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ON GENERALIZED (α, β)-DERIVATIONS AND COMMUTATIVITY IN PRIME RINGS

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ABSTRACT. Let R be a prime ring and I a nonzero ideal of R. Let α, ν, $\tau : R \to R$ be the endomorphisms and β, $\mu : R \to R$ the automorphisms. If R admits a generalized (α, β)-derivation g associated with a nonzero (α, β)-derivation δ such that $g([\mu(x), y]) = [\nu(x), y]_{\alpha,\tau}$ for all $x, y \in I$, then R is commutative.

1. Preliminaries

Throughout, R will represent an associative ring, and $Z(R)$ will be its center. Let $x, y \in R$. The commutator $xy - yx$ will be denoted by $[x, y]$. Let α and β be the endomorphisms of R. For any $x, y \in R$, we set $[x, y]_{\alpha,\beta} = x\alpha(y) - \beta(y)x$. We will also use the identities $[xy, z] = [x, z]y + x[y, z]$, $[x, yz] = [x, y]z + y[x, z]$ and $[x, yz]_{\alpha,\beta} = \beta(y)[x, z]_{\alpha,\beta} + [x, y]_{\alpha,\beta}\alpha(z)$.

Recall that R is prime if $xRy = \{0\}$ implies that either $x = 0$ or $y = 0$. An additive map $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. For some fixed $a \in R$, the map $d_a : R \to R$ given by $d_a(x) = [a, x]$ for all $x \in R$ is a derivation which is said to be an inner derivation.

An additive map $f_{a,b} : R \to R$ is called a generalized inner derivation if $f_{a,b}(x) = ax + xb$ for some fixed $a, b \in R$. It is immediate to see that if $f_{a,b}$ is a generalized inner derivation, then we have, for all $x, y \in R$,

$$f_{a,b}(xy) = f_{a,b}(x)y + xd_{-b}(y),$$

where d_{-b} is an inner derivation. Following this observation and M. Brešar [2], an additive map $f : R \to R$ is called a generalized derivation associated with d if there exists a derivation $d : R \to R$ such that $f(xy) =$

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$f(x)y+xd(y)$ for all $x, y \in R$. Other properties of generalized derivations were given by B. Hvala [4] and T. K. Lee [5], etc. Generally, we do not mention the derivation d associated with a generalized derivation f; rather we prefer to call f simply a generalized derivation. We can easily check that the notion of generalized derivation covers the notions of a derivation and a left multiplier (i.e., $f(xy) = f(x)y$ for all $x, y \in R$).

Let α and β be the endomorphisms of R. An additive map $\delta : R \to R$ is called an (α, β)-derivation if $\delta(xy) = \delta(x)\alpha(y) + \beta(x)\delta(y)$ holds for all $x, y \in R$. An $(1,1)$-derivation is called simply a derivation, where $1 : R \to R$ is an identity map. For some fixed $a \in R$, the map $\delta_a : R \to R$ given by $\delta_a(x) = [a, x]_{\alpha,\beta}$ for all $x \in R$ is an (α, β)-derivation which will be said to be an (α, β)-inner derivation. An additive map $g_{a,b} : R \to R$ will be called a generalized (α, β)-inner derivation if $g_{a,b}(x) = a\alpha(x) + \beta(x)b$ for some fixed $a, b \in R$ and all $x \in R$. A simple computation yields that if $g_{a,b}$ is a generalized (α, β)-inner derivation, then we have, for all $x, y \in R$,

$$g_{a,b}(xy) = g_{a,b}(x)\alpha(y) + \beta(x)\delta_{-b}(y),$$

where δ_{-b} is an (α, β)-inner derivation. In this viewpoint, an additive map $g : R \to R$ will be called a generalized (α, β)-derivation associated with δ if there exists an (α, β)-derivation $\delta : R \to R$ such that

$$g(xy) = g(x)\alpha(y) + \beta(x)\delta(y) \text{ for all } x, y \in R.$$

An $(1,1)$-generalized derivation is called simply a generalized derivation, where $1 : R \to R$ is an identity map. As before, we will not mention the (α, β)-derivation δ associated with a generalized (α, β)-derivation g; rather we will prefer to call g simply a generalized (α, β)-derivation.

2. Main results

There exist various results concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivation of R. For example, M. N. Daif and H. E. Bell [3] established that if in a semiprime ring R there exists a nonzero ideal I of R and a derivation d such that $d([x, y]) = [x, y]$ for all $x, y \in I$, then $I \subseteq Z(R)$. Recently, M. A. Quadri et al. [7] proved that the Daif and Bell's result obtained by replacing a generalized derivation instead of the derivation in a prime ring, is still true. The purpose of this paper is to extend this result to a generalized (α, β)-derivation.

