n}}e^{-\frac{3}{64n^2}} \\ &\quad \left(\frac{3}{2}\right)^{l_1}\left(\frac{3}{16n}\right)^{l_2}\left(\frac{9}{4}\right)^{d}\left(\frac{3}{4n}\right)^{t}\left(\frac{3}{64n^2}\right)^{q} \\ &= O(1)\left(\frac{3}{2}\right)^{l_1}\left(\frac{3}{16n}\right)^{l_2}\left(\frac{9}{4}\right)^{d}\left(\frac{3}{4n}\right)^{t}\left(\frac{3}{64n^2}\right)^{q}. \end{aligned}$$

Since

$$
\begin{aligned}
0 &< \frac{\left(\frac{n}{n-k_1}\right)^{n-k_1+1/2} \frac{1}{e^{k_1}-e^{k_2}} \left(\frac{2n}{2n-k_2}\right)^{2n-k_2+1/2}}{\left(\frac{4n}{4n-2k_2}\right)^{4n-2k_2+1/2}} \\
&= \frac{e^{k_2-k_1}}{4^{k_2}} \frac{(2n-k_2)^{2n-k_2}}{(2n-2k_1)^{n-k_1+1/2}(2n)^{n+k_1-k_2-1/2}} \qquad (2.9)\\
&\leq \frac{e^{k_2-k_1}}{4^{k_2}} \frac{(2n)^{n-k_1+1/2}}{(2n-2k_1)^{n-k_1+1/2}} \\
&= \frac{e^{k_2-k_1}}{4^{k_2}} \frac{1}{e^{-k_1+k_1^2/n-k_1/2n}} < 1,
\end{aligned}
$$

the formula (2.7) is obtained. So we are done. □

3. Asymptotic number of general 4-regular graphs

We have the following relationship between $g(n, l_1, l_2, d, t, q)$ and $N(l_1, l_2, d, t, q)$ by shrinking the 4-vertex sets V_i of configurations to single vertices for graphs. Here we need an explanation for this formula. In a configuration, the number of ways to form the adjacencies in a 4-subset V_i for each loop, double loop and in a pair of 4-subsets V_j, V_k for each double edge, triple and quartet for some i, j and k, is

$$(2\cdot 6)^{l_1} 3^{l_2} (6^2\cdot 2\cdot 4)^{l_3} (4^2\cdot 6)^{l_4} (4!)^{l_5} (4!)^{n-l_1-l_2-2l_3-2l_4-2l_5} 4^{l_3}.$$

In order to get $g(n, l_1, l_2, d, t, q)$, we need to divide $N(l_1, l_2, d, t, q)$ by this quantity.

PROPOSITION 1.

$$
\begin{aligned}
N(l_1, l_2, d, t, q) = {} & g(n, l_1, l_2, d, t, q)(2\cdot 6)^{l_1} 3^{l_2} \\
& (6^2\cdot 2\cdot 4)^{l_3} (4^2 6)^{l_4} (4!)^{l_5} (4!)^{n-l_1-l_2-2l_3-2l_4-2l_5} 4^{l_3}.
\end{aligned} \qquad (3.1)
$$

By substituting equations (2.4) and (2.5) in (3.1), we have the following corollaries.

COROLLARY 2. *For l_1, l_2, d, t and $q = o(\sqrt{n})$, respectively,*

$$
\begin{aligned}
g(n, l_1, l_2, d, t, q) = {} & (1+o(1)) \frac{e^{-15/4}}{l_1! l_2! d! t! q!} \frac{(4n)!}{2^{2n}(2n)!(4!)^n} \\
& 3^{l_1} \left(\frac{3}{2n}\right)^{l_2} \left(\frac{9}{2}\right)^{d} \left(\frac{9}{2n}\right)^{t} \left(\frac{9}{8n^2}\right)^{q}.
\end{aligned} \qquad (3.2)
$$

It can be shown that for large l_1, l_2, d, t and q:

COROLLARY 3.

$$g(n,l_1,l_2,d,t,q) = O(1)\frac{1}{l_1!l_2!d!t!q!}\frac{(4n)!}{2^{2n}(2n)!(4!)^n} 3^{l_1}\left(\frac{3}{2n}\right)^{l_2}\left(\frac{9}{2}\right)^{d}\left(\frac{9}{2n}\right)^{t}\left(\frac{9}{8n^2}\right)^{q}. \tag{3.3}$$

From the equations (3.2) and (3.3), if we consider the contributions of the double loops, triple edges and quartet edges to the total number of general 4-regular graphs, it is less than one from the time when n is just greater than 5 and the total number of them tends to zero when n tends to ∞. So we can say double loops, triple edges and quartet edges are negligible.

