Bull. Korean Math. Soc. **43** (2006), No. 1, pp. 77–80

RELATIVE INTEGRAL BASES OVER A RAY CLASS FIELD

SoYoung Choi

Abstract. Let K be a number field, $K_{\mathfrak{n}}$ its ray class field with conductor $\mathfrak{n}$ and L a Galois extension of K containing $K_{\mathfrak{n}}$. We prove that $L/K_{\mathfrak{n}}$ has a relative integral basis (RIB) under certain simple condition. Also we reduce the problem of the existence of a RIB to a quadratic extension of $K_{\mathfrak{n}}$ under some condition.

1. Introduction

Let L be an algebraic number field, K be a subfield of it. Let $\mathcal{O}_L$ and $\mathcal{O}_K$ be the rings of integers in L and K, respectively. If $\mathcal{O}_L$ is free as $\mathcal{O}_K$-module, then we say that L/K has a relative integral basis (RIB). Artin in [1] raised the problem : when does L/K has a relative integral basis ?

XianKe Zhang and FuHua Xu in [5] proved the existence of relative integral bases for extensions of n-cyclic number fields under some conditions. Mario Daberkow and Michael Pohst in [2] studied relative integral bases in relative quadratic extensions. Elena Soverchia in [4] showed the following : Let H be the Hilbert class field of an algebraic number field K and L be a Galois extension of K containing H. If the order of $Gal(L/H)$ is odd or if the 2-Sylow subgroups of $Gal(L/H)$ are not cyclic, then L/H has a relative integral basis.

It is natural to investigate analogues of Soverchia's work for more general class fields of K. Let K be an algebraic number field and $K_{\mathfrak{n}}$ be its ray class field with conductor $\mathfrak{n}$ and with genus number 1 over K. Let L/K be a Galois extension containing $K_{\mathfrak{n}}$. We suppose that L/K is unramified at all primes $\mathcal{B}$ dividing $\mathfrak{n}\mathcal{O}_{K_{\mathfrak{n}}}$. For the convenience, we

Received September 7, 2004.
2000 Mathematics Subject Classification: 11R04, 11R29, 11R37.
Key words and phrases: relative integral bases, ray class field.
This work was supported by the Korea Research Foundation Grant.(KRF-2005-214-M01-2005-000-10100-0).

assume that $\mathfrak{n}$ is an integral divisor. We denote the discriminant of a field basis of $L/K_{\mathfrak{n}}$ by $\triangle$. In this paper, we will prove that $L/K_{\mathfrak{n}}$ has a relative integral basis if h is an odd number or if $\triangle$ is contained in $K_{\mathfrak{n}}^2$. We also we reduce the problem of the existence of a RIB to a quadratic extension of $K_{\mathfrak{n}}$ if h is an even number and $\triangle$ is not contained in $K_{\mathfrak{n}}^2$, where h be the class number of the field $K_{\mathfrak{n}}$ (Theorem 4). We emphasize that our results (with respect to the ray class field $K_{\mathfrak{n}}$ of K) generalize Soverchia's results (with respect to the Hilbert class field H of K).

2. Relative integral basis over $K_{\mathfrak{n}}$

To prove our theorem, we need some lemmas.
We denote the relative discriminant of a field extension E/F by $d(E|F)$.

LEMMA 1. *Let E/F an extension of number fields. Then there exists a non-zero fractional ideal $\mathcal{B}$ in F such that $d(E|F) = \Diamond\mathcal{B}^2$, where $\Diamond$ is the discriminant of a field basis of E/F. Moreover, E/F has a RIB if and only if $\mathcal{B}$ is principal.*

Proof. See [1]. □

LEMMA 2. *Let E/K be a Galois extension of number fields containing $K_{\mathfrak{n}}$. Suppose that E is unramified at all primes $\mathcal{B}$ dividing $\mathfrak{n}\mathcal{O}_{K_{\mathfrak{n}}}$. Then $d(E|K_{\mathfrak{n}})$ is stable under the action of $Gal(K_{\mathfrak{n}}/K)$.*

