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A CURVATURE-LIKE TENSOR FIELD ON A SASAKIAN MANIFOLD

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Abstract. We investigate a curvature-like tensor defined by (3.1) in Sasakian manifold of dimension$\geq$ 5, and show that this tensor satisfies some properties. Especially, we determine compact Sasakian manifolds with vanishing this tensor and improve some theorems concerning contact conformal curvature tensor and spectrum of Laplacian acting on $p(0 \leq p \leq 2)$-forms on the manifold by using this tensor component.

1. Introduction

In their paper ([3]), S. Funabashi, H. S. Kim, J. S. Pak and the present author have determined a new tensor field on a Kähler manifold which is traceless component of the conformal curvature tensor. Moreover, this tensor is invariant under concircular change. In particular, On a $2n$-dimensional Kähler manifold the traceless component of the conformal curvature tensor field $\overset{*}{C}_{dcb}{}^{a}$ is given by

(1.1)

$$\begin{aligned}
&\overset{*}{C}_{dcb}{}^{a}\\
&= R_{dcb}{}^{a} + \frac{1}{2n}(R_{d}{}^{a}g_{cb} - R_{c}{}^{a}g_{db} + \delta_{d}^{a}R_{cb} - \delta_{c}^{a}R_{db}\\
&\quad - S_{d}{}^{a}\phi_{cb} + S_{c}{}^{a}\phi_{db} - \phi_{d}{}^{a}S_{cb} + \phi_{c}{}^{a}S_{db} + 2S_{dc}\phi_{b}{}^{a} + 2\phi_{dc}S_{b}{}^{a})\\
&\quad + \frac{(n+2)s}{4n^2(n+1)}(\phi_{d}{}^{a}\phi_{cb} - \phi_{c}{}^{a}\phi_{db} - 2\phi_{dc}\phi_{b}{}^{a})\\
&\quad - \frac{(2n^2+5n+6)s}{4n^2(2n-1)(n+1)}(\delta_{d}^{a}g_{cb} - \delta_{c}^{a}g_{db}) - \frac{2(n-2)}{n(2n-1)}(\delta_{d}^{a}R_{cb} - \delta_{c}^{a}R_{db}),
\end{aligned}$$

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where here and in the sequel the indices a, b, c, d run over the range $\{1, 2, \ldots, 2n\}$ and we use the Einstein convention with respect to this index system. We denote by $g_{ba}, R_{dcb}{}^{a}, R_{ba}, s$ and ϕ_b^a local components of g, the curvature tensor, the Ricci tensor, the scalar curvature and ϕ of M, respectively.

In this paper, we define a new tensor field on a Sasakian manifold, which is constructed from $\overset{*}{C}_{dcb}{}^{a}$ by using the Boothby-Wang's fibration ([2]), and study some properties of this new tensor field. Especially, we determine compact Sasakian manifolds with vanishing this tensor and improve some theorems concerning contact conformal curvature tensor and spectrum of Laplacian acting on $p(0 \leq p \leq 2)$-forms on the manifold by using this tensor component.

2. Preliminaries

Let M be a $(2n+1)$-dimensional differential manifold of class C^∞ covered by a system of coordinate neighborhoods $\{U \,;\, x^h\}$ in which there are given a tensor field $\phi_i{}^h$ of type (1,1), a vector field ξ^h and 1-form η_j satisfying

$$\phi_j{}^h \phi_h{}^i = -\delta_j^i + \eta_j \xi^i, \quad \phi_j{}^i \xi^j = 0, \quad \eta_i \phi_j{}^i = 0, \quad \eta_i \xi^i = 1, \tag{2.1}$$

where here and in the sequel the indices h, i, j, k, l run over the range $\{1, 2, \ldots, 2n+1\}$ and we use the Einstein convention with respect to this index system. Such a set (ϕ, ξ, η) of a tensor field ϕ, a vector field ξ and a 1-form η is called an almost contact structure and a manifold with an almost contact structure an almost contact manifold. Suppose that there is given, in an almost contact manifold, a Riemannian metric g_{ji} such that

$$g_{kh} \phi_j{}^k \phi_i{}^h = g_{ji} - \eta_j \eta_i, \; \eta_i = g_{ih} \xi^h, \tag{2.2}$$

then the almost contact structure is said to be metric and the manifold is called an almost contact metric manifold. In an almost contact metric manifold, the tensor field $\phi_{ji} = \phi_j{}^h g_{hi}$ is skew-symmetric. If an almost contact metric structure satisfies

$$\phi_{ji} = \frac{1}{2}(\partial_j \eta_i - \partial_i \eta_j),$$

then the almost contact metric structure is called a contact structure. A manifold with a normal contact structure is called a Sasakian manifold.

Let M^{2n+1} be a Sasakian manifold with Sasakian structure $(\phi_j{}^i, \xi^i, g_{ji})$. Now we assume that the structure vector ξ^i is regular. Then, as is well known by Boothby and Wang ([2]), we can define a fibred space $\{M^{2n+1}, M^{2n}, g, \pi\}$ with invariant Riemannian metric, where π is the projection

$$\pi\ :\ M^{2n+1} \longrightarrow M^{2n},$$

M^{2n} being the base manifold. Moreover, the base manifold M^{2n} is a Kähler manifold with a Kähler structure $\{\phi_b{}^a, g_{ba}\}$.

