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HYERS-ULAM STABILITY OF A CLOSED OPERATOR IN A HILBERT SPACE

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Dedicated to Professor Sin-Ei Takahasi on his 60th birthday (Kanreki)

ABSTRACT. We give some necessary and sufficient conditions in order that a closed operator in a Hilbert space into another have the Hyers-Ulam stability. Moreover, we prove the existence of the stability constant for a closed operator. We also determine the stability constant in terms of the lower bound.

1. Introduction

It seems that S. M. Ulam [16, Chapter VI] first raised the stability problem of functional equations: "For what metric groups G is it true that an ε-automorphism of G is necessarily near to a strict automorphism?" An answer has been given in the following way. Let E_1, E_2 be two real Banach spaces and $f\colon E_1 \to E_2$ be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$, the set of all real numbers, for each fixed $x \in E_1$. In 1941, D. H. Hyers [3] gave an answer to the problem above as follows. If there exists an $\varepsilon \geq 0$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in E_1$, then there exists a unique linear mapping $T\colon E_1 \to E_2$ such that $\|f(x) - T(x)\| \leq \varepsilon$ for every $x \in E_1$. This result is called the *Hyers-Ulam stability* of the additive Cauchy equation $g(x+y) = g(x) + g(y)$.

In 1978, Th. M. Rassias [9] introduced the new functional inequality and succeeded to extend the result of Hyers' by weakening the condition for the Cauchy difference to be unbounded: If there exist an $\varepsilon \geq 0$ and

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$0 \leq p < 1$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E_1$, then there exists a unique linear mapping $T\colon E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{|2 - 2^p|}\,\|x\|^p$$

for every $x \in E_1$. Since then several mathematicians were attracted to this result of Rassias and investigated a number of stability problems of functional equations. This stability phenomenon that was introduced and proved by Th. M. Rassias in his 1978 paper is called the *Hyers-Ulam-Rassias stability.* In 1991, Z. Gajda [1] solved the problem for $1 < p$, which was raised by Th. M. Rassias: In fact, the result of Rassias is valid for $1 < p$; Moreover, Z. Gajda gave an example that a similar stability result does not hold for $p = 1$. Another example was given by Th. M. Rassias and P. Šemrl [13, Theorem 2].

The second author, S. Miyajima and S. -E. Takahasi [7] introduced the notion of the Hyers-Ulam stability of a mapping between two normed linear spaces as follows:

DEFINITION 1.1. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed linear spaces and f be a (not necessarily linear) mapping from X into Y. We say that f has the Hyers-Ulam stability if there exists a constant $K \geq 0$ with the following property:

> For any $v \in f(X)$, the range of f, $\varepsilon \geq 0$ and $u \in X$ with $\|f(u) - v\|_Y \leq \varepsilon$, there exists a $u_0 \in X$ such that $f(u_0) = v$ and $\|u - u_0\|_X \leq K\varepsilon$.

We call such $K \geq 0$ a HUS constant for f, and denote by K_f the infimum of all HUS constants for f. If, in addition, K_f becomes a HUS constant for f, then we call it the HUS constant for f.

Roughly speaking, if f has the Hyers-Ulam stability, then to each "ε-approximate solution" u of the equation $f(x) = v$ there corresponds an exact solution u_0 of the equation in a $K\varepsilon$-neighborhood of u.

In [7, 8], the second author, S. Miyajima and S. -E. Takahasi obtained some stability results for particular linear differential operators D: the n-th order linear differential operator with constant coefficients and the first order linear differential operator with a continuous function as coefficient. In fact, they gave a characterization in order that D have the Hyers-Ulam stability. Among other things, for the first order linear differential operator D, the three authors above with H. Takagi

[8, 15] proved that the infimum K_D becomes the minimum of all HUS constants: Moreover, they described K_D completely.

