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ON CONSTRUCTING REPRESENTATIONS OF THE SYMMETRIC GROUPS

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ABSTRACT. Let G be a symmetric group. In this paper we describe a method that for a certain irreducible character χ of G it finds a subgroup H such that the restriction of χ on H has a linear constituent with multiplicity one. Then using a well known algorithm we can construct a representation of G affording χ.

We recall some facts about characters of the symmetric group S_n. For a general reference see, for example, [3]. A *partition* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ of a natural number n is a weakly decreasing sequence $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_l > 0$ of integers with $|\lambda| = \sum_{i=1}^{l} \lambda_i = n$, for short we write $\lambda \vdash n$. The numbers λ_i are the parts of λ. We also write the partition exponentially as $\lambda = (l_1^{a_1}, \dots, l_m^{a_m})$, $l_1 > \cdots > l_m > 0$, when we have a_i parts of size l_i.

Since the number of irreducible characters of a group is equal to the number of conjugacy classes, which in the case of S_n is the number of partitions of n, the irreducible characters of S_n are labelled by partitions of n. We denote the irreducible character labelled by the partition λ by $[\lambda]$, so $\mathrm{Irr}(S_n) = \{[\lambda] \mid \lambda \vdash n\}$.

If G is a finite group and χ is an irreducible character of G, an efficient and simple method to construct representations of finite groups has been presented in [2]. This method is applicable whenever G has a subgroup H such that χ_H has a linear constituent with multiplicity 1. We call such a subgroup H a χ-subgroup. In practice this algorithm is quite fast when H has a small order, but can be very slow for a large H. For using this method to construct representations of G, we need to find a χ-subgroup for each irreducible character χ of G.

If G is a simple group or a covering group of a simple group, then a χ-subgroup for each nontrivial irreducible character χ of G of degree < 32 has been found in [1]. Also if λ is a partition of n and λ' is the conjugate

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partition of λ, it is a well known result that the Young subgroups S_λ and $S_{\lambda'}$ are $[\lambda]$-subgroups (see [3, Theorem 2.1.3]). In this article we describe a method to find a $[\lambda]$-subgroup for some certain irreducible characters $[\lambda]$ of S_n which is smaller than Young subgroups S_λ and $S_{\lambda'}$.

A *Young diagram* $Y(\lambda)$ corresponding to λ, is a collection of squares $(i,j) \in \mathbb{Z}^2$, such that $1 \leqslant j \leqslant \lambda_i$. Each $(i,j) \in Y(\lambda)$ is called a *node* of λ. A node (i, λ_i) is called *removable* (for λ) if $\lambda_i > \lambda_{i+1}$. A node $(i, \lambda_i + 1)$ is called *addable* if $i = 1$ or $i > 1$ and $\lambda_i < \lambda_{i-1}$. We denote by

$$\lambda_A = \lambda \backslash \{A\} = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots, \lambda_l)$$

a partition of $n-1$ obtained by removing a removable node $A = (i, \lambda_i)$ from λ. Also

$$\lambda^B = \lambda \cup \{B\} = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \ldots, \lambda_l)$$

is a partition of $n+1$ obtained by adding an addable node $B = (i, \lambda_{i+1})$ to λ.

The following theorem shows the behavior of irreducible characters of S_n when we restrict them to S_{n-1}. For proof see [3, p.59].

THEOREM 1 (Branching Theorem). *If $\lambda \vdash n$, then*

$$[\lambda]_{S_{n-1}} = \sum_{\text{removable nodes } A} [\lambda \backslash \{A\}]$$

This theorem shows the constituents of $[\lambda]_{S_{n-1}}$ and using that recursively we can find all constituents of $[\lambda]_{S_{n-r}}$ for $r \geqslant 1$. The following theorem describes these constituents directly.

THEOREM 2. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$ and $r \geqslant 1$. Then $[\mu] = [\mu_1, \mu_2, \ldots, \mu_s]$ is a constituent of $[\lambda]_{S_{n-r}}$ if and only if μ is a partition of $n-r$ and $\mu_i \leqslant \lambda_i$ for $i = 1, \ldots, s$.*

Proof. If $[\mu] = [\mu_1, \mu_2, \ldots, \mu_s]$ is a constituent of $[\lambda]_{S_{n-r}}$, then μ is a partition of $n-r$. The Branching Theorem implies $\mu_i \leqslant \lambda_i$ for $i = 1, 2, \ldots, s$.