3. A curvature-like tensor field H

We define a tensor field $H_{kji}{}^h$ on M^{2n+1} by the lift of the $\overset{*}{C}_{dcb}{}^a$. Hence we have by definition

$$H_{kji}{}^h = \overset{*}{C}_{dcb}{}^a E_k{}^d E_j{}^c E_i{}^b E^h{}_a.$$

Now, transvecting (1.1) with $E_k{}^d E_j{}^c E_i{}^b E^h{}_a$, we can easily see that

(3.1)

$$\begin{aligned}
&H_{kji}{}^h\\
&= R_{kji}{}^h + \frac{1}{2n}(\delta_k^h R_{ji} - \delta_j^h R_{ki} + R_k{}^h g_{ji} - R_j{}^h g_{ki} - R_k{}^h \eta_j \eta_i\\
&\quad + R_j{}^h \eta_k \eta_i - \eta_k \xi^h R_{ji} + \eta_j \xi^h R_{ki} - \phi_k{}^h S_{ji} + \phi_j{}^h S_{ki} - S_k{}^h \phi_{ji}\\
&\quad + S_j{}^h \phi_{ki} + 2\phi_{kj} S_i{}^h + 2S_{kj}\phi_i{}^h)\\
&\quad + \{\frac{2n^2-n-2}{2n(n+1)} + \frac{(n+2)s}{4n^2(n+1)}\}(\phi_k{}^h \phi_{ji} - \phi_j{}^h \phi_{ki} - 2\phi_{kj}\phi_i{}^h)\\
&\quad - \{\frac{2n^2-7n-6}{2n(n+1)(2n-1)} + \frac{(2n^2+5n+6)s}{4n^2(n+1)(2n-1)}\}(\delta_k^h g_{ji} - \delta_j^h g_{ki})\\
&\quad - \{\frac{10n^2+19n+6}{2n(n+1)(2n-1)} - \frac{(2n^2+5n+6)s}{4n^2(n+1)(2n-1)}\}(\delta_k^h \eta_j \eta_i - \delta_j^h \eta_k \eta_i)\\
&\quad - \frac{2(n-2)}{n(2n-1)}(\delta_k^h R_{ji} - \delta_j^h R_{ki} - R_{ji}\eta_k \xi^h + R_{ki}\eta_j \xi^h)\\
&\quad - \{\frac{8n^3+2n^2+3n+6}{2n(n+1)(2n-1)} - \frac{(2n^2+5n+6)s}{4n^2(n+1)(2n-1)}\}(\eta_k \xi^h g_{ji} - g_{ki}\eta_j \xi^h),
\end{aligned}$$

which is constructed from the traceless part of the conformal curvature tensor field (1.1) in Kähler manifold by using the Boothby-Wang's fibration ([2]), where $s = R_{ji}g^{ji}$ denotes the scalar curvature of M^{2n+1},

$R_j{}^h = R_{ji}g^{ih}$ and $S_j{}^h = S_{ji}g^{ih}$. From now on, we'll call this tensor *the horizontal lift of the traceless part of the conformal curvature tensor.*

By $R = (R_{kji}{}^h), R_1 = (R_{ji})$ and s we denote the Riemannian curvature tensor, the Ricci tensor and the scalar curvature tensor, respectively. For a tensor field T on M we denote by $\|T\|$ the norm of T with respect to g. Let M be a $(2n+1)$-dimensional Sasakian manifold with a normal contact metric structure (ϕ, g, ξ, η). Thus, $\phi = (\phi_j{}^i)$, $\xi = (\xi^i)$, $\eta = (\eta_j)$ are tensor field of type (1,1), (1,0), (0,1), respectively, and $g = (g_{ji})$ is a Riemannian metric, on M such that

$$\phi_s{}^i\phi_j{}^s = -\delta_j^i + \eta_j\xi^i,\ \eta_s\xi^s = 1,\ \phi_s{}^i\xi^s = 0,\ \eta_s\phi_i{}^s = 0,$$
$$g_{ts}\phi_j{}^t\phi_i{}^s = g_{ji} - \eta_j\eta_i,\ \eta_i = \xi^s g_{si}$$
$$\nabla_k\phi_j{}^i = \eta_j\delta_k^i,\ \nabla_j\eta_i = \phi_{ji},$$

where $\phi_{ji} = \phi_j{}^s g_{si}$ and ∇_k denotes the operator of covariant differentiation with respect to the Levi-Civita connection. Then we have ([9])

(3.2)
$$R_{kjit}\xi^t = \eta_k g_{ji} - \eta_j g_{ki},$$
$$R_{kjts}\phi_i{}^t\phi_h{}^s = R_{kjih} - g_{kh}g_{ji} + g_{ki}g_{jh} + \phi_{kh}\phi_{ji} - \phi_{ki}\phi_{jh},$$
$$\frac{1}{2}R_{tsji}\phi^{ts} = R_{jt}\phi_i{}^t + (2n-1)\phi_{ji} = R_{tjsi}\phi^{ts},$$

$$(3.3)\qquad R_{it}\xi^t = 2n\eta_i,\ R_{ts}\phi_j{}^t\phi_t{}^s = R_{ji} - 2n\eta_j\eta_i,\ S_{ji} = -S_{ij},$$

where $\phi^{ji} = \phi_t{}^i g^{jt}$, $R_{kjih} = R_{kji}{}^t g_{th}$ and $S_{ji} = \phi_j{}^t R_{ti}$.

Define on M a tensor field $Q = (Q_{ji})$ by

$$Q_{ji} = R_{ji} - (\frac{s}{2n} - 1)g_{ji} - (2n + 1 - \frac{s}{2n})\eta_j\eta_i.$$

With (3.3), we can easily verify the following equation.

$$(3.4)\qquad \|Q\|^2 = \|R_1\|^2 - \frac{1}{2n}s^2 + 2s - 2n(2n+1).$$

A Sasakian manifold is said to be η-*Einstein* if $Q = 0$. For any η-Einstein Sasakian manifold of dimension ≥ 5, s is constant. Any 3-dimensional Sasakian manifold is η-Einstein, but in this case s may not be constant.