H. Takagi, the second author and S. -E. Takahasi [14] considered a bounded linear operator T from a Banach space X into another Banach space Y. To display their result, we need some terminology. Let $\ker T$ be the kernel of T. Define the induced one-to-one linear operator $\widehat{T}$ from the quotient Banach space $X/\ker T$ into Y by

$$\widehat{T}(u + \ker T) \stackrel{\text{def}}{=} Tu \qquad \forall u \in X.$$

Now, their result reads as follows:

THEOREM A ([14, Theorem 2]). *Let X and Y be two Banach spaces and T be a bounded linear operator from X into Y. Then the following statements are equivalent :*

(i) *T has the Hyers-Ulam stability.*
(ii) *T has closed range.*
(iii) *$\widehat{T}^{-1}$ from $T(X)$ onto $X/\ker T$ is bounded.*

Moreover, if one of (hence all of) the conditions (i), (ii), *and* (iii) *is true, then $K_T = \|\widehat{T}^{-1}\|$.*

Theorem A states that $K_T = \|\widehat{T}^{-1}\|$ is valid whenever T has the Hyers-Ulam stability. However, the equality only means that the *infimum* of all HUS constants for T is $\|\widehat{T}^{-1}\|$. In other words, even if we restrict ourselves to a bounded linear operator T between two Banach spaces, we do not know whether the *minimum* of all HUS constants for T exists or not. O. Hatori, K. Kobayashi, H. Takagi and S. -E. Takahasi with the second author [2, Example] proved that the infimum of all HUS constants for a bounded linear operator between two Banach spaces need not be a HUS constant: That is, the minimum of all HUS constants does not exist in general.

In this paper, we are concerned with a closed operator T defined on a linear subspace $\mathcal{D}(T)$ of a Hilbert space G into a Hilbert space H. We first give some necessary and sufficient conditions in order that T have the Hyers-Ulam stability: In fact, Theorem A is valid for a closed operator T from $\mathcal{D}(T) \subset G$ into H. Moreover, we prove that T has the Hyers-Ulam stability if and only if T is lower semibounded. Among other things, we show that the infimum of all HUS constants for T is also a HUS constant: Namely, the minimum of all HUS constants do exist. We also describe the HUS constant K_T for T in terms of the lower bound of T.

2. Preliminaries

From now on, by an operator we shall mean a *non-zero* linear operator. Let G and H be Hilbert spaces with the norm $\|\cdot\|_G$ and $\|\cdot\|_H$, respectively. An operator T with a domain $\mathcal{D}(T) \subset G$ into H is said to be *closed* if its graph $\{(u, Tu) : u \in \mathcal{D}(T)\}$ is a closed subspace in the product Hilbert space $G \times H$. In other words, if $u_n \in \mathcal{D}(T)$ and $Tu_n \in H$ converge to $u_0 \in G$ and $v_0 \in H$, respectively, then $u_0 \in \mathcal{D}(T)$ and $v_0 = Tu_0$ holds. We remark that a bounded operator T from $\mathcal{D}(T) = G$ into H is a closed operator.

First, we note the notion of the Hyers-Ulam stability of a closed operator T. Indeed, the linearity of T can make the condition simple.

REMARK 2.1. Let T be a closed operator from $\mathcal{D}(T) \subset G$ into H. Recall that T is said to have the Hyers-Ulam stability if and only if there exists a constant $K > 0$ with the following property:

(a) For any $v \in T(\mathcal{D}(T))$, $\varepsilon \geq 0$ and $u \in \mathcal{D}(T)$ with $\|Tu - v\|_H \leq \varepsilon$ there exists a $u_0 \in \mathcal{D}(T)$ such that $Tu_0 = v$ and $\|u - u_0\|_G \leq K\varepsilon$.

We excluded the case where $K = 0$. In fact, if the condition (a) were true for $K = 0$, then taking $v = 0$, we would have $Tu = 0$ for every $u \in \mathcal{D}(T)$: This contradicts the hypothesis that an operator means *non-zero*. Now the linearity of T implies that the condition (a) is equivalent to

(b) For any $\varepsilon \geq 0$ and $u \in \mathcal{D}(T)$ with $\|Tu\|_H \leq \varepsilon$ there exists a $u_0 \in \mathcal{D}(T)$ such that $Tu_0 = 0$ and $\|u - u_0\|_G \leq K\varepsilon$.

Put $\ker T \overset{\text{def}}{=} \{u \in \mathcal{D}(T) : Tu = 0\}$. The condition (b) is equivalent to

(c) For any $u \in \mathcal{D}(T)$ there exists a $u_0 \in \ker T$ such that $\|u - u_0\|_G \leq K\|Tu\|_H$.

Next, we define a lower semiboundedness of a closed operator.