Suppose $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$ is a partition of $n-r$ and $\mu_i \leqslant \lambda_i$ for $i = 1, 2, \ldots, s$, we show $[\mu]$ is a constituent of $[\lambda]_{S_{n-r}}$. We prove it by induction on r.

Let $r = 1$. Then μ is a partition of $n-1$ and $\mu_i \leqslant \lambda_i$ implies that $\mu_i = \lambda_i - 1$ for some i and $\mu_j = \lambda_j$ for $j \neq i$. Thus according to the Branching Theorem $[\mu]$ is a constituent of $[\lambda]_{S_{n-1}}$.

Suppose the theorem is true for $r \geqslant 1$, and $[\mu] = [\mu_1, \mu_2, \ldots, \mu_s]$ is a constituent of $[\lambda]_{S_{n-r}}$ for all partitions μ of $n-r$ such that $\mu_i \leqslant \lambda_i$

for $i = 1, 2, \ldots, s$. Let $\tau = [\tau_1, \tau_2, \ldots, \tau_p]$ be a partition of $n - (r + 1)$ and $\tau_i \leqslant \lambda_i$ for $i = 1, 2, \ldots, p$. Since $n - (r + 1) < n$, if $\tau_{i-1} = \tau_i$ for all $i = 2, \ldots, p$, then $\tau_1 + 1 \leqslant \lambda_1$ and $\mu = (\tau_1 + 1, \tau_2, \ldots, \tau_p)$ is a partition of $n - r$ which its parts are less than or equal corresponding parts of λ. Otherwise there exists i such that $\tau_{i-1} > \tau_i$ and $\tau_i + 1 \leqslant \lambda_i$. If we define $\mu = (\tau_1, \ldots, \tau_i + 1, \ldots, \tau_p)$ then μ is a partition of $n - r$ and its parts are less than or equal corresponding parts of λ. Hence using the assumption we have $[\mu]$ is a constituent of $[\lambda]_{S_{n-r}}$. Using the Branching Theorem $[\tau]$ is a constituent of $[\mu]_{S_{n-(r+1)}}$ so it is a constituent of $[\lambda]_{S_{n-(r+1)}}$. This proves the theorem. □

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ then we denote $\hat{\lambda} = (\lambda_2, \ldots, \lambda_l)$. We say the partition λ of n has *level* k if $\hat{\lambda} \vdash k$. Similarly we say the corresponding irreducible character $[\lambda]$ of S_n has level k. The principal character $\mathbf{1} = [n]$ is a character of level 0.

A *Young graph* $\mathcal{Y}$ corresponding to λ, is a graph which the vertices are Young diagrams. Two diagrams λ and μ, are connected by an edge going from μ to λ, if $|\lambda| = |\mu| + 1$ and λ can be obtained by adding an addable node to μ.

THEOREM 3. *Suppose* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$ *of level* k *and* $r \geqslant 1$. *If* $[\mu]$ *is a constituent of* $[\lambda]_{S_{n-r}}$ *of level* t *and* $n - r - t \geqslant \lambda_2$, *then*

$$\langle [\lambda]_{S_{n-r}}, [\mu] \rangle = \binom{r}{k-t} \langle [\hat{\lambda}]_{S_t}, [\hat{\mu}] \rangle.$$

Proof. Using the Branching Theorem recursively, the multiplicity of $[\mu]$ in $[\lambda]_{S_{n-r}}$ equals to the number of possibilities to fill with nodes the rows of the Young diagram of μ to reach to the Young diagram of λ in the corresponding Young graph. Therefore to add nodes to go from $\mu = (n - r - t, \mu_2, \ldots, \mu_s)$ to λ, we have $\langle [\hat{\lambda}]_{S_t}, [\hat{\mu}] \rangle$ possibilities to fill rows 2 to l with $k - t$ nodes (independent of what happens in the first row, by assumption that $n - r - t \geqslant \lambda_2$). In between these $k - t$ additions in rows 2 to l, we add the $r + t - k$ necessary nodes in the first row. Therefore the number of these possibilities to fill μ to reach λ is

$$\binom{(k-t) + (r+t-k)}{r+t-k} \langle [\hat{\lambda}]_{S_t}, [\hat{\mu}] \rangle = \binom{r}{k-t} \langle [\hat{\lambda}]_{S_t}, [\hat{\mu}] \rangle. \qquad \square$$

COROLLARY 4. *Let* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$ *of level* k.

(1) *If* $\lambda_1 - r \geqslant \lambda_2$ *and* $[\mu] = [\lambda_1 - r, \lambda_2, \ldots, \lambda_l]$ *then* $\langle [\lambda]_{S_{n-r}}, [\mu] \rangle = 1$ *and* $[\mu]$ *is the only constituent of* $[\lambda]_{S_{n-r}}$ *of level* k.