Proof. Let $\mathfrak{D}_{E/K}$ (respectively, $\mathfrak{D}_{E/K_{\mathfrak{n}}}$) be the different of $\mathcal{O}_E$ over K (respectively, $K_{\mathfrak{n}}$). Since $\mathfrak{D}_{E/K}$ (respectively, $\mathfrak{D}_{E/K_{\mathfrak{n}}}$) is stable under the action of $Gal(E/K)$ (respectively, $Gal(E/K_{\mathfrak{n}})$), $N^E_{K_{\mathfrak{n}}}\mathfrak{D}_{E/K_{\mathfrak{n}}} = d(E|K_{\mathfrak{n}})$ and $K_{\mathfrak{n}}$ is unramified at any prime $\mathfrak{p}$ in K which is below a prime dividing $d(E|K_{\mathfrak{n}})$, we have $d(E|K_{\mathfrak{n}})=\mathfrak{p}_1^{t_1}\cdots\mathfrak{p}_r^{t_r}$ for some prime ideals $\mathfrak{p}_i$ in K and some integers t_i. Hence $d(E|K_{\mathfrak{n}})$ is stable under the action of $\mathrm{Gal}(K_{\mathfrak{n}}/K)$. □

LEMMA 3. *Suppose that the genus number of $K_{\mathfrak{n}}$ over K is equal to 1. Then every ideal of $K_{\mathfrak{n}}$ prime to $\mathfrak{n}$ and stable under the action of $Gal(K_{\mathfrak{n}}/K)$ is principal.*

Proof. Let H be the Hilbert class field of $K_{\mathfrak{n}}$ and $G = Gal(H/K)$. Since the genus number of $K_{\mathfrak{n}}$ over K is equal to 1, we have $Gal(H/K_{\mathfrak{n}}) = G'$ and $Gal(K_{\mathfrak{n}}/K)=G/G'$, where G' is the commutator subgroup of G. Let $I_{\mathfrak{n}}(K)$ (respectively, $I(K_{\mathfrak{n}})$) be the ideal group generated by all fractional ideals in K prime to $\mathfrak{n}$ (respectively, by all fractional ideals in $K_{\mathfrak{n}}$) and $P_{\mathfrak{n},1}(K)$ (respectively, $P(K_{\mathfrak{n}})$) be the subgroup of $I_{\mathfrak{n}}(K)$ generated by the principal ideals $\beta\mathcal{O}_K$ with $\beta \in \mathcal{O}_K$ and $\beta \equiv 1 \bmod \mathfrak{n}\mathcal{O}_K$ (respectively, of $I(K_\mathfrak{n})$ generated by the principal ideals in $K_\mathfrak{n}$) where $\mathcal{O}_K$ is the ring of integers in K. Naturally, we obtain a chain of maps

$$G \xrightarrow[\text{map}]{\text{natural}} \frac{G}{G'} \xrightarrow{[\cdot,K]^{-1}} \frac{I_\mathfrak{n}(K)}{P_{\mathfrak{n},1}(K)} \xrightarrow[\text{map}]{\text{natural}} \frac{I(K_\mathfrak{n})}{P(K_\mathfrak{n})} \xrightarrow{[\cdot,K_\mathfrak{n}]} G',$$

where $[\cdot,K]$ and $[\cdot,K_\mathfrak{n}]$ are Artin maps. A brief check of the coset representatives shows that this chain of maps is a transfer V of G into G'. By the principal ideal theorem of group theory, $V(\sigma) = 1$ for all $\sigma \in G$. This implies our assertion. □

THEOREM 4. *Let K be an algebraic number field and $K_\mathfrak{n}$ be its ray class field with conductor $\mathfrak{n}$ and with genus number 1 over K. Let L/K be a Galois extension containing $K_\mathfrak{n}$. We suppose that L/K is unramified at all primes $\mathcal{B}$ dividing $\mathfrak{n}\mathcal{O}_{K_\mathfrak{n}}$ and that $\mathfrak{n}$ is an integral divisor. Let h be the class number of the field $K_\mathfrak{n}$. Then we have the following:*