DEFINITION 2.1. Let T be a closed operator from $\mathcal{D}(T) \subset G$ into H. We say that T is lower semibounded if there exists a positive constant $\gamma > 0$ such that

$$\|Tv\|_H \geq \gamma\|v\|_G \qquad \forall v \in \mathcal{D}(T) \cap (\ker T)^{\perp}. \tag{2.1}$$

Here, $(\ker T)^{\perp}$ stands for the orthogonal complement of the kernel $\ker T$ of T: More precisely, $(\ker T)^{\perp}$ is the set of all $x \in G$ which are orthogonal to every $u \in \ker T$. We put

$$\begin{aligned}\gamma(T) &\stackrel{\text{def}}{=} \sup\{\gamma > 0 : \|Tv\|_H \geq \gamma \|v\|_G,\ \forall v \in \mathcal{D}(T) \cap (\ker T)^\perp\} \\ &= \inf\{\|Tv\|_H / \|v\|_G : v \in \mathcal{D}(T) \cap (\ker T)^\perp, v \neq 0\}. \qquad (2.2)\end{aligned}$$

We call $\gamma(T)$ the lower bound of T.

If T is a closed operator from $\mathcal{D}(T) \subset G$ into H, then it is easy to see that $\ker T$ is a closed subspace of G since T is a closed operator. In particular, if P is the orthogonal projection from G onto $\ker T$, then $x - Px \in (\ker T)^\perp$ for every $x \in G$.

LEMMA 2.1. *Let T be a closed operator from $\mathcal{D}(T) \subset G$ into H. If T is lower semibounded, then T has the Hyers-Ulam stability with a HUS constant $\gamma(T)^{-1}$.*

Proof. Suppose that T is lower semibounded with the lower bound $\gamma(T) > 0$. By definition, we have

$$\|Tv\|_H \geq \gamma(T)\|v\|_G \qquad \forall v \in \mathcal{D}(T) \cap (\ker T)^\perp. \qquad (2.3)$$

Let P be the orthogonal projection from G onto $\ker T$. Fix $u \in \mathcal{D}(T)$ arbitrarily, and put $u_0 \stackrel{\text{def}}{=} Pu \in \ker T$. Since $u - u_0 \in \mathcal{D}(T) \cap (\ker T)^\perp$, we have from (2.3) that

$$\|u - u_0\|_G \leq \gamma(T)^{-1}\|T(u - u_0)\|_H = \gamma(T)^{-1}\|Tu\|_H.$$

By Remark 2.1, this implies that T has the Hyers-Ulam stability with a HUS constant $\gamma(T)^{-1}$. □

DEFINITION 2.2. Let T be a closed operator from $\mathcal{D}(T) \subset G$ into H. We define the induced one-to-one operator $\widetilde{T}$ from $\mathcal{D}(T) \cap (\ker T)^\perp \subset G$ into H by

$$\widetilde{T}v \stackrel{\text{def}}{=} Tv \qquad \forall v \in \mathcal{D}(T) \cap (\ker T)^\perp.$$

Since T is closed, so is $\widetilde{T}$.

REMARK 2.2. Suppose that T is a closed operator from $\mathcal{D}(T) \subset G$ into H. The induced operator $\widetilde{T}$ as in Definition 2.2 is corresponding to $\widehat{T}$ as in (iii) of Theorem A.

To see this, we remark that the orthogonal complement $(\ker T)^\perp$ of $\ker T$ is isometrically isomorphic to the quotient Banach space $G/\ker T$ with the quotient norm $\|\cdot\|_q$: Indeed, $x + \ker T \mapsto Qx$ $(x \in G)$ gives a one-to-one onto correspondence between $G/\ker T$ and $(\ker T)^{\perp}$, where Q denotes the orthogonal projection from G onto $(\ker T)^{\perp}$; Since

$$\begin{aligned} \|x+\ker T\|_q &= \inf\{\|x+y\|_G : y \in \ker T\} \\ &= \|Qx\|_G \qquad \forall x \in G, \end{aligned} \tag{2.4}$$

$G/\ker T$ is isometrically isomorphic to $(\ker T)^{\perp}$ as a Banach space. If, in addition, we define an *inner product* $< \cdot, \cdot >$ on $G/\ker T$ by

$$< x+\ker T, y+\ker T > \stackrel{\text{def}}{=} < Qx, Qy >_G \qquad \forall x, y \in G, \tag{2.5}$$

then $G/\ker T$ becomes an inner product space. Here, $< \cdot, \cdot >_G$ denotes the inner product on the Hilbert space G. It follows from (2.4) and (2.5) that

$$< x+\ker T, x+\ker T > = < Qx, Qx >_G = \|x+\ker T\|_q{}^2$$

for every $x \in G$. Consequently, $G/\ker T$ is isomorphic to $(\ker T)^{\perp}$ as a Hilbert space.