Define on M a tensor field $T = (T_{kjih})$ by

(3.5)
$$T_{kjih} = R_{kjih} - \frac{k+3}{4}(g_{kh}g_{ji} - g_{ki}g_{jh}) - \frac{k-1}{4}(\phi_{kh}\phi_{ji} - \phi_{ki}\phi_{jh} - 2\phi_{kj}\phi_{ih} - g_{kh}\eta_j\eta_i + g_{ki}\eta_j\eta_h - \eta_k\eta_h g_{ji} + \eta_k\eta_i g_{jh}),$$

where $k = \frac{s-n(3n+1)}{n(n+1)}$. By using (4.1), (4.2) and (4.4), we can easily verify the following equation.

$$(3.6)\quad \|T\|^2 = \|R\|^2 - \frac{2}{n(n+1)}s^2 + \frac{4(3n+1)}{n+1}s - \frac{4n(2n+1)(3n+1)}{n+1}.$$

A Sasakian manifold of dimension ≥ 5 is of constant ϕ-sectional curvature if and only if $T = 0$ ([6]). Since $H_{kjih} = H_{kji}{}^t g_{th}$, we also consider the curvature-like tensor H_{kjih} from (3.1). The tensor field H satisfies the following identities:

(3.7)
$$H_{kjih} = H_{ihkj} = -H_{jkih} = -H_{kjhi},$$
$$H_{kjih} + H_{jikh} + H_{ikjh} = 0,$$
$$g^{kh}H_{kjih} = 0,\ \xi^h H_{kjih} = 0,\ \phi^{kh}H_{kjih} = 0.$$

Using (3.1), (3.2), (3.3) and (3.4), we can check that

(3.8)
$$\|H\|^2 = \|R\|^2 + \frac{8(n-5)(n-1)}{n^2(2n-1)}\|R_1\|^2 - \frac{2(4n^3 - 11n^2 - 2n + 10)}{n^3(n+1)(2n-1)}s^2 + \frac{4(6n^4 + 3n^3 - 21n^2 - 4n + 20)}{n^2(n+1)(2n-1)}s - \frac{4(2n+1)(6n^4 + 3n^3 - 21n^2 - 4n + 20)}{n(n+1)(2n-1)}.$$

And by using (3.4), we have

(3.9)
$$\|H\|^2 = \|R\|^2 - \frac{2}{n(n+1)}s^2 + \frac{4(3n+1)}{n+1}s - \frac{4n(2n+1)(3n+1)}{n+1} + \frac{8(n-5)(n-1)}{n^2(2n-1)}\|Q\|^2.$$

Moreover by (3.6), we have the following equation

$$\|H\|^2 = \|T\|^2 + \frac{8(n-5)(n-1)}{n^2(2n-1)}\|Q\|^2. \tag{3.10}$$

By the way it is already shown in Theorem 3.3 that $H = 0$ and $Q = 0$ if and only if $T = 0$.

We next recall definition and fundamental properties of D-homothetic deformation due to S. Tanno ([10]), where D denotes the distribution defined by a contact form.

D-homothetic deformation $g \to *g$ is defined by

$$*g_{ji} = \alpha g_{ji} + \alpha(\alpha - 1)\eta_j\eta_i,$$

for a positive constant α. From $*g_{ji}$ we have ([10])

(3.11)

$$\begin{aligned}
*g^{kj} &= \alpha^{-1}g^{kj} - \alpha^{-2}(\alpha-1)\xi^k\xi^j,\\
*R_{kji}{}^h &= R_{kji}{}^h + (\alpha-1)(\phi_{ki}\phi_j{}^h - \phi_{ji}\phi_k{}^h + 2\phi_{kj}\phi_i{}^h)\\
&\quad + (\alpha-1)^2(\eta_j\delta_k^h - \eta_k\delta_j^h)\eta_i \quad + (\alpha-1)\{\eta_k(g_{ji} - \eta_i\delta_j^h)\\
&\quad - \eta_j(g_{ki}\eta^h - \eta_i\delta_k^h) + \eta_i(\eta_j\delta_k^h - \eta_k\delta_j^h)\}\\
*R_{kj} &= R_{kj} - 2(\alpha-1)g_{kj} + 2(\alpha-1)(n\alpha + n + 1)\eta_k\eta_j,\\
*s &= \alpha^{-1}s - 2n\alpha^{-1}(\alpha-1).
\end{aligned}$$

If (ϕ, ξ, η, g) is a Sasakian structure, then $(*\phi, *\xi, *\eta, *g)$ is also a Sasakian structure, where we put

$$*\phi = \phi,\ *\xi = \alpha^{-1}\xi,\ *\eta = \alpha\eta,\ *g = \alpha g + \alpha(\alpha-1)\eta\phi\eta \tag{3.12}$$

for a positive constant α ([10]). In this case it is said that $M(\phi, \xi, \eta, g)$ is D-homothetic to $M(*\phi, *\xi, *\eta, *g)$, then by using (3.11) and (3.12) we have

(3.13)

$$\begin{aligned}
*\phi_{kj} &= \alpha\phi_{kj},\\
*R_k^{\ i} &= \alpha^{-1}R_k^{\ i} - 2\alpha^{-1}(\alpha-1)\delta_k^i + 2(n+1)\alpha^{-1}(\alpha-1)\eta_k\xi^i,\\
*S_{kj} &= S_{kj} - 2(\alpha-1)\phi_{kj},\\
*S_k^{\ i} &= \alpha^{-1}S_k^{\ i} - 2\alpha^{-1}(\alpha-1)\phi_k^{\ i}.
\end{aligned}$$

Taking account of (3.11), (3.12) and (3.13), we can easily verify that

$$*H_{kji}{}^h = H_{kji}{}^h,$$

where $*H_{kji}{}^h$ denotes the horizontal lift of the traceless part of the contact conformal curvature tensor field with respect to $(*\phi, *\xi, *\eta, *g)$.