(1) *If h is an odd number or if $\triangle$ is contained in $K_\mathfrak{n}^2$, then $L/K_\mathfrak{n}$ has a RIB.*
(2) *If h is an even number and if $\triangle$ is not contained in $K_\mathfrak{n}^2$, then for the field $M = K_\mathfrak{n}(\sqrt{\triangle})$, $L/K_\mathfrak{n}$ has a RIB if and only if $M/K_\mathfrak{n}$ has a RIB.*

Proof. Let $\mathcal{B}$ be a fractional ideal in $K_\mathfrak{n}$ such that $d(L|K_\mathfrak{n}) = \triangle\mathcal{B}^2$. Lemma 2 and Lemma 3 imply that $d(L|K_\mathfrak{n})$ is principal. Hence if h is an odd number, then $\mathcal{B}$ is principal. Suppose that $\triangle$ is contained in $K_\mathfrak{n}^2$. Then $\sqrt{\triangle}\mathcal{B}$ is stable under the action of $Gal(K_\mathfrak{n}/K)$. By Lemma 2 and Lemma 3, $\sqrt{\triangle}\mathcal{B}$ is principal. This implies that $\mathcal{B}$ is principal. Now we assume that h is an even number and that $\triangle$ is not contained in $K_\mathfrak{n}^2$. We let $\mathcal{D}$ be a fractional ideal in $K_\mathfrak{n}$ such that $d(M|K_\mathfrak{n}) = 4\triangle\mathcal{D}^2$. From Lemma 2, $\mathcal{D}\mathcal{B}^{-1}$ is stable under the action of $Gal(K_\mathfrak{n}/K)$. Hence $\mathcal{D}\mathcal{B}^{-1}$ is principal. These and Lemma 1 prove the assertions. □

REMARK. Replacing H in [4, Lemma 2.2] by $K_\mathfrak{n}$, we obtain the following equivalent statements: the order of $Gal(L/K_\mathfrak{n})$ is odd or the 2-Sylow subgroup of G are not cyclic if and only if $\triangle$ is contained in $K_\mathfrak{n}^2$.

EXAMPLE. For any prime p, let $\zeta_p = e^{\frac{2\pi i}{p}}$ and K the rational number field. Then K_p is the maximal real subfield $\mathbb{Q}(\zeta_p+\zeta_p^{-1})$ of p-th cyclotomic number field $\mathbb{Q}(\zeta_p)$ and satisfies the conditions in Theorem 4. Indeed, in narrow sense the field $\mathbb{Q}(\zeta_p)$ has genus number 1 over K from the genus number formula

$$g(\mathbb{Q}(\zeta_p)) = \frac{e(p)}{[\mathbb{Q}(\zeta_p) : K]},$$

given in [3, p.53], where $e(p)$ denotes the ramification index of the prime p in $\mathbb{Q}(\zeta_p)/K$. Thus the field K_p has genus number 1 over K.

References

[1] E. Artin, *Questions de base minimale dans la théorie des nombres algébriques*, Colloques Internationauz du Centre National de la Recherche Scientifique, no. 24, Centre National de la Recherche Scientifique, Paris, 1950.

[2] M. Daberkow and M. Pohst, *On integral bases in relative quadratic extensions*, Math. Comp. **65** (1996), no. 213, 319–329.

[3] M. Ishida, *The genus fields of algebraic number fields*, Lecture Notes in Mathematics Vol 555, Springer-Verlag, Berlin-New York, 1976.

[4] E. Soverchia, *Relative integral bases over a Hilbert class field*, J. Number Theory **97** (2002), no. 2, 199–203.

[5] X. Zhang and F. Xu, *Existence of integral bases for relative extensions of n-cyclic number fields*, J. Number Theory **60** (1996), no. 2, 409–416.

FR 6.1 Mathematik, Universität des Saarlandes, Postfach 151150, D-66041 Saarbrücken, Germany
E-mail: young@math.uni-sb.de & young@math.kaist.ac.kr