LEMMA 2.2. *Let T be a closed operator from $\mathcal{D}(T) \subset G$ into H, $\widetilde{T}$ be the induced operator as in Definition 2.2. Each of the following two statements implies the other:*

(i) *$\widetilde{T}^{-1}$ is bounded.*
(ii) *T is lower semibounded.*

If, in addition, one of the conditions (i) *and* (ii) *is true, then we have $\|\widetilde{T}^{-1}\| = \gamma(T)^{-1}$.*

Proof. Put $\widetilde{H} \stackrel{\text{def}}{=} \{Tv \in H : v \in \mathcal{D}(T) \cap (\ker T)^{\perp}\}$. Note that the inverse operator $\widetilde{T}^{-1}$ from $\widetilde{H}$ into G is well-defined since $\widetilde{T}$ is an injection. If we assume that $1/0$ means ∞, then we obtain

$$\begin{aligned} &\sup_{w \in \widetilde{H}\setminus\{0\}} \frac{\|\widetilde{T}^{-1}w\|}{\|w\|} \\ =\ &\sup\left\{\frac{\|v\|_G}{\|Tv\|_H} : v \in \mathcal{D}(T) \cap (\ker T)^{\perp}, v \neq 0\right\} \\ =\ &\left[\inf\left\{\frac{\|Tv\|_H}{\|v\|_G} : v \in \mathcal{D}(T) \cap (\ker T)^{\perp}, v \neq 0\right\}\right]^{-1}. \end{aligned}$$

It follows that $\widetilde{T}^{-1}$ is bounded if and only if T is lower semibounded. In this case, the identity above with (2.2) shows that $\|\widetilde{T}^{-1}\| = \gamma(T)^{-1}$. □

3. Main results

THEOREM 3.1. *Let T be a closed operator from $\mathcal{D}(T) \subset G$ into H, $\widetilde{T}$ be the induced operator as in Definition 2.2. The following assertions are equivalent :*

(i) *T has the Hyers-Ulam stability.*
(ii) *T has closed range.*
(iii) *$\widetilde{T}^{-1}$ is bounded.*
(iv) *T is lower semibounded.*

Moreover, if one of the conditions above is true, then $K_T = \|\widetilde{T}^{-1}\| = \gamma(T)^{-1}$.

Proof. We shall prove that (i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (ii).

(i) $\Rightarrow$ (iv). Suppose that T has the Hyers-Ulam stability. By (c) of Remark 2.1, there exists a constant $K > 0$ with the following property: For any $u \in \mathcal{D}(T)$ there exists a $u_0 \in \ker T$ such that $\|u - u_0\|_G \leq K\|Tu\|_H$. Pick $u \in \mathcal{D}(T) \cap (\ker T)^\perp$ arbitrarily. By hypothesis, there exists a $u_0 \in \ker T$ such that $\|u - u_0\|_G \leq K\|Tu\|_H$. Since $u \in (\ker T)^\perp$ and since $u_0 \in \ker T$, we get

$$\|u\|_G{}^2 \leq \|u\|_G{}^2 + \|u_0\|_G{}^2 = \|u - u_0\|_G{}^2.$$

Since u was arbitrary, we thus obtain

$$\|Tu\|_H \geq K^{-1}\|u\|_G \qquad \forall u \in \mathcal{D}(T) \cap (\ker T)^\perp. \tag{3.6}$$

This implies that T is lower semibounded.

(iv) $\Rightarrow$ (i) and (iv) $\Leftrightarrow$ (iii). These are direct consequences of Lemma 2.1 and 2.2, respectively.

Although the equivalence of (iii) and (ii) is well-known, here we give a proof.