(2) *If* $\lambda_1 - k \geqslant \lambda_2$ *then* $\langle [\lambda]_{S_{\lambda_2+k-1}}, [\mu] \rangle > 1$ *for all constituents* $[\mu]$ *of* $[\lambda]_{S_{\lambda_2+k-1}}$.

Proof. Since $[\mu]$ has level k so $[\hat{\lambda}]_{S_k} = [\hat{\mu}]$ and $\lambda_1 - r \geqslant \lambda_2$ implies $n - r - k \geqslant \lambda_2$. Using the Theorem 3 we obtain $\langle [\lambda]_{S_{n-r}}, [\mu] \rangle = 1$. By the Theorem 2, $[\mu]$ is the only constituent of $[\lambda]_{S_{n-r}}$ of level k and all the other constituents have level less than k.

If we consider $n - r = \lambda_2 + k - 1$ then $\lambda_1 - k \geqslant \lambda_2$ implies $r > k$. We show $[\lambda]_{S_{\lambda_2+k-1}}$ has no constituent of level k. Indeed if $[\mu]$ is a constituent of level k then $\mu_i = \lambda_i$ for $i \geqslant 2$ and $\mu_1 \geqslant \mu_2$ results $|\mu| \geqslant \lambda_2 + k$ which is a contradiction with $\mu \vdash \lambda_2 + k - 1$. Since $r > k$ and $[\lambda]_{S_{\lambda_2+k-1}}$ has no constituent of level k, the Theorem 3 proves (2). $\square$

If λ is a partition of n of level k then it is easy to see that $S_{\lambda'} \leqslant S_{\lambda_2+k}$. In addition if $\lambda_1 - k \geqslant \lambda_2$ then $S_{\lambda'}$ is smaller than S_λ and using the Corollary 4, S_{λ_2+k-1} does not contain any $[\lambda]$-subgroup.

Suppose λ is a partition of n of level k such that $\lambda_1 - k \geqslant \lambda_2$. We describe a method to find a $[\lambda]$-subgroup smaller than the Young subgroup $S_{\lambda'}$.

According to the Corollary 4, $[\lambda]_{S_{\lambda_2+k}}$ has a constituent of level k with multiplicity one. Suppose ρ^0 denotes this irreducible character of S_{λ_2+k}, which is corresponding to the partition $(\lambda_2, \lambda_2, \ldots, \lambda_l)$. Then $[\lambda]_{S_{\lambda_2+k}} = \rho^0 + \sum m_i \psi_i$ where ψ_i's are irreducible characters of S_{λ_2+k} of levels less than k. Now we search among the subgroups of S_{λ_2+k} which are smaller than $S_{\lambda'}$ and are not subgroups of S_{λ_2+k-1}. If H is a subgroup of S_{λ_2+k} and φ is a linear character of H such that $\langle \rho^0_H, \varphi \rangle = 1$ and $\langle (\psi_i)_H, \varphi \rangle = 0$ for all i, then $\langle [\lambda]_H, \varphi \rangle = 1$ which it means H is a $[\lambda]$-subgroup.

If such a subgroup does not exist in S_{λ_2+k}, then the procedure can be restarted with ρ^1 in place of ρ^0, where ρ^1 is the constituent of level k with multiplicity one of $[\lambda]_{S_{\lambda_2+1+k}}$, corresponding to the partition $(\lambda_2 + 1, \lambda_2, \ldots, \lambda_l)$. That is, searching among the subgroups of S_{λ_2+1+k} which are smaller than $S_{\lambda'}$ and are not subgroups of S_{λ_2+k}. If H is a subgroup of S_{λ_2+1+k} and φ is a linear character of H such that $\langle \rho^1_H, \varphi \rangle = 1$ and φ is not a constituent of the restriction of the other constituents of $[\lambda]_{S_{\lambda_2+1+k}}$ on H, then $\langle [\lambda]_H, \varphi \rangle = 1$ which it implies H is a $[\lambda]$-subgroup. If such a subgroup does not exist in S_{λ_2+1+k} the procedure continues.

Practically in most of the cases S_{λ_2+k} contains a $[\lambda]$-subgroup smaller than $S_{\lambda'}$ and this method stops after the first pass.

For example if $[\lambda] = [8, 2, 1^4]$ is an irreducible character of S_{14}, then $|S_{\lambda'}| = 1440$. Using this method S_8 contains a $[\lambda]$-subgroup H of order 240.

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