COROLLARY 4. *Double loops, triple edges and quartet edges are negligible in* $g(n,l_1,l_2,d,t,q)$.

If we sum up the values of $g(n,l_1,l_2,d,t,q)$ from zero to ∞ for each l_1,l_2,d,t and q using equation (3.2) and equation (3.3), we have the asymptotic number of general 4-regular graphs on n vertices which is the equation (1.2):

COROLLARY 5.

$$g(n) = (1+o(1))\frac{e^{15/4}(4n)!}{2^{2n}(2n)!(4!)^n}. \tag{3.4}$$

Proof. When l_1, l_2, d, t and $q = o(\sqrt{n})$, respectively, by using equation (3.2), it easily can be seen that

$$\sum_{l_1,l_2,d,t,q>o} g(n,l_1,l_2,d,t,q) = (1+o(1))\frac{e^{15/4}(4n)!}{2^{2n}(2n)!(4!)^n}. \tag{3.5}$$

When one of l_1,l_2,d,t and q is not equal to $o(\sqrt{n})$, we have thirty one subcases to consider. We omit the detailed calculations except one case. Explanations for other cases can be done similarly. Let us consider the following case: all $l_1 \geq 0$, $l_2 \geq 0, t \geq 0$, $q \geq 0$ and $d \geq \frac{\sqrt{n}}{w_n}$ for some w_n, where w_n go to infinity very slowly. Since

$$\sum_{d \geq \frac{\sqrt{n}}{w_n}} \frac{2^d}{d!} = o(1),$$

it can be shown that, from the equation (3.3),

$$\sum_{l_1\geq 0, l_2\geq 0, t\geq 0, q\geq 0, d\geq \frac{\sqrt{n}}{w_n}} g(n,l_1,l_2,d,t,q) = \mathrm{O}(1)\frac{(4n)!}{2^{2n}(2n)!(4!)^n}\mathrm{o}(1) = \mathrm{o}(1)\frac{(4n)!}{2^{2n}(2n)!(4!)^n}. \tag{3.6}$$

For other terms, we have the same results. Therefore we have

$$\begin{aligned} g(n) &= (1+\mathrm{o}(1))\frac{e^{15/4}(4n)!}{2^{2n}(2n)!(4!)^n} + \mathrm{o}(1)\frac{31(4n)!}{2^{2n}(2n)!(4!)^n} \\ &= (1+\mathrm{o}(1))\frac{e^{15/4}(4n)!}{2^{2n}(2n)!(4!)^n}. \end{aligned} \tag{3.7}$$

This is the asymptotic number of general 4-regular graphs which is not necessarily connected. Now we are going to investigate it more in detail to figure out the asymptotic behavior of $g(n)$ with given connectivities. □

4. Asymptotic numbers of general 4-regular graphs with given connectivities

Let $g_1(n)$ be the number of connected general 4-regular graphs of order n. It is well known that $g(n)$ and $g_1(n)$ are related by the following sum:

$$g(n) = \sum_{k=1}^{n} \binom{n}{k} \frac{k}{n} g_1(k) g(n-k) \tag{4.1}$$

or

$$g_1(n) = g(n) - \sum_{k=1}^{n-1} \binom{n}{k} \frac{k}{n} g_1(k) g(n-k), \tag{4.2}$$

where $g(0) = 1$. To show that almost all general 4-regular graphs are connected, *i.e.*, $g(n) \sim g_1(n)$, we need to show that

$$\sum_{k=1}^{n-1} \binom{n}{k} \frac{k}{n} \frac{g_1(k)g(n-k)}{g(n)} = \mathrm{o}(1), \tag{4.3}$$

which is the sum in the equation (4.2) divided by $g(n)$. Since $k \cdot g_1(k) < n \cdot g(k)$, it is enough to show that

$$\sum_{k=1}^{n-1} \binom{n}{k} \frac{g(k)g(n-k)}{g(n)} = o(1). \tag{4.4}$$

By using equation (3.4) and some simple estimates, we find the left side of equation (4.4) is

$$O(1) \sum_{k=1}^{n/2} \frac{\sqrt{n}}{\sqrt{k}\sqrt{n-k}} \left[\frac{k}{e(n-k)}\right]^k. \tag{4.5}$$

This sum can be estimated by splitting it into two parts according as $k \leq \log n$ or $k > \log n$. We find that for $1 \leq k \leq \log n$, the value of the sum is $O(n^{-2})$ and for $\log n < k \leq n/2$ it is $O(n^{-1}(\log n)^{-1/2})$. Therefore we have the following theorem:

THEOREM 3. *Almost all general 4-regular graphs are connected.*

For convenience, let

$$F(n) = \frac{(4n)!}{2^{2n}(2n)!(4!)^n}. \tag{4.6}$$

Then the equation (3.4) can be written

$$g(n) \sim F(n)e^{\frac{15}{4}}. \tag{4.7}$$

Let $gl(n)$ be the number of general 4-regular graphs with at least 1 loop. It follows from equation (3.2) with $l_1 = 0$ that the number of general 4-regular graphs with no loops is asymptotic to $F(n)e^{\frac{3}{4}}$. But from the equation (3.3), we have $O(1)F(n)$ instead of $F(n)e^{\frac{3}{4}}$. Hence the results we have from now on based on the assumption that l_1, l_2, d, t and $q = o(\sqrt{n})$, respectively. Therefore

$$g(n) \sim gl(n) + F(n)e^{\frac{3}{4}}. \tag{4.8}$$

Hence the number of general 4-regular graphs with at least 1 loop is expressed as follows:

PROPOSITION 2.

$$gl(n) \sim F(n)(e^{\frac{15}{4}} - e^{\frac{3}{4}}). \tag{4.9}$$

Let $gnlnd(n)$ be the number of general 4-regular graphs with no loops and no double edges. Then we have from equation (3.2) by substituting $l_1 = 0$, $l_2 = 0$ and $d = 0$,

$$gnlnd(n) \sim F(n)e^{-\frac{15}{4}}. \tag{4.10}$$

A general 4-regular graph with at least one loop has $\kappa(G) = 1$ or $\kappa(G) = 2$. Since a general 4-regular graph is either with at least one loop or loopless, if we show that almost all loopless general 4-regular graphs are 3-connected, then almost all general 4-regular graph with at least one loop has $\kappa(G) = 1$ or $\kappa(G) = 2$, asymptotically. Therefore it can be said that the asymptotic number of general 4-regular graphs with $\kappa(G) = 1$ or $\kappa(G) = 2$ is $F(n)(e^{\frac{15}{4}} - e^{\frac{3}{4}})$ by proposition 2. In any graph, there is a even number of odd vertices, we have the following fact:

LEMMA 1. *There is no (general) 4-regular graphs which has a bridge.*

Proof. If we have a bridge uv in a 4-regular graph G, $G - uv$ has a component with one vertex with degree 3 and others have with degree 4. But this is impossible.

Moreover, by the same reasoning to the lemma above, odd number of edges regardless of which they are part of single, double, triple or quartet edges cannot make a general 4-regular graph apart. It reduces the number of cases to consider when we prove the following theorem. In fact, there are 5 types of loopless connected general 4-regular graphs with $\kappa(G) = 1$ or $\kappa(G) = 2$ to consider in the following theorem (see Figure 2.) □

THEOREM 4. *Almost all general 4-regular graphs without loops are 3-connected.*

Proof. It is enough to show that all loopless general 4-regular graphs with connectivity 1 and 2 are negligible. There are 5 types of such graphs to consider as illustrated in Figure 2. In the figure, the graphs H_1 through H_5 (except the dotted edges and vertices on them) are disconnected general 4-regular graphs. Let G be a loopless general 4-regular graphs with connectivity 1 or 2. The graph G can be constructed from one of $H_i^{\prime}s$ in Figure 2 by adding the dotted edges and the vertices on them for some $i = 1, \ldots, 5$. Let H_1 is a disconnected general 4-regular graph with $n-1$ vertices without loops rooted at two edges which belong to different components and may be a part of a single, double, or triple edge. The graph H_1 will be converted to a connected general 4-regular graph with n vertices and no loops, rooted at new vertex by deleting two root edges, adding a vertex and joining it to four vertices which were the end vertices of two root edges deleted from H_1. Let H_2 is a disconnected general 4-regular graph with $n-2$ vertices without loops rooted at two edges which belong to different components and may be a part of a single, double, or triple edge. The graph H_2 will be converted to a connected general 4-regular graph with n vertices and no loops, rooted at a double edge by inserting a vertex on each root edges and joining them by a double edge. Let H_3 is a disconnected general 4-regular graph with $n-2$ vertices without loops rooted at four edges which belong to different components and may be a part of a single, double, or triple edge. The graph H_3 will be converted to a connected general 4-regular graph with n vertices and no loops, rooted at two vertices by deleting four root edges, adding two vertices and joining them to four vertices each which were the end vertices of four root edges deleted from H_3. Let H_4 is a disconnected graph with n vertices without loops rooted at two edges which belong to different components. The graph H_4 will be converted to a connected general 4-regular graph with n vertices and no loops, rooted at two single edges by deleting two root edges in H_4 and joining two vertices which were incident to root edges in different components in order to make a connected graph rooted at the two single edges. And let H_5 is a disconnected graph with $n-4$ vertices without loops rooted at four edges(two or three edges also can be chosen for roots but we examine only the case of four root edges because it is the hardest one to estimate. But we give a diagram with two root edges in the Figure 2). The graph H_5 will be converted to a connected general 4-regular graph with n vertices and no loops, rooted at two double edges by inserting four vertices on root edges in H_5 and joining each two new vertices in different components by a double edge in order to make a connected graph rooted at the two double edges.