THEOREM 3.1. *$H_{kji}{}^h$ with respect to (ϕ, ξ, η, g) coincide with the one with respect to $*H_{kji}{}^h$ with respect to $(*\phi, *\xi, *\eta, *g)$.*

COROLLARY 3.2. *A Sasakian manifold with vanishing a curvature-like tensor H is D-homothetic to a Sasakian manifold with vanishing tensor H.*

THEOREM 3.3. *A $(2n+1)$-dimensional Sasakian manifold is of constant ϕ-holomorphic sectional curvature if and only if the manifold is η-Einsteinian and $H_{kji}{}^h$ vanishes everywhere.*

Proof. It is clear from (3.1) that $H_{kji}{}^h = 0$ implies

$$\begin{aligned}
&R_{kji}{}^h \\
&= -\frac{1}{2n}(\delta_k^h R_{ji} - \delta_j^h R_{ki} + R_k^h g_{ji} - R_j^h g_{ki} - R_k^h \eta_j \eta_i + R_j^h \eta_k \eta_i \\
&\quad - \eta_k \xi^h R_{ji} + \eta_j \xi^h R_{ki} - \phi_k^h S_{ji} + \phi_j^h S_{ki} - S_k^h \phi_{ji} + S_j^h \phi_{ki} \\
&\quad + 2\phi_{kj} S_i^h + 2S_{kj}\phi_i^h) \\
&\quad - \{\frac{2n^2-n-2}{2n(n+1)} + \frac{(n+2)s}{4n^2(n+1)}\}(\phi_k^h \phi_{ji} - \phi_j^h \phi_{ki} - 2\phi_{kj}\phi_i^h) \\
&\quad + \{\frac{2n^2-7n-6}{2n(n+1)(2n-1)} + \frac{(2n^2+5n+6)s}{4n^2(n+1)(2n-1)}\}(\delta_k^h g_{ji} - \delta_j^h g_{ki}) \\
&\quad + \{\frac{10n^2+19n+6}{2n(n+1)(2n-1)} - \frac{(2n^2+5n+6)s}{4n^2(n+1)(2n-1)}\}(\delta_k^h \eta_j \eta_i - \delta_j^h \eta_k \eta_i) \\
&\quad + \frac{2(n-2)}{n(2n-1)}(\delta_k^h R_{ji} - \delta_j^h R_{ki} - R_{ji}\eta_k \xi^h + R_{ki}\eta_j \xi^h) \\
&\quad + \{\frac{8n^3+2n^2+3n+6}{2n(n+1)(2n-1)} - \frac{(2n^2+5n+6)s}{4n^2(n+1)(2n-1)}\}(\eta_k \xi^h g_{ji} - g_{ki}\eta_j \xi^h).
\end{aligned}$$

If the manifold is η-Einstein, we have $R_{ji} = (\frac{s}{2n} - 1)g_{ji} + (2n+1-\frac{s}{2n})\eta_j\eta_i$ and $S_{ji} = (\frac{s}{2n} - 1)\phi_{ji}$. Inserting those equations back into the above equation, we obtain

$$\begin{aligned}
R_{kji}{}^h = &\frac{1}{4}(k+3)(\delta_k^h g_{ji} - \delta_j^h g_{ki}) + \frac{1}{4}(k-1)(\eta_k \eta_i \delta_j^h - \eta_j \eta_i \delta_k^h \\
&+ g_{ki}\eta_j \xi^h - g_{ji}\eta_k \xi^h + \phi_k^h \phi_{ji} - \phi_j^h \phi_{ki} - 2\phi_{kj}\phi_i^h),
\end{aligned}$$

where $k = \frac{s-3n^2-n}{n(n+1)}$. This equation means that the manifold is of constant ϕ-holomorphic sectional curvature k. Conversely, if the manifold is of constant ϕ-holomorphic sectional curvature, then $R_{ji} = (\frac{s}{2n} - 1)g_{ji} + (2n+1-\frac{s}{2n})\eta_j\eta_i$ and consequently $S_{ji} = (\frac{s}{2n} - 1)\phi_{ji}$. Substituting those equations into (3.3), we can see that $H_{kji}{}^h = 0$. □

We suppose that the curvature-like tensor $H_{kji}{}^h$ coincides with the contact conformal curvature tensor $C_{0,kji}^{\ h}$(for the definition of $C_{0,kji}^{\ h}$, see [4]). Then it follows that

$$R_{ji} = (\frac{s}{2n} - 1)g_{ji} + (2n+1-\frac{s}{2n})\eta_j\eta_i.$$

Conversely, if M^{2n+1} is η-Einstein,

$$\begin{aligned} H_{kji}{}^h &= C_{0,kji}^{\ h} = R_{kji}{}^h \\ &= \frac{1}{4}(k+3)(\delta_k^h g_{ji} - \delta_j^h g_{ki}) \\ &\quad + \frac{1}{4}(k-1)(\eta_k\eta_i\delta_j^h - \eta_j\eta_i\delta_k^h + g_{ki}\eta_j\xi^h - g_{ji}\eta_k\xi^h \\ &\quad + \phi_k^h\phi_{ji} - \phi_j^h\phi_{ki} - 2\phi_{kj}\phi_i^h). \end{aligned}$$

Thus we have

THEOREM 3.4. *If $H_{kji}{}^h$ coincides with $C_{0,kji}^{\ h}$ then M^{2n+1} is η-Einstein and vise versa.*

On the other hand, J. C. Jeong, J. D. Lee, G. H. Oh and J. S. Pak have obtained the following theorem.