In this section, let α, ν and τ be endomorphisms of R and β, μ be automorphisms of R. We first need the next well-known lemma.

LEMMA 2.1 ([6]). *Let R be a prime ring containing a nonzero commutative right ideal of R. Then R is commutative.*

Our main theorem is as follows:

THEOREM 2.2. *Let R be a prime ring and I a nonzero ideal of R. If R admits a generalized (α, β)-derivation g associated with a nonzero (α, β)-derivation δ such that $g([\mu(x), y]) = [\nu(x), y]_{\alpha,\tau}$ for all $x, y \in I$, then R is commutative.*

Proof. We replace y by zy in the defining equation

$$g([\mu(x), y]) = [\nu(x), y]_{\alpha,\tau} \tag{2.1}$$

to obtain

$$g(z[\mu(x), y] + [\mu(x), z]y) = \tau(z)[\nu(x), y]_{\alpha,\tau} + [\nu(x), z]_{\alpha,\tau}\alpha(y)$$

for all $x, y, z \in I$ which implies that

$$\begin{aligned} &g(z)\alpha([\mu(x), y]) + \beta(z)\delta([\mu(x), y]) \\ &+g([\mu(x), z])\alpha(y) + \beta([\mu(x), z])\delta(y) \\ &= \tau(z)[\nu(x), y]_{\alpha,\tau} + [\nu(x), z]_{\alpha,\tau}\alpha(y) \quad \text{for all } x, y, z \in I. \end{aligned} \tag{2.2}$$

By employing (2.1), we see that the relation (2.2) is reduced to

$$\begin{aligned} &g(z)\alpha([\mu(x), y]) + \beta(z)\delta([\mu(x), y]) + \beta([\mu(x), z])\delta(y) \\ &= \tau(z)[\nu(x), y]_{\alpha,\tau} \quad \text{for all } x, y, z \in I. \end{aligned} \tag{2.3}$$

If we substitute $y\mu(x)$ for y in (2.3), then we get

$$\begin{aligned} &g(z)\alpha([\mu(x), y])\alpha(\mu(x)) + \beta(z)\delta([\mu(x), y])\alpha(\mu(x)) \\ &+\beta(z)\beta([\mu(x), y])\delta(\mu(x)) + \beta([\mu(x), z])\delta(y)\alpha(\mu(x)) \\ &+\beta([\mu(x), z])\beta(y)\delta(\mu(x)) \\ &= \tau(z)\tau(y)[\nu(x), \mu(x)]_{\alpha,\tau} + \tau(z)[\nu(x), y]_{\alpha,\tau}\alpha(\mu(x)) \\ &= \tau(z)\tau(y)g([\mu(x), \mu(x)]) + \tau(z)[\nu(x), y]_{\alpha,\tau}\alpha(\mu(x)) \\ &= \tau(z)[\nu(x), y]_{\alpha,\tau}\alpha(\mu(x)), \end{aligned}$$

that is,

$$\begin{aligned} &g(z)\alpha([\mu(x), y])\alpha(\mu(x)) + \beta(z)\delta([\mu(x), y])\alpha(\mu(x)) \\ &+\beta(z)\beta([\mu(x), y])\delta(\mu(x)) + \beta([\mu(x), z])\delta(y)\alpha(\mu(x)) \\ &+\beta([\mu(x), z])\beta(y)\delta(\mu(x)) \\ &= \tau(z)[\nu(x), y]_{\alpha,\tau}\alpha(\mu(x)) \quad \text{for all } x, y, z \in I. \end{aligned} \tag{2.4}$$

Right-multiplicating by $\alpha(\mu(x))$ in (2.3) and comparing (2.4) with the result, we obtain

$$\{\beta(z)\beta([\mu(x),y])+\beta([\mu(x),z])\beta(y)\}\delta(\mu(x))=0 \text{ for all } x,y,z\in I$$

which is equivalent to

$$(2.5)\qquad \{z[\mu(x),y]+[\mu(x),z]y\}\beta^{-1}(\delta(\mu(x)))=0 \text{ for all } x,y,z\in I.$$

Replacing z by wz ($w\in R$) in (2.5) and using (2.5), we have

$$(2.6)\qquad [\mu(x),w]zy\beta^{-1}(\delta(\mu(x)))=0 \text{ for all } x,y,z\in I,\ w\in R.$$

Let $z=z\beta^{-1}(\delta(\mu(x)))$ and $y=y[\mu(x),w]z$ $(x,y,z\in I, w\in R)$ in (2.6). Then we obtain

$$[\mu(x),w]z\beta^{-1}(\delta(\mu(x)))y[\mu(x),w]z\beta^{-1}(\delta(\mu(x)))=0$$

for all $x,y,z\in I$, $w\in R$ and the primeness of I yields

$$[\mu(x),w]z\beta^{-1}(\delta(\mu(x)))=0$$

for all $x,z\in I$, $w\in R$.