The number of G obtained from H_1 has an upper bound

$$UH_1 = \binom{2n-2}{2}\left(g(n-1) - g_1(n-1)\right)n, \tag{4.11}$$

since $g(n-1) - g_1(n-1)$ is the number of disconnected general 4-regular graphs. We need $\binom{2n-2}{2}$ to choose 2 edges for roots of H_1 and n for labelling a new inserted vertex. Similarly, the number of G obtained from H_2, H_3, H_4, and H_5 have upper bounds

$$UH_2 = \binom{2n-4}{2}\left(g(n-2) - g_1(n-2)\right)n(n-1), \tag{4.12}$$

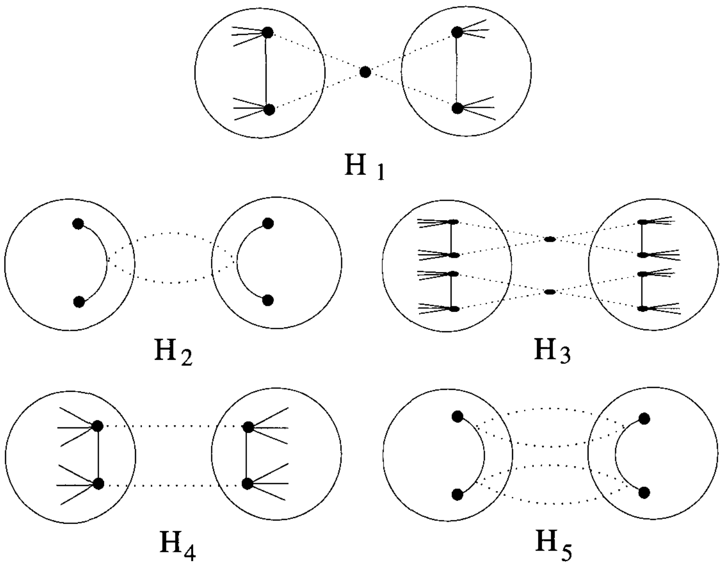

FIGURE 2. Graphs H_1, H_2, H_3, H_4, and H_5

$$UH_3 = 2\binom{2n-4}{4}\left(g(n-2) - g_1(n-2)\right)n(n-1), \tag{4.13}$$

$$UH_4 = 2\binom{2n}{2}\left(g(n) - g_1(n)\right) \tag{4.14}$$

and

$$UH_5 = 2\binom{2n-8}{4}\left(g(n-4) - g_1(n-4)\right)n(n-1)(n-2)(n-3), \tag{4.15}$$

respectively. We want to show that the sum of numbers in equations (4.11), (4.12), (4.13), (4.14) and (4.15) tend to 0, when it is divided by $g(n)$, the total number of general 4-regular graphs with n vertices. We process this work by dividing each of these equations by $g(n)$. With the help of the Stirling's formula, tedious calculations give us what we want as follows:

$$\frac{UH_1}{g(n)} + \frac{UH_2}{g(n)} + \frac{UH_3}{g(n)} + \frac{UH_4}{g(n)} + \frac{UH_5}{g(n)} \longrightarrow 0. \tag{4.16}$$

So the proof is completed. Here we present a proof for $\frac{UH_1}{g(n)} \to 0$. One can do other cases similarly. This proof is also similar to that of Theorem 3.