THEOREM 3.5 ([4]). *A necessary and sufficient condition in order that $C_{0,kji}^{\ h}$ coincides with C-Bochner curvature tensor $C_{kji}{}^h$(for details of $C_{kji}{}^h$ see [6]) is that M^{2n+1} is η-Einstein.*

But if the tensor $H_{kji}{}^h$ with C-Bochner curvature tensor $C_{kji}{}^h$ are coincide, M is not η-Einstein. And finally we can say the following fact.

COROLLARY 3.6. *If M^{2n+1} is η-Einstein then $C_{0,kji}^{\ h}$, $C_{kji}{}^h$ and $H_{kji}{}^h$ are coincides but the converse doesn't hold.*

4. Spectrum of the Laplacian

Let M be a compact Sasakian manifold of real dimension $m(=2n+1)$ and denote by Δ the Laplacian acting on p-forms on M, $0 \le p \le m$. Then we have the spectrum for each p:

$$Spec^p(M,g) = \{0 \le \lambda_{0,p} \le \lambda_{1,p} \le \lambda_{2,p} \le \cdots \uparrow +\infty\},$$

where each eigenvalue $\lambda_{\alpha,p}$ is repeated as many as times as its multiplicity indicates. Furthermore, the Minakshisundaram-Pleijel-Gaffney's formula for $Spec^p(M,g)$ is given by

$$\sum_{\alpha=0}^{\infty} exp(-\lambda_{\alpha,p}t) \sim (4\pi t)^{\frac{-n}{2}} \sum_{\alpha=0}^{\infty} a_{\alpha,p}t^{\alpha} \quad \text{as } t \to 0^+,$$

where the constants $A_{\alpha,p}$ are spectral invariant. Especially, for $p = 0$, we have

$$a_{0,0} = \int_M dM = Vol(M,g), \tag{4.1}$$

$$a_{1,0} = \frac{1}{6}\int_M s\, dM, \tag{4.2}$$

$$a_{2,0} = \frac{1}{360}\int_M \{2\|R\|^2 - 2\|R_1\|^2 + 5s^2\}dM, \tag{4.3}$$

where dM denotes the natural volume element of (M,g) ([1]). For $p = 1$, we have

$$a_{0,1} = mVol(M,g), \tag{4.4}$$

$$a_{1,1} = \frac{m-6}{6}\int_M s\, dM, \tag{4.5}$$

$$a_{2,1} = \frac{1}{360}\int_M \{2(m-15)\|R\|^2 - 2(m-90)\|R_1\|^2 + 5(m-12)s^2\}dM \tag{4.6}$$

([11]). For $p = 2$, we have

$$a_{0,2} = \frac{m(m-1)}{2}Vol(M,g), \tag{4.7}$$

$$a_{1,2} = \frac{m^2 - 13m + 24}{12}\int_M s\, dM, \tag{4.8}$$

$$a_{2,2} = \frac{1}{720}\int_M \{2(m^2 - 31m + 240)\|R\|^2 - 2(m^2 - 181m + 1080)\|R_1\|^2 + 5(m^2 - 25m + 120)s^2\}dM \tag{4.9}$$

([8, 12, 13]).

We next introduce the following lemma provided by Tanno ([11]) for later use.

LEMMA 4.1 ([11]). *Let (M,g) and (M',g') be compact orientable Riemannian manifolds with $Vol(M,g) = Vol(M',g')$ and $\int_M s\, dM = \int_{M'} s'\, dM'$. If s' =constant, then $\int_M s^2\, dM \geq \int_{M'} s'^2\, dM'$ with equality if and only if s = constant = s'.*

A straightforward computation by using (3.4), (3.9) and (4.3) yields

$$\begin{aligned} a_{2,0} = \frac{1}{180}\int_M \{\|H\|^2 - b_0(n)\|Q\|^2 - c_0(n)s + d_0(n)\}dM \\ + \frac{e_0(n)}{360}\int s^2\, dM, \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} b_0(n) &= \frac{2n^3+7n^2-48n+40}{n^2(2n-1)} < 0 \quad \text{for } n=2; \\ c_0(n) &= \frac{2(7n+3)}{n+1}, \quad d_0(n) = \frac{2n(2n+1)(5n+1)}{n+1}\ ; \\ e_0(n) &= \frac{5n^2+4n+3}{n(n+1)} > 0\ ; \end{aligned}$$

Thus we have

THEOREM 4.2. *Let M and M' be compact Sasakian manifolds. Assume that $Spec^0 M = Spec^0 M'$. Then $dim M = dim M' = m(= 2n+1)$, and*

(a) for $m = 5$, M is of constant ϕ-holomorphic sectional curvature if and only if so is M', and s' = constant = s;

(b) when M and M' are η-Einstein and $m \geq 5$, M is of constant ϕ-holomorphic sectional curvature if and only if so is M', and $s' = s$.

Proof. Our assumption $Spec^0 M = Spec^0 M'$ implies $a_{0,0} = a'_{0,0}$ and $a_{1,0} = a'_{1,0}$. Hence (4.1) and (4.2) yield

$$Vol(M) = Vol(M'), \quad \int_M s\, dM = \int_{M'} s'\, dM'. \tag{4.11}$$

Moreover, since $a_{2,0} = a'_{2,0}$, it follows from (4.10) that

$$\begin{aligned} &\int_M \{\|H\|^2 - b_0(n)\|Q\|^2\}dM + \frac{e_0(n)}{2}\int s^2\, dM \\ &= \int_{M'} \{\|H'\|^2 - b_0(n)\|Q'\|^2\}dM + \frac{e_0(n)}{2}\int s'^2\, dM', \end{aligned} \tag{4.12}$$

(*a*) For $n = 2$, if M' is of constant holomorphic sectional curvature, then $H' = 0$ and $Q' = 0$ and consequently (4.12) gives

$$\int_M \{\|H\|^2 - b_0(n)\|Q\|^2\}dM + \frac{e_0(n)}{2}(\int s^2\, dM - \int s'^2\, dM') = 0.$$

Since $s' = constant$, Lemma 4.1 implies $\int s^2\, dM \geq \int s'^2\, dM'$ and consequently $H = 0$ and $Q = 0$. By means of Theorem 3.3 M is of constant ϕ-holomorphic sectional curvature.