(iii) $\Rightarrow$ (ii). Suppose $\widetilde{T}^{-1}$ is bounded. We shall show that if Tu_n ($u_n \in \mathcal{D}(T)$) converges to an element, say $w \in H$, then $w = Tu_0$ for some $u_0 \in \mathcal{D}(T)$. Let Q be the orthogonal projection from G onto $(\ker T)^\perp$. Since $\widetilde{T}^{-1}$ is bounded,

$$\|v\|_G \leq \|\widetilde{T}^{-1}\| \, \|Tv\|_H \qquad \forall v \in \mathcal{D}(T) \cap (\ker T)^\perp. \tag{3.7}$$

Note that $Qu_n \in \mathcal{D}(T)$ since $\mathcal{D}(T)$ is a linear space, which contains $\ker T$. It follows from (3.7) that

$$\begin{aligned} \|Qu_n - Qu_m\|_G &\leq \|\widetilde{T}^{-1}\| \, \|TQ(u_n - u_m)\|_H \\ &= \|\widetilde{T}^{-1}\| \, \|Tu_n - Tu_m\|_H, \end{aligned}$$

and hence $\{Qu_n\}$ is a Cauchy sequence of $(\ker T)^{\perp}$. Since $(\ker T)^{\perp}$ is closed, Qu_n converges to an element, say $v_0 \in (\ker T)^{\perp}$. Because T is a closed operator, we get $v_0 \in \mathcal{D}(T)$ and $w = Tv_0$.

(ii) $\Rightarrow$ (iii). Suppose that T has closed range. That is, the range

$$\widetilde{H} = \{Tv \in H : v \in \mathcal{D}(T) \cap (\ker T)^{\perp}\}$$

of T is a closed subspace of H. Since $\widetilde{T}^{-1}$ is a closed operator from the Hilbert space $\widetilde{H}$ into G, it follows from the closed graph theorem that $\widetilde{T}^{-1}$ is bounded.

Now, suppose that one of the conditions (i), (ii), (iii) and (iv) is true. We show that the infimum K_T of all HUS constants for T satisfies $K_T = \|\widetilde{T}^{-1}\| = \gamma(T)^{-1}$: By Lemma 2.2, it is enough to prove that $K_T = \gamma(T)^{-1}$. To do this, fix a HUS constant K_0 for T arbitrarily. A quite similar argument to (i) $\Rightarrow$ (iv) shows that $\|Tu\|_H \geq {K_0}^{-1}\|u\|_G$ for every $u \in \mathcal{D}(T) \cap (\ker T)^{\perp}$. By the definition of the lower bound, we get $\gamma(T) \geq {K_0}^{-1}$. Since K_0 was arbitrary, we obtain $\gamma(T)^{-1} \leq K_T$. Recall that $\gamma(T)^{-1}$ is a HUS constant for T, by Lemma 2.1, and hence $K_T \leq \gamma(T)^{-1}$. We now obtain $K_T = \gamma(T)^{-1}$, and the proof is complete. □

COROLLARY 3.2. *Let T be a closed operator from $\mathcal{D}(T) \subset G$ into H. If T has the Hyers-Ulam stability, then K_T is the HUS constant for T.*

Proof. By Theorem 3.1, we see that $K_T = \gamma(T)^{-1}$. Since $\gamma(T)^{-1}$ is a HUS constant for T, by Lemma 2.1, we conclude that K_T is the HUS constant for T. □

The authors believe that Corollary 3.2 is interesting since the infimum K_S of all HUS constants for a *bounded* operator S between two Banach spaces need not be a HUS constant (cf. [2, Example]): In other words, although the infimum K_S exists, K_S is not necessarily the minimum.

We recall that every closed operator T from $\mathcal{D}(T) \subset G$ into H can be regarded as a *bounded* operator from a Hilbert space into H. In fact, put $G_0 \overset{\text{def}}{=} \mathcal{D}(T)$ as a set. We define

$$(3.8) \qquad < u, v >_{G_0} \overset{\text{def}}{=} < u, v >_G + < Tu, Tv >_H \qquad \forall u, v \in G_0,$$

which becomes an inner product on G_0. Here $< \cdot, \cdot >_G$ and $< \cdot, \cdot >_H$ denote the inner product on G and H, respectively. Since T is a closed operator, we see that G_0 is complete with respect to the induced norm $\|u\|_{G_0} \overset{\text{def}}{=} \sqrt{< u, u >_{G_0}}$ for every $u \in G_0$. Hence G_0 is a Hilbert space. We now consider the operator T_0 from G_0 into H defined by

$$(3.9) \qquad T_0 u \overset{\text{def}}{=} Tu \qquad \forall u \in G_0.$$

Then T_0 is a well-defined bounded operator since

$$\|T_0u\|_H{}^2 \leq \|u\|_G{}^2 + \|T_0u\|_H{}^2 = \|u\|_{G_0}{}^2 \qquad \forall u \in G_0$$

by (3.8) and (3.9).