For any fixed $w\in R$, again using the fact that I is prime, we have for all $x\in I$, either $[\mu(x),w]=0$ or $\delta(\mu(x))=0$. This means that I is the union of its additive subgroups $A=\{x\in I:[\mu(x),w]=0\}$ and $B=\{x\in I:\delta(\mu(x))=0\}$. Since a group cannot be the union of two proper subgroups and δ is nonzero, we get $A=I$, i.e., $[\mu(x),w]=0$ for all $x\in I$.

Indeed, suppose that $B=I$, that is, $\delta(\mu(x))=0$ for all $x\in I$. Then we see that for all $x\in I$ and $y\in R$,

$$\begin{aligned}0=\delta(\mu(xy)) &= \delta(\mu(x)\mu(y))\\ &= \delta(\mu(x))\alpha(\mu(y))+\beta(\mu(x))\delta(\mu(y))\\ &= \beta(\mu(x))\delta(\mu(y))\\ &= (\beta\circ\mu)(x)\delta(\mu(y)),\end{aligned}$$

from which we obtain $x(\beta\circ\mu)^{-1}(\delta(\mu(y)))=0$ for all $x\in I$ and $y\in R$. Since I is prime, it follows that $(\beta\circ\mu)^{-1}(\delta(\mu(y)))=0$ and hence $\delta(\mu(y))=0$ holds for all $y\in R$ which implies that $\delta=0$. This contradicts that δ is nonzero.

Now $w\in R$ was arbitrary and so we see that $[\mu(x),w]=0$ holds for all $x\in I$ and $w\in R$ which gives $\mu(I)\subseteq Z(R)$. Since $\mu(I)$ is a nonzero ideal of R, Lemma 2.1 guarantees that R is commutative. The proof of the theorem is completed. □

COROLLARY 2.3 ([7, Theorem 2.1]). *Let R be a prime ring and I a nonzero ideal of R. If R admits a generalized derivation g associated with a nonzero derivation δ such that $g([x,y]) = [x,y]$ for all $x, y \in I$, then R is commutative.*

Proof. Putting $\alpha = \beta = \mu = \nu = \tau = 1$ in Theorem 2.2 guarantees the conclusion of the corollary, where $1 : R \to R$ is an identity map. □

H. E. Bell and M. N. Daif [1] showed that if a 2-torsion-free prime ring R admits a nonzero derivation d satisfying $d(xy) = d(yx)$ for all $x, y \in R$, then R is commutative.

Here we improve this result.

COROLLARY 2.4. *Let R be a prime ring and I a nonzero ideal of R. If R admits a generalized (α, β)-derivation g associated with a nonzero (α, β)-derivation δ such that $g(xy) = g(yx)$ for all $x, y \in I$, then R is commutative.*

Proof. Setting $\mu = 1$ and $\nu = 0$, respectively, in Theorem 2.2, we obtain the result of the corollary, where $1 : R \to R$ is an identity map and $0 : R \to R$ is a zero map. □

The following example shows that in the assumption of Corollary 2.4, if we replace the prime ring by a semiprime ring, then R may not be commutative.

EXAMPLE 2.5. Let R_1 be a noncommutative prime ring and R_2 a commutative prime ring. Then $R = R_1 \oplus R_2$ is a semiprime ring. Suppose that α_2 and β_2 are two endomorphisms of R_2 with $\alpha_2 \neq \beta_2$. Then $\alpha_2 - \beta_2$ defines a nonzero (α_2, β_2)-derivation on R_2. From this, it follows that a map $\delta : R \to R$ defined by $\delta(x_1, x_2) = (0, (\alpha_2 - \beta_2)(x_2))$ for all $(x_1, x_2) \in R$, is a nonzero (α, β)-derivation on R, where α is an endomorphism of R defined by $\alpha(x_1, x_2) = (0, \alpha_2(x_2))$ and β is an endomorphism of R given by $\beta(x_1, x_2) = (0, \beta_2(x_2))$.

Let us define a map $\gamma : R \to R$ by $\gamma(x_1, x_2) = (0, a\alpha_2(x_2))$, $a \in R_2$. Then it is easy to see that $g = \alpha - \beta + \gamma$ is a generalized (α, β)-derivation associated with a nonzero (α, β)-derivation δ such that $g(xy) = g(yx)$ for all $x, y \in R$. However, R is not commutative.

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