(*b*) If $Q = Q' = 0$, then s and s' are both constants for $n \geq 2$. Thus (4.11) gives $s = s'$, which together with (4.12) implies

$$\int_M \|H\|^2\, dM = \int_{M'} \|H'\|^2\, dM'.$$

Hence we have our assertions. □

We next consider the case of $p = 1$. In this case it follows from (3.4), (3.9) and (4.6) that

(4.13)

$$a_{2,1} = \frac{1}{180}\int_M \{2(n-7)\|H\|^2 - b_1(n)\|Q\|^2 - c_1(n)s + d_1(n)\}dM + e_1(n)\int_M s^2 dM,$$

where

$$b_1(n) = \frac{4n^4 - 164n^3 - 119n^2 + 752n - 560}{n^2(2n-1)} < 0 \quad \text{for } 1 \leq n \leq 41\ ;$$

$$c_1(n) = \frac{2(10n^2 + 7n + 61)}{n+1}, \qquad d_1(n) = \frac{2n(2n+1)(10n^2 + 7n + 61)}{n+1};$$

$$e_1(n) = \frac{(n-3)(2n+1)(5n-11)}{n(n+1)} > 0 \qquad \text{for } n \neq 3;$$

Thus we have

THEOREM 4.3. *Let M and M' be compact Sasakian manifolds. Assume that $Spec^1 M = Spec^1 M'$. Then $dim M = dim M' = m(= 2n+1)$, and*

(*a*) *for $17 \leq m \leq 83$, M is of constant ϕ-holomorphic sectional curvature if and only if so is M', and $s' = constant = s$;*

(*b*) *when M and M' are η-Einstein, $m \geq 15$ and $m \neq 15$, M is of constant ϕ-holomorphic sectional curvature if and only if so is M', and $s' = s$.*

Proof. Our assumption $Spec^1 M = Spec^1 M'$ implies $a_{0,1} = a'_{0,1}$ and $a_{1,1} = a'_{1,1}$. Hence (4.4) and (4.5) yield

$$Vol(M) = Vol(M'), \quad \int_M s\, dM = \int_{M'} s'\, dM'. \tag{4.14}$$

Moreover, since $a_{2,1} = a'_{2,1}$, it follows from (4.13) that

(4.15)
$$\int_M \{2(n-7)\|H\|^2 - b_1(n)\|Q\|^2\}dM + 180e_1(n)\int s^2\, dM$$
$$= \int_{M'} \{2(n-7))\|H'\|^2 - b_1(n)\|Q'\|^2\}dM + 180e_1(n)\int s'^2\, dM'.$$

(a) For $8 \le n \le 41$, if M' is of constant ϕ-holomorphic sectional curvature, then $H' = 0$ and $Q' = 0$ and consequently (4.15) gives

$$\int_M \{2(n-7)\|H\|^2 - b_1(n)\|Q\|^2\}dM$$
$$+ 180e_1(n)(\int s^2\, dM - \int s'^2\, dM') = 0.$$

Since $s' = constant$, Lemma 4.1 implies $\int s^2\, dM \ge \int s'^2\, dM'$ and consequently $H = 0$ and $Q = 0$. By means of Theorem 3.3 M is of constant ϕ-holomorphic sectional curvature.

(b) When $Q = Q' = 0$ and $n \ge 2$, s and s' are both constants. Thus (4.14) gives $s = s'$, which together with (4.15) implies

$$\int_M \|H\|^2\, dM = \int_{M'} \|H'\|^2\, dM',$$

provided $n \ne 7$. Hence we have our assertions. □

THEOREM 4.4. *Let M and M' be compact Sasakian manifolds. Assume that $Spec^0 M = Spec^0 M'$ and $Spec^1 M = Spec^1 M'$. Then $dim M = dim M' = m(= 2n+1)$, and*

(a) for $m = 5, 7, 9$, M is of constant ϕ-holomorphic sectional curvature if and only if so is M', and $s' = constant = s$;

(b) for $m \ge 5$, M is η-Einstein if and only if so is M', and $s' = s$.

Proof. Our assumption $Spec^0 M = Spec^0 M'$ implies $a_{0,0} = a'_{0,0}$ and $a_{1,0} = a'_{1,0}$. Hence (4.1) and (4.2) yield

$$Vol(M) = Vol(M'), \quad \int_M s\, dM = \int_{M'} s'\, dM'. \tag{4.16}$$

Moreover, the assumptions $Spec^0 M = Spec^0 M'$ and $Spec^1 M = Spec^1 M'$ give $a_{2,0} = a'_{2,0}$ and $a_{2,1} = a'_{2,1}$, from which together with (4.3) and (4.6), we have

$$\int_M (5\|R\|^2 + 13s^2) dM = \int_{M'} (5\|R'\|^2 + 13s'^2) dM'. \tag{4.17}$$