Next, we are concerned with the Hyers-Ulam stability of T_0. Moreover, we describe the HUS constant K_{T_0}.

THEOREM 3.3. *Let T be a closed operator from $\mathcal{D}(T) \subset G$ into H. Let T_0 be a bounded operator from G_0 into H defined by* (3.9). *The following assertions are equivalent :*

(i) *T has the Hyers-Ulam stability.*
(ii) *T_0 has the Hyers-Ulam stability.*

Moreover, if one of the conditions (i) *and* (ii) *is true, then the HUS constants K_T, K_{T_0} and the lower bounds $\gamma(T)$, $\gamma(T_0)$ are connected with the following relations:*

$$K_T = \gamma(T)^{-1}, \; K_{T_0} = \gamma(T_0)^{-1} \quad \textit{and} \tag{3.10}$$

$$K_{T_0}{}^2 = K_T{}^2 + 1. \tag{3.11}$$

Proof. (i) $\Rightarrow$ (ii). Suppose that T has the Hyers-Ulam stability with a HUS constant K. We prove that for any $u \in G_0$ there exists a $u_0 \in \ker T_0$ such that $\|u - u_0\|_{G_0} \leq \sqrt{K^2+1}\,\|Tu\|_H$. To do this, pick $u \in G_0$ arbitrarily. Recall that $G_0 = \mathcal{D}(T)$, by definition, and that $\ker T = \ker T_0$. Since T is assumed to have the Hyers-Ulam stability, there exists a $u_0 \in \ker T_0$ such that $\|u - u_0\|_G \leq K\,\|Tu\|_H$. Adding the term $\|Tu\|_H{}^2 = \|Tu - Tu_0\|_H{}^2$ to the both sides of the last inequality, we obtain

$$\|u - u_0\|_G{}^2 + \|Tu - Tu_0\|_H{}^2 \leq (K^2+1)\,\|Tu\|_H{}^2.$$

It follows from (3.8) and (3.9) that $\|u - u_0\|_{G_0} \leq \sqrt{K^2+1}\,\|T_0u\|_H$, which implies that T_0 has the Hyers-Ulam stability with a HUS constant $\sqrt{K^2+1}$. In particular, $K_{T_0} \leq \sqrt{K^2+1}$, and so $K_{T_0} \leq \sqrt{K_T{}^2+1}$.

(ii) $\Rightarrow$ (i). If T_0 has the Hyers-Ulam stability with a HUS constant K_0, then to each $u \in G_0$ there corresponds a $u_0 \in \ker T_0$ such that

$$\sqrt{\|u - u_0\|_G{}^2 + \|T_0u\|_H{}^2} = \|u - u_0\|_{G_0} \leq K_0\,\|T_0u\|_H,$$

which implies that

$$\|u - u_0\|_G \leq \sqrt{K_0{}^2 - 1}\,\|Tu\|_H. \tag{3.12}$$

Hence T has the Hyers-Ulam stability with a HUS constant $\sqrt{{K_0}^2 - 1}$. We especially obtain $K_T \le \sqrt{{K_0}^2 - 1}$, and hence $K_T \le \sqrt{{K_{T_0}}^2 - 1}$.

Suppose one of (hence both of) the conditions (i) and (ii) is true. By Theorem 3.1 and Corollary 3.2, we see that $K_T = \gamma(T)^{-1}$ and $K_{T_0} = \gamma(T_0)^{-1}$ are the HUS constants for T and T_0, respectively. As proved above, $K_{T_0} \le \sqrt{{K_T}^2 + 1}$ and $K_T \le \sqrt{{K_{T_0}}^2 - 1}$. Consequently,

$$ {K_{T_0}}^2 \le {K_T}^2 + 1 \le ({K_{T_0}}^2 - 1) + 1 = {K_{T_0}}^2, $$

and hence ${K_{T_0}}^2 = {K_T}^2 + 1$. □

REMARK 3.1. If we apply Theorem 3.1, then we obtain other equivalent conditions in order that T_0 have the Hyers-Ulam stability: Moreover, K_{T_0} can be described by the induced operator $\widetilde{T}_0$.

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