It can be seen that

$$(4.17)\quad \begin{aligned} \frac{UH_1}{g(n)} &= \frac{\binom{2n-2}{2}\left(g(n-1)-g_1(n-1)\right)n}{g(n)} \\ &= O(n^3)\sum_{k=3}^{(n-1)/2}\binom{n-1}{k}\frac{g(k)g(n-k-1)}{g(n)}. \end{aligned}$$

By the equation (1.2) and Stirling's formula, it is obtained

$$(4.18)\quad \begin{aligned} \frac{UH_1}{g(n)} = O(n^3)\sum_{k=3}^{(n-1)/2}&\frac{1}{\sqrt{k(n-k-1)(n-1)}} \\ &\frac{1}{e(n-k-1)}\left[\frac{k}{e(n-k-1)}\right]^k. \end{aligned}$$

This sum can be estimated by splitting it into two parts according as $3 \le k \le \log n$ or $\log n < k \le (n-1)/2$. We find that for $3 \le k \le \log n$, the value of the sum is $O(\frac{(\log n)^3}{n^2})$ and for $\log n < k \le (n-1)/2$, it is $O((\log n)^{-1/2})$. So we are done. □

Again note that the connectivity of a general 4-regular graph with at least one loop is 1 or 2. Here we claim that almost all general 4-regular graphs with at least one loop has $\kappa(G) = 2$. Let A be the number of general 4-regular graphs with at least one loop and $\kappa(G) = 1$. In order to prove this claim, we need to calculate the following ratio:

$$(4.19)\quad \frac{A}{g(n)} = \frac{A}{F(n)e^{\frac{15}{4}}} \longrightarrow 0.$$

To find the upper bound of A, we need to consider two cases(which are very similar to the first and second cases in the proof of Theorem 4).

Let L_1 is a disconnected general 4-regular graph with $n-1$ vertices with loops rooted at two edges which belong to different components and may be a part of a loop, single, double, or triple edge. The graph L_1 will be converted to a connected general 4-regular graph with n vertices rooted at new vertex by deleting two root edges, adding a vertex and joining it to four vertices which were the end vertices of two root edges deleted from L_1. Let L_2 is a disconnected general 4-regular graph with $n-2$ vertices with loops rooted at two edges which belong to different components and may be a part of a loop, single, double, or triple edge. The graph L_2 will be converted to a connected general 4-regular graph with n vertices rooted at a double edge by inserting a vertex on each root edges and joining them by a double edge. Therefore, A also has

<table>
<tr><td colspan="2">4-regular graphs with connectivity 1 are negligible</td></tr>
<tr><td colspan="2">Almost all 4-regular graphs with at least one loop has connectivity 2
$F(n)(e^{15/4}- e^{3/4}) \sim 95.0213\%$ k(G) = 2</td></tr>
<tr><td>No loops and no doubls
$F(n)e^{-15/4} \sim 0.0553\%$
k(G) = 4</td><td>No loops but with at least one double
$F(n)(e^{3/4} - e^{-15/4}) \sim 4.9233\%$
k(G) = 3</td></tr>
</table>

FIGURE 3. Summarization

the same upper bound $UH_1 + UH_2$, which are the upper bound of the number of general 4-regular graphs constructed from type H_1 and H_2 in the Theorem. The ratio is

$$\frac{UH_1 + UH_2}{F(n)e^{15/4}} \tag{4.20}$$

and it is already shown that it converges to zero as in the proof of Theorem 4. That means the number of general 4-regular graphs with $\kappa(G) = 1$ which have at least one loop is negligible. So we have the following corollaries:

COROLLARY 6. *Almost all general 4-regular graphs are 2-connected.*

COROLLARY 7. *Almost all general 4-regular graphs with at least one loop has connectivity 2. So the number of general 4-regular graphs with $\kappa(G) = 2$ is*

$$F(n)(e^{15/4} - e^{3/4}).$$

COROLLARY 8. *Asymptotic number of 3-connected general 4-regular graphs is*

$$F(n)e^{3/4}.$$

Now from Theorem 4, almost all general 4-regular graphs with no loops are 3-connected. If a general 4-regular graphs has a double edge, then it can not have connectivity 4. So in a 4-connected general 4-regular graph, there are no loops and no double edges. Note that a general 4-regular graphs with no loops, no doubles, no triples is just a 4-regular graph with $\kappa(G) = 4$, whose number was found by Wormald [8]. So we have the following corollaries. From the equation (4.10), we have

COROLLARY 9. *Asymptotic number of general 4-regular graphs with* $\kappa(G) = 4$ *is*

$$F(n)e^{-15/4},$$

which is approximately 0.0553% *of general 4-regular graphs. And that is the percentage of 4-regular graph out of general 4-regular graphs.*

By the corollaries above 8 and 9, we have the following:

COROLLARY 10. *Asymptotic number of general 4-regular graphs with* $\kappa(G) = 3$ *is*

$$F(n)(e^{3/4} - e^{-15/4}),$$

which is approximately 4.9233% *of general 4-regular graphs.*

All the results are summarized in Figure 3.

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