It follows from (3.9) and (4.17) that

(4.18)

$$\begin{aligned}&\int_M \{5\|H\|^2 - b_{0,1}(n)\|Q\|^2 - c_{0,1}(n)s + d_{0,1}(n) + e_{0,1}(n)s^2\} dM \\ = &\int_{M'} \{5\|H'\|^2 - b_{0,1}(n)\|Q'\|^2 - c_{0,1}(n)s' + d_{0,1}(n) + e_{0,1}(n)s'^2\} dM',\end{aligned}$$

where

$$\begin{aligned} b_{0,1} &= \frac{40(n-1)(n-5)}{n^2(2n-1)} < 0 \quad \text{for } n = 2,3,4\ ; \\ c_{0,1} &= \frac{20(3n+1)}{n+1}, \quad d_{0,1} = \frac{20(2n+1)(3n+1)}{n+1}\ ; \\ e_{0,1} &= \frac{13n^2 + 13n + 10}{n(n+1)} > 0. \end{aligned}$$

(*a*) For $n = 2, 3, 4$, if M' is of constant ϕ-holomorphic sectional curvature, then $H' = 0$, $Q' = 0$ and (4.16) and consequently (4.18) gives

$$\int_M \{5\|H\|^2 - b_{0,1}(n)\|Q\|^2\} dM + e_1(n)\left(\int s^2\, dM - \int s'^2\, dM'\right) = 0.$$

Since $s' = constant$, Lemma 4.1 implies $\int s^2\, dM \geq \int s'^2\, dM'$ and consequently $H = 0$ and $Q = 0$. By means of Theorem 3.3 M is of constant ϕ-holomorphic sectional curvature.

(b) When $Q = Q' = 0$ and $n \geq 2$, s and s' are both constants. Thus (4.16) gives $s = s'$, which together with (4.18) implies

$$\int_M \|H\|^2 \, dM = \int_{M'} \|H'\|^2 \, dM'.$$

Hence we have our assertions. □

Finally we consider the case of $p = 2$. In this case it follows from (3.4), (3.9) and (4.9) that

(4.19)

$$a_{2,2} = \frac{1}{720}\int_M \{4(n-7)(2n-15)\|H\|^2 - b_2(n)\|Q\|^2 - c_2(n)s + d_2(n) + e_2(n)s^2\}dM,$$

where

$$b_2(n) = \frac{4(4n^5 - 344n^4 + 751n^3 + 1862n^2 - 6200n + 4200)}{n^2(2n-1)} < 0 \quad \text{for} \quad 2 \leq n \leq 83\,;$$

$$c_2(n) = \frac{8(10n^3 + 7n^2 + 301n - 240)}{n+1}$$

$$d_2(n) = \frac{8n(2n+1)(10n^3 + 7n^2 + 301n - 240)}{n+1}$$

$$e_2(n) = \frac{2(10n^4 - 107n^3 + 310n^2 - 147n - 30)}{n(n+1)} > 0, \text{ for } n \geq 1 \ (n \neq 5);$$

Thus we have

THEOREM 4.5. *Let M and M' be compact Sasakian manifolds. Assume that $Spec^2 M = Spec^2 M'$. Then $dim M = dim M' = m (= 2n+1)$, and*

(a) for $m = 15$, M is of constant ϕ-holomorphic sectional curvature if and only if so is M', $s' = constant = s$;

(b) when M and M' are η-Einstein, $m \geq 5$ and $m \neq 15$, M is of constant ϕ-holomorphic sectional curvature if and only if so is M', and $s' = s$.

Proof. Our assumption $Spec^2 M = Spec^2 M'$ implies $a_{0,2} = a'_{0,2}$ and $a_{1,2} = a'_{1,2}$. Hence (4.7) and (4.8) yield

$$Vol(M) = Vol(M'), \quad \int_M s \, dM = \int_{M'} s' \, dM'. \tag{4.20}$$

Moreover, since $a_{2,2} = a'_{2,2}$, it follows from (4.19) that

(4.21)

$$\int_M 4(n-7)(2n-15)\|H\|^2 - b_2(n)\|Q\|^2 + e_2(n)s^2 \, dM$$
$$= \int_{M'} \{4(n-7)(2n-15)\|H'\|^2 - b_2(n)\|Q'\|^2 + e_2(n)s'^2 \, dM'.$$

(a) is trivial.

(b) When $Q = Q' = 0$ and $n \geq 2$, s and s' are both constants. Thus (4.20) gives $s = s'$, which together with (4.21) implies

$$\int_M \|H\|^2 \, dM = \int_{M'} \|H'\|^2 \, dM',$$

provided $n \neq 7$. Hence we have our assertions. □

In the above Theorem 4.5 (a), if $m \neq 15$ then J. S. Pak, J. C. Jeong and W. -T. Kim have obtained the following result.

THEOREM 4.6 ([7]). *Let M and M' be compact Sasakian manifolds. Assume that $Spec^2 M = Spec^2 M'$. Then $dim M = dim M' = m (= 2n+1)$. For $m = 3, 5, 7, 9, 11$ and 13, or $17 \leq m \leq 187$, M is of constant ϕ-sectional curvature k if and only if M' is of constant ϕ-sectional curvature $k' = k$.*

THEOREM 4.7. *Let M and M' be compact Sasakian manifolds. Assume that $Spec^0 M = Spec^0 M'$ and $Spec^2 M = Spec^2 M'$. Then $dim M = dim M' = m (= 2n+1)$, and*

(a) for $m = 7, 9$, M is of constant ϕ-holomorphic sectional curvature if and only if so is M', $s' =$ constant $= s$.

(b) for $m \geq 5$, M is η-Einstein if and only if so is M', $s' = s$;

Proof. Our assumption $Spec^0 M = Spec^0 M'$ yields $a_{0,0} = a'_{0,0}$ and $a_{1,0} = a'_{1,0}$. Hence it follows from (4.1) and (4.2) that

$$Vol(M) = Vol(M'), \quad \int_M s\, dM = \int_{M'} s'\, dM'. \tag{4.22}$$

Moreover, the assumptions $Spec^0 M = Spec^0 M'$ and $Spec^2 M = Spec^2 M'$ give $a_{2,0} = a'_{2,0}$ and $a_{2,2} = a'_{2,2}$, from which together with (4.3) and (4.6), we have

$$\begin{aligned}\int_M \{(5m-28)\|R\|^2 + (13m-80)s^2\}dM \\ = \int_{M'} \{(5m-28)\|R'\|^2 + (13m-80)s'^2\}dM.\end{aligned} \tag{4.23}$$

It follows from (3.4), (4.23) and $m = 2n+1$ that

(4.24)

$$\begin{aligned}&\int_M \{(10n-23)\|H\|^2 - b_{0,2}(n)\|Q\|^2 - c_{0,2}(n)s + d_{0,2}(n) \\ &+ e_{0,2}(n)s^2\}dM \\ = &\int_{M'} \{(10n-23)\|H'\|^2 + b_{0,2}(n)\|Q'\|^2 - c_{0,2}(n)s' + d_{0,2}(n) \\ &+ e_{0,2}(n)s'^2\}dM',\end{aligned}$$

where

$$\begin{aligned} b_{0,2}(n) &= \frac{8(n-5)(n-1)(10n-23)}{n^2(2n-1)} < 0 \qquad \text{for } n = 3,\ 4\ ; \\ c_{0,2}(n) &= \frac{4(3n+1)(10n-23)}{n+1} \\ d_{0,2}(n) &= \frac{4n(2n+1)(3nb+1)(10n-23)}{n+1} \\ e_{0,2}(n) &= \frac{26n^3 - 15n^2 - 114n - 46}{n(n+2)} > 0, \quad \text{for} \quad n \geq 3\ ; \end{aligned}$$

(a) For $n = 3, 4$, if M' is of constant ϕ-holomorphic sectional curvature, then $H' = 0$ and $Q' = 0$ and consequently (4.24) gives

$$\begin{aligned}\int_M \{(10n-23)\|H\|^2 - b_{0,2}(n)\|Q\|^2\}dM \\ + e_{0,2}(n)(\int s^2\, dM - \int s'^2\, dM') = 0.\end{aligned}$$

Since $s' = constant$, Lemma 4.1 implies $\int s^2\, dM \geq \int s'^2\, dM'$ and consequently $H = 0$ and $Q = 0$. By means of Theorem 3.3 M is of constant ϕ-holomorphic sectional curvature.

(*b*) When $Q = Q' = 0$ and $n \geq 2$, s and s' are both constants. Thus (4.22) gives $s = s'$, which together with (4.24) implies

$$\int_M \|H\|^2\, dM = \int_{M'} \|H'\|^2\, dM'.$$

Hence we have our assertions □

THEOREM 4.8. *Let M and M' be compact Sasakian manifolds. Assume that $Spec^1 M = Spec^1 M'$ and $Spec^2 M = Spec^2 M'$. Then $dim M = dim M' = m (= 2n+1)$. Especially, for $m \geq 5$ and $m \neq 15$, M is η-Einstein if and only if so is M', $s' = s$;*

Proof. Our assumption $Spec^1 M = Spec^1 M'$ yields $a_{0,1} = a'_{0,1}$ and consequently it follows from (4.4) that $Vol(M) = Vol(M')$. Since $Spec^2 M = Spec^2 M'$, $a_{1,2} = a'_{1,2}$ yields $\int_M s\, dM = \int_{M'} s'\, dM'$. Summing up, we have

$$Vol(M) = Vol(M'), \quad \int_M s\, dM = \int_{M'} s'\, dM'. \tag{4.25}$$

Moreover, the assumptions $Spec^1 M = Spec^1 M'$ and $Spec^2 M = Spec^2 M'$ give $a_{2,1} = a'_{2,1}$ and $a_{2,2} = a'_{2,2}$, from which together with (4.6) and (4.9), we have

(4.26)
$$\begin{aligned}&\int_M \{(5m^2 - 51m - 360)\|R\|^2 + (13m^2 - 147m + 360)s^2\}dM \\ &= \int_{M'} \{(5m^2 - 51m - 360)\|R'\|^2 + (13m^2 - 147m + 360)s'^2\}dM.\end{aligned}$$

It follows from (3.9), (4.26) and $m = 2n+1$ that

(4.27)
$$\begin{aligned}&\int_M \{2(n-7)(10n+29)\|H\|^2 - b_{1,2}(n)\|Q\|^2 - c_{1,2}(n)s + d_{1,2}(n) \\ &+ e_{1,2}(n)s^2\}dM \\ &= \int_{M'} \{2(n-7)(10n+29)\|H'\|^2 - b_{1,2}(n)\|Q'\|^2 - c_{1,2}(n)s' + d_{1,2}(n) \\ &+ e_{1,2}(n)s'^2\}dM',\end{aligned}$$

where

$$
\begin{aligned}
b_{1,2}(n) &= \frac{16(n-5)(n-1)(n-7)(10n+29)}{n^2(2n-1)} < 0 \qquad \text{for } n = 6\ ; \\
c_{1,2}(n) &= \frac{8(3n+1)(n-7)(10n+29)}{n+1} \\
d_{1,2}(n) &= \frac{8n(2n+1)(3n+1)(n-7)(10n+29)}{n+1} \\
e_{1,2}(n) &= \frac{2(26n^4 - 69n^3 - 109n^2 + 144n - 406)}{n(n+2)} > 0, \quad \text{for } n \geq 4.
\end{aligned}
$$

When $Q = Q' = 0$ and $n \geq 2$, s and s' are both constants. Thus (4.25) gives $s = s'$, which together with (4.27) implies

$$\int_M \|H\|^2 \, dM = \int_{M'} \|H'\|^2 \, dM',$$

provided $n \neq 7$. Hence we have our assertions. □

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